Posters
Algebra
Matrix valuation pseudo ring

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Abstract
Let \( R \) be a ring and \( V \) be a matrix valuation on \( R \), \( M(R) \) and \( GL(R) \) will represent, an associative ring with unit, the set of all square matrices, and the set of invertible matrices over \( R \) respectively. It is shown that, there exits a correspondence between matrix valuations on \( R \) and some special subsets \( \sum(MVPR) \) of the set of all square matrices over \( R \), analogous there exists a natural bijection between the matrix valuations on \( R \) and valuated epic \( R \)-fields.

Keywords: Matrix valuation, Simple valuation, Matrix valuation pseudo-Epic field

Mathematics Subject Classification: 13A18

1 Introduction
In the classical commutative and non-commutative field theory, the equivalence of valuations and valuation rings are considered as a natural base for the study of these notions. For any ring \( R \) we denote by \( M(R) \) the set of all square matrices. This will include in particular the unique \( 0 \times 0 \) matrix over \( R \). On \( M(R) \) we have two operations, the diagonal sum, the determinantal sum. By \( \Gamma \) we shall again denote an abelian ordered group, written additively.

2 Matrix valuations and their pseudo

Definition 2.1. A matrix valuation on a ring \( R \) is a function \( V \) on \( M(R) \) with values in \( \Gamma \cup \{\infty\} \) such that:

\( MV.1 \) \( V(A) = \infty \) for any non-full matrix \( A \) over \( R \).

\( MV.2 \) \( V(A \oplus B) = V(A) \oplus V(B) \) for all \( A, B \in M(R) \) such that \( A \nabla B \) is defined.

\( MV.3 \) \( V(A \nabla B) \geq \min\{V(A), V(B)\} \), for all \( A, B \in M(R) \) such that \( A \nabla B \) is defined.

\( MV.4 \) \( V(A) \) is unchanged if any row or column is multiplied by \((-1)\).

\( MV.5 \) \( V(1) = 0 \).

\( MV.6 \) If \( A \nabla B \) is defined and \( V(A) \neq V(B) \), then \( V(A \nabla B) = \min\{V(A), V(B)\} \).

\( MV.7 \) \( V(A) \) remains unchanged under any permutation of rows (or columns).
\[
V \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = V \begin{bmatrix} A & 0 \\ D & B \end{bmatrix} = V(A) + V(B)
\]

for any \(C, D\) of appropriate size.

**MV.9** \(V(AB) = V(A) + V(B)\) for square matrices \(A, B\) of the same order.

**Example 2.2.** Let \(K\) be a skew field with an abelian valuation \(\nu\), then \(\nu\) may be extended to a matrix valuation on \(K\) by the equation \(V(A) = \nu(\text{Det}A)\).

**Remark 2.3.** Let \(V\) be a matrix valuation on a division ring \(D\) and let \(\sum_V = \{A \in M(D) : V(A) \geq 0\}\), this subset of \(M(D)\) has the following properties:

1. It contains all non-full matrices.
2. \(1 \in \sum_V\).
3. \(A, B \in \sum_V \implies A \oplus B \in \sum_V\).
4. \(A, B \in \sum_V\) and \(A\nabla B\) is defined \(\implies A\nabla B \in \sum_V\).
5. \(A \oplus I \in \sum_V \implies A \in \sum_V\).
6. \(A \in \text{GL}(D) \implies A \in \sum_V\) or \(A^{-1} \in \sum_V\).
7. \(A \oplus B \in \sum_V \implies B \oplus A \in \sum_V\).
8. \(A, B\) are of the same size and \(A \oplus B \in \sum_V \implies AB \in \sum_V\).
9. \(A \in \sum_V \iff B \oplus A \oplus B^{-1} \in \sum_V\) for all \(B \in \text{GL}(D)\).

we define:

**Definition 2.4.** For any ring \(R\) if a set \(\sum\) has the first five conditions in above, we call it is a matrix pseudo ring (briefly MPR). Reasonably any matrix pseudo ring with the sixth condition in (Remark 2.2) could be called a matrix valuation pseudo ring (MVPR). To sum up, we have proved that:

**Proposition 2.5.** There is a 1-1 correspondence between matrix valuations and MVPR on a division ring \(D\).

**Proposition 2.6.** There is a 1-1 correspondence between matrix valuations and pseudo matrix valuation rings on any ring embeddable in a division ring.

The following theorem states a generalization of the above propositions.

**Theorem 2.7.** There is a 1-1 correspondence between matrix valuations and MVPRS on any ring.

**Proof.** Let \(R\) be any ring and \(\sum\) be its MVPR, which is a proper subset of \(M(R)\). \(\sum\) contains all of the non-full matrices, hence it contains the least matrix ideal containing non-full matrices. In other words, this matrix ideal is a proper matrix ideal. The other side is clear.
3 Simple valuation on matrices

Definition 3.1. Let \( \Gamma \) be a totally ordered abelian group. A simple valuation on \( A = M_n(D) \) is a function \( \mu : A \rightarrow \Gamma \cup \{\infty\} \) satisfying:

1. \( \mu(XY) = \mu(X) + \mu(Y) \)
2. \( \mu(X\nabla Y) \geq \min\{\mu(X), \mu(Y)\} \), for all \( X, Y \in A \) such that \( X\nabla Y \) is defined.
3. \( \mu(X) \) is unchanged if any row or column of \( X \) is multiplied by \((-1)\).
4. \( \mu(X) = \infty \) for any singular matrix \( X \in A \).

Theorem 3.2. Let \( R \) be a simple ring and \( D \) be a division ring. Then \( R \) and \( M_n(D) \) are isomorphic.

Theorem 3.3. Let \( D \) be a division ring with an abelian valuation \( \nu \). Then \( \nu \) may be extended to a simple valuation \( \mu \) on \( M_n(D) \), for each \( n \geq 1 \), by the equation \( \mu(X) = \nu(DetX) \), \( X \in A \), where \( Det \) denotes the dieudonne’ determinant, together with the rule \( \mu(X) = \infty \) when \( X \) is singular. Moreover, the correspondence \( \nu \leftrightarrow \mu \) is bijection between abelian valuations on \( D \) and simple valuation on \( A \).

Remark 3.4. Finally matrix valuation to express holderi valuation and expect that we can made a simple valuation into holderi valuation.

References


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Tensor product of multiplication modules

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Abstract
In this paper, we shall consider some properties are conserved under tensor product operations. We investigate some results on faithful multiplication free R-modules. All rings are commutative with identity and all modules are unital.

Keywords: Tensor Product, Faithful Multiplication Module, Cancellation Module.

Mathematics Subject Classification: 15A69

1 Introduction
Tensor product is one of the most important subject of diverse branches of mathematics especially in multilinear algebras. Moreover it is one of fundamental factors in homologic algebras. In addition tensor product indicate a critical role in quantum theory as well mathematics.

Definition 1.1. An R-module M is called a multiplication module if for every submodule N of M there exists an ideal I of R such that N=IM.

Definition 1.2. An R-module M is called a cancellation module if for ideals I and J of R, IM=JM implies that I=J.

Definition 1.3. For an R-module M, the set
$$\text{Ann}_R(M) = \{a \in R | aM = \{0\}\}$$
is called annihilator of M.

Definition 1.4. An R-module M is called a faithful module, if \(\text{Ann}_R(M) = 0\).

Definition 1.5. Let N be a submodule of R-module M, then the set
$$\langle N:R \rangle M = \{r \in R : rM \subseteq N\}$$
is called colon of N.

Definition 1.6. A submodule N of R-module M is called prime submodule if N ≠ M and for r ∈ R and m ∈ M, we have
$$rm \in N \Rightarrow r \in (N:R)M \text{ or } m \in M.$$
Theorem 1.7. Let \( M \) and \( M' \) are faithful multiplication \( R \)-module. If \( N \) be a prime submodule of \( M \) and \( K \) be a prime submodule of \( M' \), then there exist unique prime ideals \( P \supseteq \text{Ann}(M) \) and \( Q \supseteq \text{Ann}(M') \) of \( R \) where \( N \otimes M \cong PM \otimes QM' \).

Proof. [2]

Corollary 1.8. Let \( M \) and \( M' \) are faithful multiplication free \( R \)-module and \( N \leq M \), \( K \leq M' \) where \( N = IM \) and \( K = JM' \), then

\[
\text{rad}(N \otimes K) = \sqrt{(N \otimes K : M \otimes M')(M \otimes M')}
\]

\[
\cong \sqrt{(N : M) \otimes (K : M')(M \otimes M')} = \sqrt{I \otimes J(M \otimes M')}.
\]

Proof. We note that \( M \otimes M' \) is a cancellation multiplication \( R \)-module, hence for any submodule \( N \otimes K \),

\[
N \otimes K = (N \otimes K : M \otimes M')(M \otimes M')
\]

Since \( M \) and \( M' \) are faithful multiplication \( R \)-modules, then

\[
(N \otimes K : M \otimes M') \cong (N : M) \otimes (K : M')
\]

Now since \( M \otimes M' \) is free, hence \( \text{rad}(I(M \otimes M')) = \sqrt{I(M \otimes M')} \) for every ideal \( I \) of \( R \).

2 Main Result

Theorem 2.1. Let \( M \) and \( M' \) are faithful multiplication free \( R \)-module and \( N \leq M \), \( K \leq M' \) where \( N = IM \) and \( K = JM' \), then

i) \( \text{rad}(N \otimes M') \cong (\text{rad}N) \otimes M' \).

ii) \( \text{rad}(M \otimes K) \cong M \otimes (\text{rad}K) \)

Proof. i) Since \( M \otimes M' \) is a faithful multiplication free \( R \)-module, then

\[
\text{rad}(N \otimes M') = \text{rad}(IM \otimes M') \cong \text{rad}(I(M \otimes M')) = \sqrt{I(M \otimes M')}
\]

\[
\cong \sqrt{IM \otimes M'} = (\text{rad}N) \otimes M'.
\]

ii) similarly,

\[
\text{rad}(M \otimes K) = \text{rad}(M \otimes JM') \cong \text{rad}(J(M \otimes M')) = \sqrt{J(M \otimes M')}
\]

\[
\cong M \otimes \sqrt{JM'} = M \otimes \text{rad}(K).
\]

Corollary 2.2. Let \( M \) and \( M' \) are faithful multiplication free \( R \)-module and \( N \leq M \), \( K \leq M' \), where \( M = \oplus_{\alpha \in \Gamma} M_{\alpha} \) and \( N = \oplus_{\alpha \in \Gamma} N_{\alpha} \) where \( N_{\alpha} \) is a submodule of \( M_{\alpha} \) for any \( \alpha \in \Gamma \), then

\[
\text{rad} \left( \bigoplus N_{\alpha} \otimes M' \right) \cong \text{rad} \left( \bigoplus (N_{\alpha} \otimes M') \right) \cong \bigoplus \text{rad}(N_{\alpha} \otimes M')
\]

\[
\cong \bigoplus (\text{rad}N_{\alpha}) \otimes M'
\]
References


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Introducing a large non-commuting subset of general linear group using maximal torus

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Abstract
Let $V$ be an $n$-dimensional vector space over a finite field $F$ with $|F| = q$ and let $G = GL(n,q)$ be a $n$-dimensional general linear group. A subset $N$ of $G$ is said to be non-commuting if $xy \neq yx$ for any two distinct elements $x, y \in N$. Using maximal torus we introduce a large non-commuting subset of $n$-dimensional general linear group.

Keywords: general linear group, maximal torus

Mathematics Subject Classification: 20D60

1 Introduction
Let $G$ be a non-abelian group and $Z(G)$ be its center. We call a subset $N$ of $G$ non-commuting if $xy \neq yx$ for any distinct elements $x, y \in N$. If $|N| \geq |M|$ for any other non-commuting subset $M$, then $N$ is said to be a maximal non-commuting subset. The cardinality of such a subset is denoted by $\omega(G)$. By a famous result of Neumann [5] answering a question of P.Erdős, we know that the finiteness of $\omega(G)$ in $G$ implies the finiteness of the factor group $\frac{G}{Z(G)}$. In [2], it was proved that $\omega(GL(3,q)) = q^6 + q^5 + 3q^4 + 3q^3 + q^2 - q - 1$. In this paper we consider maximal non-commuting subset in general linear groups of dimension $n$ over a finite field of order $q$, and obtain lower bounds for their cardinality.

Definition 1.1. Let $g \in GL(n,q)$ where $q = p^k$, $p$ a prime, and $|g| = q^n - 1$. Then $\langle g \rangle$ is called a Singer cycle subgroup of $G$.

Definition 1.2. Let $V$ be a vector space over a finite field $F$ with dimension $n$. We call $V = V_{n_1} \oplus V_{n_2} \oplus \ldots \oplus V_{n_k}$ an $(n_1, n_2, \ldots, n_k)$-decomposition if $(n_1, n_2, \ldots, n_k)$ is a partition of $n$ and for $i = 1, 2, \ldots, k$, $V_{n_i}$ is a subspace of $V$ of dimension $n_i$.

Definition 1.3. Let $V$ be an $n$-dimensional vector space over a finite field $F$ with size $q$. An element $g$ of $GL(n,q)$ is called an $(n_1, n_2, \ldots, n_k)$-Singer generator if there is an $(n_1, n_2, \ldots, n_k)$-decomposition $V = V_{n_1} \oplus V_{n_2} \oplus \ldots \oplus V_{n_k}$ of $V$ such that $g = g_{n_1} g_{n_2} \ldots g_{n_k}$, where for each $i$, $\langle g_{n_i} \rangle$ is a Singer cycle subgroup of $GL(V_{n_i})$, or if $n_1 = 1$ then $g_{n_1}$ has eigenvalue 1, and if $n_1 = n_i$ with $i \neq j$, then $c_{g_{n_i}}(t) \neq c_{g_{n_j}}(t)$, where $c_{g_{n_i}}(t)$ is the characteristic polynomial for $g_{n_i}$ on $V_{n_i}$. We call $\Pi_{i=1}^k(g_{n_i})$ the $(n_1, n_2, \ldots, n_k)$-maximal torus corresponding to $g$.

Definition 1.4. Let $n$ be a natural number. We define a partition of $n$ by $l_k = (1,1,1,\ldots,1,n-k)$ so that the first $k$ elements are 1 and the last is $n - k$, with $k = 0, 1, 2, \ldots, n - 1$.

Lemma 1.5. Let $G = GL(n,q)$, with $q = p^k \geq n + 1$ and suppose that $g \in G$ is an $l_k$-Singer generator, where $k = 0, 1, 2, \ldots, n - 1$, with $g = g_{n_1} g_{n_2} \ldots g_{n_k} g_{n-k}$. Then $C_G(g) = \Pi_{i=1}^k(g_{n_i}) \times \langle g_{n-k} \rangle$. We know $C_G(g)$ is a $l_k$ maximal torus corresponding to $g$. 

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**Definition 1.6.** Let \( m \geq 2 \) be a natural number. We put \( \left\lfloor \frac{n}{m} \right\rfloor = 1 + q + q^2 + \ldots + q^{n-1} \), \( \left\lfloor \frac{n}{m} \right\rfloor = 0 \), \( \left\lfloor \frac{n}{m} \right\rfloor = 1 \) and for \( n \geq m \geq 0 \) we define 
\[ \begin{bmatrix} n \\ m \end{bmatrix} = \frac{\left\lfloor \frac{n}{m} \right\rfloor \cdot \left\lfloor n - m + 1 \right\rfloor}{\left\lfloor \frac{n}{m} \right\rfloor} . \]

**Proposition 1.7.** Let \( V \) be an \( n \)-dimensional vector space over finite filed \( F \) of size \( q \). If \( U \) is a \( k \)-dimensional subspace of \( V \), then the number of \( m \)-dimensional subspaces \( W \) of \( V \) with \( W \cap U = \emptyset \) is 
\[ q^{km} \left\lfloor \frac{n-k}{m} \right\rfloor . \]

**Proof.** See [4]

### 2 Main Result

In this section we obtain a non-commuting subset of \( GL(n,q) \).

**Theorem 2.1.** Let \( G = GL(n,q) \), where \( q = p^k \geq n+1 \). Let \( N_{1_k} \) consist of one \( 1_k \)-Singer generator element of \( G \) corresponding to each \( 1_k \)-maximal torus of \( G \). Then \( N_{1_k} \) is a non-commuting subset of size 
\[ \frac{|G|}{n(q-1)(q^{n-k}-1)} . \]

**Lemma 2.2.** Let \( G = GL(n,q) \), where \( q = p^k \geq n+1 \). If \( x \in N_{1_k} \) and \( y \in N_{1_s} \), where \( k \neq s \), then \( xy \neq yx \).

**Theorem 2.3.** Let \( G = GL(n,q) \), where \( q = p^k \geq n+1 \). Then 
\[ \omega(G) \geq \frac{|G|}{n(q^n - 1)} + \sum_{k=1}^{n-2} \frac{|G|}{k!(q-1)(q^{n-k}-1)} + \frac{|G|}{n!(q-1)^n} . \]

As an application of main Theorem, we have the following result that was proved in [2] for \( GL(3,q) \).

**Corollary 2.4.** Let \( G = GL(3,q) \), where \( q \geq 4 \). Then 
\[ \omega(G) \geq \frac{|G|}{3(q^3 - 1)} + \frac{|G|}{2(q-1)(q^2 - 1)} + \frac{|G|}{3!(q-1)^3} . \]

### References


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On generalized rough modules

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Abstract
Roughness in modules has been investigated by B. Davvaz and M. Mahdavipour. The main purpose of this paper is to introduce and discuss the concept of $T$-rough module and generalized $T$-rough modules over a commutative ring.

Keywords: Lower approximation, Upper approximation, $T$-rough set, Set-valued homomorphism, $T$-rough submodule, Commutative ring.

Mathematics Subject Classification: 16D10, 16D80

1 Introduction

The notion of rough sets has been introduced by Z. Pawlak in his papers [6,7]. It soon invoked a natural question concerning possible connection between rough sets and algebraic systems. Davvaz [1] introduced the notion of rough subring with respect to an ideal of a ring. S.B.Hosseini and at el [4,5] introduced $T$-rough ideal in a semigroup and in a commutative ring.

In this paper is discussed and introduced to set-valued homomorphism of a module, the concept $T$-rough module on a commutative ring and is proved some interesting properties.

Suppose that $U$ is a non-empty set. A partition or classification of $U$ is a family $\Theta$ of non-empty subsets of $U$ such that each element of $U$ is contained in exactly one element of $\Theta$. Recall that an equivalence relation $\theta$ on a set $U$ is a reflexive, symmetric, and transitive binary relation on $U$.

Each partition induces an equivalence relation on $U$. If $\theta$ is an equivalence relation on $U$, then for every $x \in U$, $[x]_{\theta}$ denotes the equivalence class of $\theta$ determined by $x$.

Definition 1.1. A pair $(U, \theta)$ where $U \neq \emptyset$ and $\theta$ is an equivalence relation on $U$ is called an approximation space.

Definition 1.2. For an approximation space $(U, \theta)$ by a rough approximation in $(U, \theta)$ we mean a mapping $\text{Apr} : P(U) \rightarrow P(U) \times P(U)$ defined by for every $X \in P(U)$, $\text{Apr}(X) = (\text{Apr}_{\theta}(X), \overline{\text{Apr}}_{\theta}(X))$, where

$\text{Apr}_{\theta}(X) = \{ x \in X \mid [x]_{\theta} \subseteq X \}$  ;  $\overline{\text{Apr}}_{\theta}(X) = \{ x \in X \mid [x]_{\theta} \cap X \neq \emptyset \}$

$\text{Apr}(X)$ is called a lower rough approximation of $X$ in $(U, \theta)$ whereas $\overline{\text{Apr}}(X)$ is called upper rough approximation of $X$ in $(U, \theta)$.

Definition 1.3. Given an approximation space $(U, \theta)$ a pair $(A, B)$ in $P(U) \times P(U)$ is called a rough set in $(U, \theta)$ if $(A, B) = (\text{Apr}(X), \overline{\text{Apr}}(X))$ for some $X \in P(U)$.
2 Set-valued Homomorphism and T-rough module

In this section, we define the concept of a set-valued homomorphism and give some important examples of a set-valued mapping. We show that every module homomorphism is a set-valued homomorphism. We also investigate some basic properties of a generalized lower and upper approximation operators in a module.

**Definition 2.1.** (See [3]) Let X and Y be two non-empty sets and \( \emptyset \neq B \subseteq Y \). Let \( T : X \to P^*(Y) \) be a set-valued mapping. The lower inverse and upper inverse of \( B \) under \( T \) are defined by

\[
L_T(B) = \{ x \in X \mid T(x) \subseteq B \} ; \quad U_T(B) = \{ x \in X \mid T(x) \cap B \neq \emptyset \}
\]

respectively.

**Definition 2.2.** (See [3]) Let X and Y be two non-empty sets and \( B \in P^*(Y) \). Let \( T : X \to P^*(Y) \) be a set-valued mapping. \( (L_T(B), U_T(B)) \) is called a T-rough set of \( B \).

**Proposition 2.3.** (See [3, 6, 7]) Let X and Y be two non-empty sets and \( A, B \subseteq Y \). Let \( T : X \to P^*(Y) \) be a set-valued mapping, then the following hold:

(i) \( U_T(A \cup B) = U_T(A) \cup U_T(B) \);
(ii) \( L_T(A \cap B) = L_T(A) \cap L_T(B) \);
(iii) \( A \subseteq B \) implies \( L_T(A) \subseteq L_T(B) \) and \( U_T(A) \subseteq U_T(B) \);
(iv) \( L_T(A) \cup L_T(B) \subseteq L_T(A \cup B) \) and \( U_T(A \cap B) \subseteq U_T(A) \cap U_T(B) \).

**Definition 2.4.** Let M and N be R-modules and \( T : M \to P^*(N) \) be a set-valued mapping. T is called a set-valued homomorphism if

(i) \( T(m_1 + m_2) = T(m_1) + T(m_2) \);
(ii) \( T(rm) = rT(m) \);

for all \( r \in R \) and \( m, m_1, m_2 \in M \).

It is clear that \( T(0) = \{0\} \) and \( T(-m) = -T(m) \) for all \( m \in M \). The following corollaries and proposition are clear.

**Corollary 2.5.** Let \( T : M \to P^*(N) \) be a set-valued homomorphism and \( f : Q \to M \) be a module homomorphism, then \( T \circ f \) is a set-valued homomorphism from \( Q \to P^*(N) \) such that \( U_{T \circ f}(B) = f^{-1}(U_T(B)) \) and \( L_{T \circ f}(B) = f^{-1}(L_T(B)) \) for all \( B \in P^*(N) \).

**Corollary 2.6.** Let \( T : M \to P^*(N) \) be a set-valued homomorphism and \( f : N \to Q \) be a module homomorphism, then \( T_f \) is a set-valued homomorphism from \( M \to P^*(Q) \) defined by \( T_f(m) = f(T(m)) \) such that \( L_{T_f}(A) = L_T(f^{-1}(A)) \) and \( U_{T_f}(A) = U_T(f^{-1}(A)) \) for all \( A \in P^*(Q) \).

**Proposition 2.7.** Let \( T : M \to P^*(N) \) be a set-valued homomorphism and \( S \) be a submodule of \( N \). Define \( T_S : M \to P^*(N) \) by \( T_S(m) = \{ a + S \mid a \in T(m) \} \), where \( N/S \) is the quotient module of \( N \) by \( S \). Then \( T_S \) is a set-valued homomorphism.

**Lemma 2.8.** Let \( A \in P^*(N) \) be an R-submodule of \( N \) and \( T : M \to P^*(N) \) be a set-valued homomorphism and \( L_T(A) \neq \emptyset \neq U_T(A) \), then both of them are R-submodules of \( M \).

**Theorem 2.9.** Let \( T : M \to P^*(N) \) be a set-valued homomorphism. If \( A \) be a submodule of \( N \) and \( S \) is a non-empty subset of \( R \). Then

(i) \( SL_T(A) \subseteq L_T(SA) \);
(ii) \( SU_T(A) \subseteq U_T(SA) \).
3 Generalized T-rough Module

In this section, we define a T-rough module with respect to a submodule of a module which is said to generalized T-rough module and study some their interesting properties.

**Definition 3.1.** Let $A$ be an $R$-submodule of $N$ and $B$ be a non-empty subset of $N$. Let $T : M \rightarrow P^*(N)$ be a set-valued homomorphism, then

$$L^A_T(B) = \{x \in M \mid (T(x) + A) \subseteq B\}; \quad U^A_T(B) = \{x \in M \mid (T(x) + A) \cap B \neq \emptyset\}$$

are called the generalized lower and upper approximations of $B$, respectively, with respect to the submodule $A$.

In the special case, if $A = 0$, then $L^A_T(B) = L_T(B)$ and $U^A_T(B) = U_T(B)$.

**Theorem 3.2.** Let $A$ and $B$ be $R$-submodules of $N$, then

(i) $L^A_T(B)$ is an $R$-submodule of $M$;

(ii) $U^A_T(B)$ is an $R$-submodule of $M$.

**Lemma 3.3.** Let $M$ be an $R$-module and $A, B$ be $R$-submodules of $N$ such that $A \subseteq B$ and let $S$ be a non-empty subset of $N$, then

(i) $L^B_T(S) \subseteq L^A_T(S)$;

(ii) $U^B_T(S) \subseteq U^A_T(S)$.

The following corollary follows from Lemma 4.3.

**Corollary 3.4.** Let $A, B$ be $R$-submodules of $N$ and $S$ be a non-empty subset of $N$, then

(i) $L^A_T(S) \cap L^B_T(S) \subseteq L^{A\cap B}_T(S)$;

(ii) $U^{A\cap B}_T(S) \subseteq U^A_T(S) \cap U^B_T(S)$.

**Theorem 3.5.** Suppose $S$ be a non-empty subset of $R$ and $B$ be an $R$-submodule of $N$ and $A$ be a non-empty subset of $N$. If $T : M \rightarrow P^*(N)$ be a set-valued homomorphism, then

(i) $SL^A_T(B) \subseteq L^A_T(SB)$;

(ii) $SU^A_T(B) \subseteq U^A_T(SB)$.

**Theorem 3.6.** Suppose $A, B, C$ be $R$-submodules of $N$. If $T : M \rightarrow P^*(N)$ be a set-valued homomorphism, then

(i) $L^A_T(C) + L^B_T(C) = L^{A+B}_T(C)$;

(ii) $U^A_T(C) + U^B_T(C) \subseteq U^{A+B}_T(C)$.

**Proposition 3.7.** Let $M$ be an $R$-module and $A$ is a submodule of $N$ and $B$ is non-empty subset of $N$, then

(i) $L^A_T(B^c) = (U^A_T(B))^c$;

(ii) $U^A_T(B^c) = (L^A_T(B))^c$. 
References


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Generalized rough set in semi-lattices

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Abstract

The notion of rough sets was introduced by Z. Pawlak in 1982. The main of this paper is to introduce and study set-valued homomorphism on semi-lattices and T-rough semi-lattice with respect to a set-valued homomorphism. This paper deals with T-rough set approach on semi-lattice theory. We generalize some results in T-rough set and approximation theory that those have been proved in several papers.

Keywords: Approximation space, semi-lattice, prime ideal, rough ideal, T-rough set, set-valued homomorphism.

Mathematics Subject Classification: 03G10, 06B05, 06B10

1 Introduction


The basic concepts of the theory include only orders, least upper bounds, or greatest lower bounds. In this paper, T-rough set and ideal based on semi-lattice is defined and some properties are given.

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief. Suppose that U is a non-empty set. A partition or classification of U is a family Θ of non-empty subsets of U such that each element of U is contained in exactly one element of Θ. Each partition Θ induces an equivalence relation θ on U by setting

\[ x \theta y \iff x \text{ and } y \text{ are in the same class of } \Theta. \]

Definition 1.1. [see[1].] For an approximation space \((U, \theta)\) by a rough approximation in \((U, \theta)\) we mean a mapping \(\text{Apr} : P(U) \rightarrow P(U) \times P(U)\) defined by for every \(X \in P(U)\), \(\text{Apr}(X) = (\text{Apr}(X), \overline{\text{Apr}}(X))\), where

\[ \text{Apr}(X) = \{x \in U \mid [x]_{\theta} \subseteq X\}, \overline{\text{Apr}}(X) = \{x \in U \mid [x]_{\theta} \cap X \neq \emptyset\}. \]

\(\text{Apr}(X)\) is called a lower rough approximation of \(X\) in \((U, \theta)\) whereas \(\overline{\text{Apr}}(X)\) is called upper rough approximation of \(X\) in \((U, \theta)\).

Definition 1.2. Given an approximation space \((U, \theta)\) a pair \((A, B)\) in \(P(U) \times P(U)\) is called a rough set in \((U, \theta)\) if \((A, B) = (\text{Apr}(X), \overline{\text{Apr}}(X))\) for some \(X \in P(U)\).

Definition 1.3. [see[3].] In mathematical order theory, a semi-lattice is a partially ordered set (poset) closed under one of two binary, either supremum or infimum.
Remark 1.4. We consider \( a \lor b \) for \( \sup\{a, b\} \) and \( a \land b \) for \( \inf\{a, b\} \).

Definition 1.5. [See[2].] Let \( S \) and \( M \) be two non-empty sets and \( A \in \mathcal{P}^*(M) \) where \( \mathcal{P}^*(M) \) denotes the set of all non-empty subsets of \( M \). Let \( T : S \rightarrow \mathcal{P}^*(M) \) be a set-valued mapping. The upper inverse and the lower inverse of \( A \) under \( T \) are defined by

\[
U_T(A) = \{x \in S \mid T(x) \cap A \neq \emptyset\}; \quad L_T(A) = \{x \in S \mid T(x) \subseteq A\},
\]

Definition 1.6. The pair \((L_T(A), U_T(A))\) is referred to as the generalized rough set of \( A \), induced by \( T \) or called \( T \)-rough set.

Definition 1.7. [See[3].] (i) A non-empty subset \( K \) of \( S \) is a sub-semi-lattice of the semi-lattice \((S, \land)\) if \( a \land b \in K \) for all \( a, b \in K \).

(ii) Let \( S \) be a semi-lattice and \( I \) be a nonempty subset of \( S \). \( I \) is called an ideal of \( S \), if \( x \land a \in I \) for all \( x \in S \) and \( a \in I \).

(iii) A proper ideal \( P \) of \( S \) is called a prime ideal, if \( a, b \in S \) and \( a \land b \in P \) implies that \( a \in P \) or \( b \in P \).

Definition 1.8. Let \( S \) and \( M \) be two semi-lattices and \( T : S \rightarrow \mathcal{P}^*(M) \) be a set-valued mapping. \( T \) is called a set-valued homomorphism for all \( x, y \in S \) if

\[
T(x \land y) = T(x) \land T(y) = \{a \land b \mid a \in T(x), b \in T(y)\}.
\]

Definition 1.9. Let \( S \) and \( M \) be two semi-lattices and \( T : S \rightarrow \mathcal{P}^*(M) \) be a set-valued homomorphism. The lower \( T \)-rough quotient and the upper \( T \)-rough quotient for \( A \in \mathcal{P}^*(M) \) are

\[
\frac{L_T(A)}{T} = \{T(x) \mid T(x) \subseteq A\}; \quad \frac{U_T(A)}{T} = \{T(x) \mid T(x) \land A \neq \emptyset\}.
\]

Theorem 1.10. Let \( S \) and \( M \) be two semi-lattices and \( T : S \rightarrow \mathcal{P}^*(M) \) be a set-valued homomorphism. If \( K \) is a sub-semi-lattice of \( M \) and \( L_T(K) \neq \emptyset \neq U_T(K) \), then \((L_T(K), U_T(K))\) is a \( T \)-rough sub-semi-lattice of \( S \).

Proof. Let \( x, y \in L_T(K) \), by Definition 1.5, \( T(x), T(y) \subseteq K \). Since \( K \) is a sub-semi-lattice of \( M \), we have \( T(x \land y) = T(x) \land T(y) \subseteq K \). It shows that \( x \land y \in L_T(K) \). Moreover, let \( x, y \in U_T(K) \), by Definition 1.5, \( T(x) \cap K \neq \emptyset \) and \( T(y) \cap K \neq \emptyset \). Suppose \( a \in T(x) \cap K \) and \( b \in T(y) \cap K \). Since \( K \) is a sub-semi-lattice of \( M \), we have \( a \land b \in K \) and \( a \land b \in T(x) \cap T(y) \). So that \( a \land b \in T(x \land y) \cap K \). Therefore \( T(x \land y) \cap K \neq \emptyset \). It means that \( x \land y \in U_T(K) \).

Proposition 1.11. Let \( S \) and \( M \) be two semi-lattices and \( T : S \rightarrow \mathcal{P}^*(M) \) be a set-valued homomorphism. If \( A, B \) be non-empty subsets of \( M \), then

\begin{enumerate}
  \item \( U_T(A) \land U_T(B) \subseteq U_T(A \land B) \);
  \item \( L_T(A) \land L_T(B) \subseteq L_T(A \land B) \).
\end{enumerate}

Proof. (1). Let \( z \in U_T(A) \land U_T(B) \). Then \( z = a \land b \) for some \( a \in U_T(A) \) and \( b \in U_T(B) \). Hence \( T(a) \cap A \neq \emptyset \) and \( T(b) \cap B \neq \emptyset \) and so there exist \( x \in T(a) \cap A \) and \( y \in T(b) \cap B \). Therefore \( x \land y \in A \land B \) and \( x \land y \in T(a) \land T(b) = T(a \land b) \). Thus \( x \land y \in T(a \land b) \land (A \land B) \) which implies that \( T(a \land b) \land (A \land B) \neq \emptyset \). So \( z = a \land b \in U_T(A \land B) \).

(2). The proof is similar to the proof of (1).

The following example shows the converse of above proposition are not true.
Example 1.12. (1) Let \( \mathbb{N} \) be a natural numbers set and a semi-lattice. Let \( T : \mathbb{N} \rightarrow P^*(\mathbb{N}) \) be a set-valued homomorphism with \( T(n) = \{1\} \) for all \( n \in \mathbb{N} \). And let \( A = \{2\}, B = \{1, 3\} \) Then \( A \land B = \{1, 2\}, U_T(A \land B) = \{1\}, U_T(A) = \emptyset, U_T(B) = \{1\}, U_T(A) \lor U_T(B) = \emptyset \). Therefore \( U_T(A \land B) \not\subseteq U_T(A) \land U_T(B) \).

(2) Let \( L = [0, 1] \) and \( T : L \rightarrow P^*(L) \) be a set-valued homomorphism with \( T(x) = [0, x] \) for all \( x \in L \). And let \( A = \{0, \frac{1}{2}\}, B = \{\frac{1}{3}, \frac{1}{2}\} \). Then \( L_T(A) = \{0\}, L_T(B) = \emptyset, L_T(A \land B) = \{0\}, L_T(A) \land L_T(B) = \emptyset \). Therefore \( L_T(A \land B) \not\subseteq L_T(A) \land L_T(B) \).

2 Main results

Corollary 2.1. Let \( S \) and \( M \) be two semi-lattices and \( T : S \rightarrow P^*(M) \) be a set-valued homomorphism. If \( I \) is an ideal of \( M \) and \( L_T(I) \neq \emptyset \) and \( U_T(I) \neq \emptyset \), then \( (L_T(I), U_T(I)) \) is a T-rough ideal of \( S \).

Corollary 2.2. Let \( S \) and \( M \) be two semi-lattices and \( T : S \rightarrow P^*(M) \) be a set-valued homomorphism. If \( P \) is a prime ideal of \( M \) and \( S \neq L_T(P) \neq \emptyset \) and \( S \neq U_T(P) \neq \emptyset \), then \( (L_T(P), U_T(P)) \) is a T-rough prime ideal of \( S \).

Corollary 2.3. Let \( S \) and \( M \) be two semi-lattices and \( T : S \rightarrow P^*(M) \) be a set-valued homomorphism. If \( A \in P^*(M) \) is an ideal of \( M \) and \( S \neq \frac{L_T(A)}{T} \neq \emptyset \) and \( S \neq \frac{U_T(A)}{T} \neq \emptyset \), then \( \frac{L_T(A)}{T} \) and \( \frac{U_T(A)}{T} \) is a T-rough quotient ideal of \( S \).

Corollary 2.4. Let \( S \) and \( M \) be two semi-lattices and \( T : S \rightarrow P^*(M) \) be a set-valued homomorphism. If \( (L_T(P), U_T(P)) \) be a T-rough prime ideal of \( M \) and \( \frac{L_T(P)}{T} \neq \frac{S}{T} \neq \frac{U_T(P)}{T} \neq \emptyset \), then \( \frac{L_T(P)}{T} \) and \( \frac{U_T(P)}{T} \) is a T-rough quotient prime ideal of \( S \).

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Sum of element orders on groups of order 168

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Abstract
Let $G$ be a finite group. Then we define $\psi(G) = \sum_{g \in G} o(g)$ where $o(g)$ denotes the order of $g \in G$. We prove that $\psi(PSL(2,7)) < \psi(G)$ for all nonsimple groups $G$ of order 168. This result confirms the conjecture posed in [2] for simple group $PSL(2,7)$.

Keywords: Finite groups, simple group, element orders

Mathematics Subject Classification: 20D60

1 Introduction
Let $G$ be a finite group. We define the function

$$\psi(G) = \sum_{g \in G} o(g)$$

where $o(g)$ denotes the order of $g \in G$. We ask what information about $G$ can be obtained by $\psi(G)$ and $|G|$. The starting point for the function $\psi$ is given by paper [1] which investigates the maximum of $\psi$ on the groups of the same order.

Also one can ask about the structure of the groups which have minimum sum of element orders on all groups of the same order. In [2] authors proved that:

**Theorem 1.1.** Let $G$ be a nilpotent group of order $n$. Then $\psi(G) \leq \psi(H)$ for every nilpotent group $H$ of order $n$ if and only if each Sylow subgroup of $G$ has prime exponent.

**Theorem 1.2.** Let $n$ be a positive integer such that there exists a nonnilpotent group of order $n$. Then there exists a nonnilpotent group $K$ of order $n$ with the property that $\psi(K) < \psi(H)$ for every nilpotent group $H$ of order $n$.

In other words the minimum of $\psi$ occurs in a non-nilpotent group on all groups of the same order.

Also authors in [2] conjectured that:

**Conjecture** Let $S$ be a simple group. If $G$ is a nonsimple group of order $|S|$, then $\psi(S) < \psi(G)$.

In other words if $n$ is a natural number such that there is a simple group of order $n$, then the minimum of $\psi$, on all groups of order $n$, occurs in a simple group. Here we confirm this conjecture for $PSL(2,7)$. Note that we determine $PSL(2,7)$ by their orders and sum of element orders.

2 Main Result
It is easy to check that $PSL(2,7)$ has 21 elements of order 2, 42 elements of order 4, 56 elements of order 3 and 48 elements of order 7. Therefore $\psi(PSL(2,7)) = 715.$
Lemma 2.1. Let $G$ be a group of order 168. If $G$ contains an element of order 21, then $\psi(G) > 715$.

Lemma 2.2. Let $G$ be a group of order 168. If $G$ contains no element of order 21 and $n_7 = 8$, then $\psi(G) \geq 715$.

Theorem 2.3. Let $G$ be any nonsimple group of order 168. Then $\psi(G) > \psi(PSL(2,7))$.

References


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The Dieudonné determinant and valuation on matrices and cubic matrices over skew field

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Abstract

Let $R$ be a commutative ring. We denote by $M^{(3)}_n(K)$ the set of all cubic matrices. If $\nu$ is a valuation on $R$, then function $V : M^{(3)}_n(K) \rightarrow \mathbb{R} \cup \{\infty\}, V(A) = \nu(\det(A))$ is cubic matrix valuation on $R$. In this article we show how it can be hold in the case of skew field $K$, Where $\det(A)$ is a Dieudonné determinant. In fact we prove that if $K$ be a skew field and $\det(A)$ be a Dieudonné determinant of $A$, then function $V : M^{(3)}_n(K) \rightarrow (\mathbb{R} \cup \{\infty\}, +), V(A) = \bar{\nu}(\det(A))$ will be a cubic matrix valuation on skew field $K$, where $\bar{\nu}(k.[K^*, K^*]) = \nu(k)$.

Keywords: matrix valuation, Cubic matrix valuation, Dieudonné determinant.

Mathematics Subject Classification: 12J20

1 Matrix valuations and some examples

The purpose of this article is to describe and existence proof of a cubic matrix valuation on a cubic matrix ring of skew field (noncommutative field) by using the Dieudonné determinant.

Definition 1.1. By a valuation on a skew field $K$ with group $\Gamma$ we shall understand a function $v : K \rightarrow \Gamma \cup \{\infty\}$ such that
1) $v(x) \neq 0$ for some $x \in K$
2) $v(x + y) \geq \min\{v(x), v(y)\}$
3) $v(x \cdot y) = v(x) + v(y)$

We shall usually exclude the improper valuation $v(x) = \infty$ for all $x \in K$ and call it trivial valuation when $v(x) = 0$ for $x \neq 0$. (See [1])

Example 1.2. A) Suppose $\mathbb{Z}$ be the set off all integer number, $P$ be an arbitrary prime number and $\Gamma = \mathbb{Z}$. Then function $v : \mathbb{Z} \rightarrow \Gamma \cup \{\infty\}, v(x) = n, v(0) = \infty$ in which $n$ is the power of $P$ in decomposition $X$ into primes is a valuation.
B) Given an ordered group $G$ and any field $K$, we form $K((G))$, the set of formal series $f = \sum a_g g$ ($g \in G$) such that the support $D(f) = \{g \in G \mid a_g \neq 0\}$ is well-ordered. This set can be shown to be a field, and we can define a valuation $V$ on $K((G))$ with value group $G$ by $v(f) = g_0$. If $g_0$ is the least element in the support of $f$. 

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Definition 1.3. A matrix valuation on a ring $R$ is a function $V$ on $M(R)$ with values in $\Gamma \cup \{ \infty \}$, such that
1) $V(a) = \infty$ for any non-full matrix $A$ over $R$.
2) $V(A)$ is unchanged if any row or column is multiplied by -1.
3) $V(AB) \geq \min\{V(A), V(B)\}$, for all $A, B$ such that $(AB)$ is defined ($AB$ is determinantal sum of $A$ and $B$).
4) $V(A + B) = V(A) + V(B)$, for all $A, B \in M(R)$.
5) $V(I) = 0$ (I is the identity matrix from related rank)

Theorem 1.4. Let $R$ be a commutative ring with valuation $\nu$. Then map $V$ with $V(A) = \nu(\det(A))$ from $M(R)$ is a matrix valuation on $R$. (see [2])

2 Dieudonné determinant and cubic matrix valuation

If $K$ is a skew field and let $K^*$ be the the multiplicative group $K - \{0\}$. Consider quotient group $K^* = K^*/[K^*, K^*]$ from $K^*$ with the commutator $[K^*, K^*]$. Every non-zero element $a$ from $K$ have a canonical representation such as $a = a \cdot [K^*, K^*] \in K^*$.

Definition 2.1. For each matrix $A \in M(R)$, Dieudonné determinant of matrix $A$ that denoted by $\det(A)$ is defined equal to element $x \in K$ that satisfied in the following conditions:
1) If the matrix $A'$ is obtained from the matrix $A$ by multiplying one row from the left by number $\mu$ then $\det(A') = \mu \cdot \det(A)$.
2) If the matrix $A'$ is obtained from the matrix $A$ by replacing a row from $a_i$ by a row $a_i + \mu a_j$, then $\det(A') = \det(A)$.
3) The determinant of the identity matrix equal to 1.

Remark 2.2. From the above conditions the following properties results:
A) If a row $a_i$ of matrix $A$ is replaced by a row $a_i + \mu a_j$, where $i \neq j$, $\det(A)$ will not change.
B) If matrix $A$ is non-full ($\text{rank}A < n$) then $\det A = 0$.
C) If two rows (or two columns) of matrix $A$ are replacing by each other $\det(A)$ will multiplied to $(-1)$.
D) $\det(D(\mu)) = \mu$, where $D(\mu)$ is a matrix that result from Identity matrix by replacing unit element of the last row by $\mu$.
E) If $A$ is non-full matrix and $A = B \cdot D(\mu)$ that $B \in \text{SL}_n K$ then $\det(A) = \mu$.
F) Matrix $A$ is non-full if and only if $\det(A) = 0$.
G) $\det(AB) = \det(A) \cdot \det(B)$.
H) If the multiple of the one column is adding to another column from right the determinant will not change.
K) If matrix $A$ is multiple to $\mu$ from right then the determinant will multiply to $\mu$.

Now let $\nu$ be a valuation from skew field $K$ to $(R \cup \{ \infty \}, +)$ and $V : K^* : (K^*, 0) \rightarrow (R \cup \{ \infty \}, +)$ be it’s restriction to $K^*$, in this case we have homomorphism $\nu : K^*/[K^*, K^*] \rightarrow (R \cup \{ \infty \}, +)$ where $V(k \cdot [K^*, K^*]) = \nu(k)$ and the statements in theorem 1-4 about matrices of skew field will be proved in the next theorem.

Theorem 2.3. Let $K$ be a skew field with a commutative valuation $\nu$. Then the map $V : M_n(K) \rightarrow \Gamma \cup \{0\}$, where $V(A) = \nu(\det(A)) = \nu(k)$ is the matrix valuation.

Proof. Now we proof conditions 1-5
1) If $A \in M_n(K)$ and $A$ is non-full, then from result (B) we have $\det(A) = 0$ consequently $V(A) = \nu(\det(A)) = \nu(0) = \infty$
2) If \( A, B \in M_n(K) \) and \( B_j = -A_j \) and \( B_i = A_i \) for \( i \neq j \), then by part (1) from definition 2-1 have:
\[
\det(B) = (-1)^{i+j} \det(A) = \det(A) \text{ consequently } V(A) = v(\det(B)) = v(\det(A)) = V(A)
\]
for each two matrices \( A, B \in M_n(k) \) we prove that:

(i) \( \det(A \oplus B) = \det(A) \cdot \det(B) \)

(ii) \( \det(A \nabla B) = \det(A) + \det(B) \)

if \( A, B \) are both non-full (i) is trivial. Let \( A, B \) are both be full, then we have
\( A = D(\mu), B = D(\rho) \) and \( \det(A) = \mu, \det(B) = \rho \)

and so \( \det(A + B) = \mu \cdot \rho = \det(A) \cdot \det(B) \) thus (i) is proved. For (ii) if \( A \) or \( B \) is non-full the statement is trivial otherwise there exist full matrices \( P \) and \( Q \) such that
\[
A = P \begin{pmatrix} a_{11} & a_{12} \\ a & I \end{pmatrix} Q, \quad B = P \begin{pmatrix} b_{11} & 0 \\ b & I \end{pmatrix} Q, \quad C = P \begin{pmatrix} c_{11} & 0 \\ c & I \end{pmatrix} Q.
\]

where \( c_{11} = a_{11} + b_{11} \) and \( c = a + b \). Now from result (G) and part (i) have:
\[
\begin{align*}
\det(C) &= \det(P).c_{11}.\det(I)\det(Q) = \det(P).c_{11}.\det(Q) \\
\det(A) &= \det(P).a_{11}.\det(I)\det(Q) = \det(P).a_{11}.\det(Q) \\
\det(B) &= \det(P).b_{11}.\det(I)\det(Q) = \det(P).b_{11}.\det(Q)
\end{align*}
\]

thus:
\[
\det(A) + \det(B) = (\det(P).a_{11}.\det(Q)) + (\det(P).b_{11}.\det(Q)) = \det(P).\left(a_{11} + b_{11}\right)\det(Q) = \det(P).c_{11}.\det(Q) = \det(C)
\]

consequently:
\[
\det(A \nabla B) = \det(C) = \det(A) + \det(B)
\]

so (ii) is true.

3) If \( A, B \in M_n(K) \) and \( (A \nabla B) \) is defined then by (ii):
\[
\det(A \nabla B) = \det(A) + \det(B)
\]

and consequently:
\[
V(A \nabla B) = v(\det(A) + \det(B)) \geq \min\{v(\det(A)), v(\det(B))\}
\]

4) For \( A, B \in M_n(K) \) by (i): \( \det(A \oplus B) = \det(A) \cdot \det(B) \)

and so:
\[
V(A \oplus B) = v(\det(A \oplus B)) = v(\det(A) \cdot \det(B)) = v(\det(A) + v(\det(B)) = V(A) + V(B)
\]

5) \( V(I) = v(\det(I)) = v(1) = 0 \) so the proof of theorem is complete.

Let given the numerical field \( K \). Any system from \( n^3 \) elements \( A_{i,j,k}(i,j,k = 1,2,...,n) \) of field \( K \)

that defined as coordinates \( i, j, k, \) is called a 3-dimensional (cubic) matrix of order \( n \) on \( K \) and it is denoted in abbreviated form by a symbol \( \|A_{i,j,k}\| \quad (i,j,k = 1,2,...,n) \).

**Definition 2.4.** The valuation on set of cubic matrices \( M^{(3)}(K) \) on a field \( K \) is called map \( | \cdot : M^{(3)}(K) \rightarrow \mathbb{R} \cup \{\infty\}, \) satisfying the following conditions:

1. if \( A = \|A_{i,j,k}\| \in M_n^{(3)}(K) \) and \( A \) is a singular matrix, then \( |A| = \infty \);
2. if \( A = \|A_{i,j,k}\|, B = \|B_{i,j,k}\| \in M_n^{(3)}(K), 1 \leq \lambda, 1 \leq \rho \leq n, B_{\lambda j k} = -A_{\lambda j k}, \) and \( B_{i j k} = A_{i j k} \) for \( i \neq \lambda, 1 \leq j, k \leq n, \) then \( |A| = |B|; \)
3. if \( A = \|A_{i,j,k}\|, B = \|B_{i,j,k}\| \in M_n^{(3)}(K) \) and the determinant sum \( A \nabla B \) is determined, then \( |A \nabla B| \geq \min\{|A|, |B|\}; \)
4. for any matrix \( A = \|A_{i,j,k}\| \in M_n^{(3)}(K), B = \|B_{i,j,k}\| \in M_n^{(3)}(K) \):
\[ | A \oplus B | = | A | + | B |; \]

5. \(| I | = 0, \) where \( I = I_n \in M_n^3(K) \) is an identity cubic matrix.

**Theorem 2.5.** Let \( K \) be a skew field with a commutative valuation \( v \). Then the map \( V : M_n^3(K) \to \Gamma \cup \{ \infty \} \), where \( V(A) = v(\text{det}(A)) = v(k) \) is the matrix valuation.

**References**


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A residual transcendental extension of a valuation on $K$ to $K(x_1, \ldots, x_n)$

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Abstract

Let $v$ be a valuation of a field $K$, $G_v$ its value group and $k_v$ its residue field. Let $w$ be an extension of $v$ to $K(x_1, \ldots, x_n)$. $w$ is called a residual transcendental extension of $v$ if $k_w/k_v$ is a transcendental extension. In this study a residual transcendental extension $w$ of $v$ to $K(x_1, \ldots, x_n)$ such that $\text{transdeg } k_w/k_v = n$ is defined and some considerations related with this valuation are given.

Keywords: Extensions of valuations, Residual transcendental extensions, Valued fields.

Mathematics Subject Classification[2010]: 12J20, 13A18

1 Preliminaries and notations

Let $v$ be a nontrivial valuation on $K$ having value group $G_v$ and residue field $k_v$. In this paper a residual transcendental extension $w$ of $v$ to the rational function field with $n$ variables $K(x_1, \ldots, x_n)$ is studied. It is defined by using residual transcendental extensions $w_i$ of $v$ to $K(x_i)$, for $i = 1, \ldots, n$. It is simple but it is a practical valuation. Because different extensions of $v$ to $K(x_1, \ldots, x_n)$ can be defined by using this valuation. Then a question related with this valuation can be asked: can we find an intrinsic condition to describe such a $w$?

Throughout the paper $v$ is a valuation of a field $K$ with value group $G_v$, valuation ring $O_v$ and residue field $k_v$. If $\alpha \in O_v$, then $\alpha^*$ will denote the natural image of $\alpha$ in $k_v$. If $L$ is an extension of the field $K$ and $u$ is an extension of $v$ to $L$ then $k_u$ is canonically identified with a subfield of $k_u$ and $G_v$ with a subgroup of $G_u$. We will denote by $\overline{K}$ the algebraic closure of $K$ and by $\overline{v}$ a fixed extension of $v$ to $K$. Then the residue field of $\overline{v}$ is $k_{\overline{v}} = k_v$ which is the algebraic closure of $k_v$ and the value group of $\overline{v}$ is $G_{\overline{v}} = G_v$ which is the divisible closure of $G_v$. $K(x)$ and $K(x_1, \ldots, x_n)$ are rational function fields over $K$ with one and $n$ variables respectively. If $a_1, \ldots, a_n \in k$ then the restriction of $\overline{v}$ to $K(a_1, \ldots, a_n)$ will be denoted by $v_{a_1, \ldots, a_n}$.

Let $w$ be an extension of $\overline{v}$ to $K(x)$. $w$ is called a residual transcendental (r.t.) extension of $v$ if $k_w/k_v$ is a transcendental extension.
which is defined for each \( F = \sum a_i(x-a)^i \in \overline{K}[x] \) as
\[
\overline{w}(F) = \inf_i (\overline{v}(a_i) + i\delta)
\]
(1)

is a valuation on \( \overline{K}(x) \) and it is called a valuation defined by the pair \((a, \delta) \in \overline{K} \times G_{\overline{v}}\).

If \([K(a) : K] \leq [K(b) : K]\) for every \(b \in K\) such that \(\overline{v}(b-a) \geq \delta\) then \((a, \delta)\) is called minimal pair with respect to \(K\).

If \(w\) is a r.t. extension of \(v\) to \(K(x)\) then there exists an extension \(\overline{w}\) of \(w\) to \(\overline{K}(x)\) such that \(\overline{w}\) is also an extension of \(\overline{v}\).

If \(w\) is a r.t. extension of \(v\) to \(K(x)\) then there exists a minimal pair \((a, \delta) \in \overline{K} \times G_{\overline{v}}\) of definition for \(w\) with respect to \(K\), where \(a\) is separable over \(K\). Let \(f = \text{Irr}(a, K)\) be a minimal polynomial of a respect to \(K\) and \(\gamma = w(f)\). If \(F \in K[x]\) let \(F = F_1 + F_2 x + \ldots + F_n x^n\), \(\deg F_i < \deg f_i, i = 1, \ldots, n\) be the \(f\)-expansion of \(F\). Then \(w\) is defined as:
\[
w(F) = \inf_i (\nu_0(F_i(a)) + i\gamma).
\]
(2)

(see [?, ?])

2 Main Results

Let \(w_i\) be a r.t. extension of \(v\) to \(K(x_i)\) defined by a minimal pair \((a_i, \delta_i) \in \overline{K} \times G_{\overline{v}}\) and \(f_i = \text{Irr}(a_i, K)\), \(\gamma_i = w_i(f_i)\) for \(i = 1, \ldots, n\), where \([K(a_1, \ldots, a_n) : K] = \prod_{i=1}^n [K(a_i) : K]\) Each polynomial \(F \in K[x_1, \ldots, x_n]\) can be written uniquely as:
\[
F = \sum_{i_1, \ldots, i_n} F_{i_1, \ldots, i_n} f_1^{i_1} \cdots f_n^{i_n},
\]
where \(F_{i_1, \ldots, i_n} \in K[x_1, \ldots, x_n]\); \(\deg x_i F_{i_1, \ldots, i_n} < \deg f_i\) for \(i = 1, \ldots, n\). Define
\[
w(F) = \inf_{i_1, \ldots, i_n} (\nu_{a_1, \ldots, a_n}(F_{i_1, \ldots, i_n}(a_1, \ldots, a_n)) + i_1 \gamma_1 + \ldots + i_n \gamma_n)
\]
(3)

The following theorem can be given for describing some basic properties of \(w\) (see [?]).
All of these properties are natural generalizations of the known results of [?, ?].

**Theorem 2.1.** Let \(w\) be a r.t. extension of \(v\) to \(K(x_1, \ldots, x_n)\) which is a common extension of \(w_1, \ldots, w_n\) to \(K(x_1, \ldots, x_n)\). Then

1. \(G_w = G_{v_{a_1, \ldots, a_n}} + Z\gamma_1 + \ldots + Z\gamma_n\).
2. \(\epsilon_i\) be the smallest positive integer such that \(\epsilon_i \gamma_i \in G_{v_{a_i}}\), where \(v_{a_i}\) is the restriction of \(v\) to \(K(a_i)\). Then there exists \(h_i \in K[x_i]\) such that \(\deg h_i < \deg f_i\), \(v_{a_i}(h(a_i)) = \epsilon_i \gamma_i\), \(r_i = f_i^{\epsilon_i} / h_i \in O_{w_i}\) and \(r_i^*\) is transcendental over \(k_v\) for \(i = 1, \ldots, n\) (see also [?, ?]).
3. \(k_{v_{a_1, \ldots, a_n}}\) can be canonically identified with the algebraic closure of \(k_v\) in \(k_w\) and \(k_w = k_{v_{a_1, \ldots, a_n}}(r_1^*, \ldots, r_n^*)\).

Can we find an intrinsic condition to describe such a \(w\)? Let \(w\) be a r.t. extension of \(v\) to \(K(x_1, \ldots, x_n)\) such that \(\text{transdeg}_k k_v = n\). Denote by
\[
E = \epsilon(w/v) = [G_w : G_v],
\]
\(F = f(w/v) = [k : k_v]\), where \(k\) is the algebraic closure of \(k_v\) in \(k_w\).
\(D = \text{deg}(w/v)\), the smallest positive integer for which there exist elements \(r_1, \ldots, r_n \in O_w\) such that \([K(x_1, \ldots, x_n) : K(r_1, \ldots, r_n)] = \text{deg}(w/v)\) and \(r_1^*, \ldots, r_n^*\) are algebraically independent over \(k_v\).
We know from [?] that
\[ EF \leq D. \] (4)

Firstly, let us give some special considerations for \( w \) which is defined in (3) by using \( w_i \) for \( i = 1, \ldots, n \).

**Theorem 2.2.** Under the above notations
\[ D = \deg(w/v) = \prod_{i=1}^{n} \deg(w_i/v) \]
is satisfied.

**Proof.** Since \( \deg(w_i/v) = [K(x_i) : K(r_i)] \) for \( i = 1, \ldots, n \) then
\[ D = [K(x_1, \ldots, x_n) : K(r_1, \ldots, r_n)] \]
\[ = [K(x_1, x_2, \ldots, x_n) : K(r_1, x_2, \ldots, x_n)][K(r_1, x_2, \ldots, x_n) : K(r_1, \ldots, r_n)] \]
\[ = d(w_1/v)[K(r_1, x_2, \ldots, x_n)][K(r_1, r_2, x_3, \ldots, x_n) : K(r_1, \ldots, r_n)] \]
\[ = d(w_1/v)d(w_2/v)[K(r_1, r_2, \ldots, x_n) : K(r_1, \ldots, r_n)] \]
\[ = d(w_1/v)d(w_2/v) \ldots d(w_n/v)[K(r_1, r_2, \ldots, r_n-1, x_n) : K(r_1, \ldots, r_n)] = \prod_{i=1}^{n} \deg(w_i/v) \]
is obtained. \( \square \)

One of the most important notions related to valuations is the notion of defect. The rational number \( \text{def}(w/v) \) for \( w \) can be defined as below and it can be compared with defect of \( w_i/v \) for \( i = 1, \ldots, n \) defined in [?].

\[ \text{def}(w/v) = D/EF. \] (5)

Then the following theorem is obtained:

**Theorem 2.3.** Under the above notations
\[ \deg(w/v) \geq \prod_{i=1}^{n} \deg(w_i/v) \]
is obtained.

**Proof.** According to statement of Theorem 2.1 of [?] \( \deg(w_i/v) = e_i n_i \) where \( e_i = e(\gamma_i, K(a_i)) \) and \( n_i = \text{deg}(f_i) \), then \( D = \prod_{i=1}^{n} \deg(w_i/v) = \prod_{i=1}^{n} e_i n_i \). Since \( G_w = G_{v_{a_1 \ldots a_n}} + \mathbb{Z} \gamma_1 + \ldots + \mathbb{Z} \gamma_n \) then \( E = [G_w : G_v] = \overline{e}[G_{v_{a_1 \ldots a_n}} : G_v] \), where \( \overline{e} \) is the smallest multiple of \( e_i \) for \( i = 1, \ldots, n \).

\[ F = [k_{v_{a_1 \ldots a_n}} : k_v] \leq \prod_{i=1}^{n} [k_{v_{a_i}} : k_v] = \prod_{i=1}^{n} f(w_i/v) \]

and
\[ E = \overline{e}[G_{v_{a_1 \ldots a_n}} : G_v] \leq [G_{v_{a_1 \ldots a_n}} : G_v] \prod_{i=1}^{n} e_i \leq \prod_{i=1}^{n} e_i [G_{v_{a_1 \ldots a_n}} : G_v] = \prod_{i=1}^{n} e(w_i/v). \]

Therefore
\[ \prod_{i=1}^{n} (\deg(w_i/v)/E_i F_i) \leq D/EF \]
because of \( EF \leq \prod_{i=1}^{n} E_i F_i \) where \( E_i = [G_{w_i} : G_v] \) and \( F_i = [k_{w_i} : k_v] \). \( \square \)
Now, we can give a conclusion when the equality in [?] turns to equality:

**Theorem 2.4.** Let $w$ be a r.t. extension of $v$ to $K(x_1, \ldots, x_n)$ such that $\text{transdeg}_{k_w}/k_v = n$. If $EF = D$ then $w$ is defined as in (3).

**Proof.** If $EF = D$ then $k_w$ is isomorphic to the field of rational functions in $n$ variables $K(r_1^*, \ldots, r_n^*)$ where $k$ is the algebraic closure of $k_v$ in $k_w$ [?]. $w$ is obtained by consecutive extensions of $v$. Let $u_i$ be an extension of $u_{i-1}$ on $K(x_1, \ldots, x_{i-1})$ to $K(x_1, \ldots, x_i)$ for $i = 1, \ldots, n$. Here $u_n = w$ and $u_0 = v$. Since $\text{transdeg}_{k_w}/k_v = n$ and $\text{transdeg}_{k_{u_i}}/k_{u_{i-1}} \leq 1$ then $u_i$ is a r.t. extension of $u_{i-1}$ then it must be defined by a minimal pair $(\alpha_i, \delta_i) \in \overline{K} \times G_f$. Therefore $w$ can be obtained as a common extension of $w_i$ where $w_i$ is a r.t. extension of $v$ to $K(x_i)$ defined by minimal pair $(\alpha_i, \delta_i) \in \overline{K} \times G_f$ for $i = 1, \ldots, n$. □

**References**


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Irreducible ideals and topological lattice

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Abstract
Let $R$ be ring associative with identity. In this paper, we show that if every $\alpha$-irreducible ideal is $\alpha$-strongly irreducible and $I \cap (\sum I_\alpha) = \sum (I \cap I_\alpha)$, then $R$ is a topological lattice under order-convergence, for all ideals $I, I_\alpha$ of $R$.

Keywords: Topology lattice, Lattice completely distributive, Irreducible ideal.

Mathematics Subject Classification: 13A15, 20E15

1 Introduction
Throughout the paper, all rings associative with identity. For a ring $R$, we denote by $Lat(R)$ the lattice of ideals of $R$. In [2], L. Fuchs first introduced the concept of irreducible ideals in $R$, as follows: $I \in Lat(R)$ is said to be irreducible if for $I, K \in Lat(R)$, $J \cap K = I$ implies that either $J = I$ or $K = I$. Furthermore, he established a condition under which an irreducible ideal is primary. Heinzer-Ratliff-Resh extended irreducible ideals to strongly irreducible ideals, as follows: $I \in Lat(R)$ is said to be strongly irreducible if for $J, K \in Lat(R)$, $J \cap K \subseteq I$ implies that either $J \subseteq I$ or $K \subseteq I$. Furthermore, they investigated some relations between irreducible ideals and strongly irreducible ideals. In this paper, we will extend irreducible ideals and strongly irreducible ideals to $\alpha$-irreducible ideals and $\alpha$-strongly irreducible ideals and topological lattice for any cardinals $\alpha$, and we will prove that if every $\alpha$-irreducible ideal is $\alpha$-strongly irreducible and if $I \cap (\sum I_\alpha) = \sum (I \cap I_\alpha)$ for all ideals $I, I_\alpha$ of $R$, then $R$ is a topological lattice under order-convergence.

2 Main Result
We begin with some definitions, notation and basic properties of topology lattice and irreducible ideals of $R$. In this section, will deal with various applications of lattice concepts to general topology-i.e., to the general theory of topological spaces. The ideas of general topology can be most simply introduced through the concept of a metric space.

In a metric space $M$, a sequence $\{x_n\}$ of points is said to converge to the limit point $a$ (in symbol, $x_n \rightarrow a$), if and only if $\lim_{n \rightarrow \infty} \delta(x_n, a) = 0$

Definition 2.1. A topology lattice is a lattice with a specified convergence topology, in which

$x_\alpha \rightarrow x$ and $y_\beta \rightarrow y$ imply $x_\alpha \wedge y_\beta \rightarrow x \wedge y$

and
\[ x_\alpha \to x \text{ and } y_\beta \to y \text{ imply } x_\alpha \lor y_\beta \to x \lor y. \]

**Definition 2.2.** A lattice \( L \) is called distributive if satisfying following:
\[ x \land (y \lor z) = (x \land y) \lor (x \land z). \]

Recall, from [1] that a ring \( R \) is distributive if \( \text{Lat}(R) \) is a distributive lattice, i.e., \( I, J, K \in \text{Lat}(R), I \cap (J + K) = (I \cap J) + (I \cap K). \) However, call a (complete) lattice completely distributive when it satisfied the dual extended distributive laws:
\[
\bigwedge_C \left[ \bigvee_{\gamma \in A_\gamma} x_{\gamma, \alpha_{\gamma}} \right] = \bigvee_C \left[ \bigwedge_{\phi \in \phi \in C} x_{\phi(\gamma)} \right],
\]
\[
\bigvee_C \left[ \bigwedge_{\gamma \in A_\gamma} x_{\gamma, \alpha_{\gamma}} \right] = \bigwedge_C \left[ \bigvee_{\phi \in \phi \in C} x_{\phi(\gamma)} \right],
\]
for any nonvoid family of index-sets \( A_\gamma \), one for each \( \gamma \in C \), provided \( \phi \) is the set of all functions \( \phi \) with domain \( C \) and \( \phi(\gamma) \in A_\gamma \). The poset \( L \) is a complete lattice, that is,

\[
\bigvee_{a \in A} a = \sup A, \quad \bigwedge_{a \in A} a = \inf A
\]
exist in \( L \) for every subset \( A \subseteq L \).

**Theorem 2.1.** A complete distributive lattice is a topological lattice under order-convergence if and only if it satisfies
\[ a \land (\lor x_\alpha) = \lor(a \land x_\alpha) \]
and, dually
\[ a \lor (\land x_\alpha) = \land(a \lor x_\alpha) \]

**Definition 2.3.** Let \( \alpha \) be a cardinal and let \( \Lambda \) be a set with \(|\Lambda| = \alpha \). \( M \in \text{Lat}(R) \) is said to be \( \alpha \)-irreducible if for ideals \( \{I_\lambda\}_{\lambda \in \Lambda} \) of \( R \), the equation \( \bigcap_{\lambda \in \Lambda} I_\lambda = M \) implies that there exists some \( \lambda_0 \in \Lambda \) such that \( I_{\lambda_0} = M \).

For a ring \( R \), we denote by \( \alpha \text{-} \text{Irr}(R) \) the set of all \( \alpha \)-irreducible ideals of \( R \).

**Definition 2.4.** Let \( \alpha \) be a cardinal and let \( \Lambda \) be a set with \(|\Lambda| = \alpha \). \( M \in \text{Lat}(R) \) is said to be \( \alpha \)-strongly irreducible if for ideals \( \{I_\lambda\}_{\lambda \in \Lambda} \) of \( R \), the inclusion \( \bigcap_{\lambda \in \Lambda} I_\lambda \subseteq M \) implies that there exists some \( \lambda_0 \in \Lambda \) such that \( I_{\lambda_0} \subseteq M \).

For a ring \( R \), we denote by \( \alpha \text{-} \text{SIrr}(R) \) the set of all \( \alpha \)-strongly irreducible ideals of \( R \).

**Theorem 2.2.** Let \( R \) be a ring and \( \alpha \text{-} \text{Irr}(R) = \alpha \text{-} \text{SIrr}(R) \) for any cardinal \( \alpha \). If \( I \cap (\bigcup I_\alpha) = \bigcup (I \cap I_\alpha) \), then \( R \) is a topological lattice under order-convergence, for all ideals \( I, I_\alpha \) of \( R \).
References


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A perspective to properties of multiplication and co-multiplication modules

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Abstract
Let $R$ be a commutative ring with non-zero identity. This paper is devoted to study some of properties of multiplication and co-multiplication modules. A number results concerning multiplication and co-multiplication modules are given.

Keywords: multiplication module, co-multiplication module.

Mathematics Subject Classification: 51P05

1 Introduction
Throughout this paper all rings will be commutative with non-zero identity and all modules will be unitary. Let $R$ be a ring. An $R$-module $M$ is called a multiplication module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. Let $\text{End}(M)$ be the ring of $R$-homomorphisms of $M$. The purpose of this paper is to introduce the concept of multiplication and co-multiplication $R$-modules. $M$ is said to be a co-multiplication $R$-module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = (0 :_M I)$. We have shown that a free module is a multiplication module if and only if it has rank 1. Also we have shown that a free module is a co-multiplication module if and only if it has rank 1. Furthermore, we will obtain another characterization for co-multiplication and multiplication $R$-modules.

2 multiplication modules
In this section we will provide the definitions and results of multiplication module.

Definition 2.1. $M$ is said to be a multiplication $R$-module if for any submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$.

Example 2.2. Every cyclic $R$-module is multiplication. If $R$ be a commutative ring with identity and $M = R$. Then $M$ is a multiplication $R$-module ($Z$ as a $Z$-module is multiplication module).

Proposition 2.3. Every homomorphic images of multiplication module are multiplication.

Proposition 2.4. Every epimorphism of multiplication module is an automorphism.

Proof. This proposition proved in [4].

Proposition 2.5. Let $M$ be a multiplication and $X$ be an arbitrary $R$-modules If there exists a epimorphism $f \in \text{Hom}_R(M, X \oplus X)$, then $X = 0$
Proof. This proposition proved in [5].

**Proposition 2.6.** If $M$ is a nonzero free multiplication $R$-module, then $M \cong R$

**Proof.** Since $M$ is a free $R$-module, then $M \cong \bigoplus_{X} R$ for some index set $X$. If $\vert X \vert \geq 2$, then there exists a epimorphism $f \in \text{Hom}_R(M, R \oplus R)$. Since $M$ is multiplication $R$-module, by 2.4, $R = 0$. It is a contradiction. Thus $\vert X \vert = 1$ and hence $M \cong R$.

A free module is a multiplication module if and only if it has rank 1.

**Definition 2.7.** On the other hand, a module $M$ is said to satisfy Fitting’s Lemma if for each $\varphi \in \text{End}(M)$ there exists an integer $n \geq 1$ such that $M = \text{Ker}\varphi^n \oplus \text{Im}\varphi^n$.

**Theorem 2.8.** Let $M$ be a multiplication module satisfying descending chain condition on multiplication submodule and $\varphi \in \text{End}(M)$. Then $M$ satisfies Fitting’s Lemma.

**Proof.** This theorem proved in [4].

Recall that an $R$-module $M$ is said to be indecomposable if it is not the direct sum of two nonzero submodules.

**Theorem 2.9.** Let $M$ be an indecomposable multiplication module satisfying descending chain condition on multiplication submodules and let $\varphi \in \text{End}(M)$. Then the following are equivalent.

(1) $\varphi$ is a monomorphism.
(2) $\varphi$ is an epimorphism.
(3) $\varphi$ is an automorphism.
(4) $\varphi$ is not nilpotent.

**Proof.** This theorem proved in [4].

3 co-multiplication modules

Let $M$ be an $R$-module. For each ideal $I$ of $R$ and each submodule $N$ of $M$ define $(N :_M I) = \{m \in M | ma \in N \text{ for each } a \in I\}$ and $\text{ann}_M(I) = \{m \in M | ma = 0 \text{ for each } a \in I\}$. It is clear that $(N :_M I)$ and $\text{ann}_M(I)$ are two submodule of $M$.

**Definition 3.1.** An $R$-module $M$ is called co-multiplication provided for each submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = (0 :_M N) = \text{ann}_M(I)$.

**Example 3.2.** If $p$ is a prime number, then $\mathbb{Z}(p^\infty)$ is a co-multiplication $\mathbb{Z}$-module but $\mathbb{Z}$ (as a $\mathbb{Z}$-module) is not a co-multiplication module.

It is clear that, if $M$ is a co-multiplication $R$-module, then for each submodule $N$ of $M$, $N = (0 :_M \text{ann}_R(N))$.

**Proposition 3.3.** Let $M$ be a co-multiplication $R$-module. Then every submodule of $M$ is a co-multiplication module.

**Proof.** This theorem proved in [2].

**Proposition 3.4.** Not every homomorphic image of co-multiplication module is a co-multiplication module.

**Proof.** By example 3.6 in [1].

**Proposition 3.5.** Every monomorphism of co-multiplication module is an automorphism.
Proof. This theorem proved in [1].

Lemma 3.6. Let $M$ be a co-multiplication $R$-module, $K$ and $L$ be submodules of $M$ and $f \in \text{Hom}_R(K,L)$. Then
(1) If $f$ is a monomorphism, Then $K \subseteq L$.
(2) If $f$ is an epimorphism, Then $L \subseteq K$.
(3) If $f$ is an isomorphism, Then $K = L$.

Proof. First, assume that $f$ is a monomorphism. For each $x \in K$, $\text{ann}_R(x) = \text{ann}_R(f(x))$. So $x \in f(x)R \subseteq \text{Im} f$. Hence $K \subseteq L$. For 2, let $f$ be an epimorphism. For each $y \in L$ there exists $x \in K$ such that $f(x) = y$. At the other hand $\text{ann}_R(x) \subseteq \text{ann}_R(y)$. Therefore $y \in xR \subseteq K$. Since it is true for each $y \in L$, then $L \subseteq K$. Finally, 1 and 2 imply 3.

Proposition 3.7. Let $M$ be a co-multiplication and $X$ be an arbitrary $R$-modules. If there exists a monomorphism $f \in \text{Hom}_R(X \oplus X, M)$, then $X = 0$

Proof. It is clear that $f(X \oplus 0) \cong f(0 \oplus X)$. Then by 3.5, $f(x \oplus a) = f(0 \oplus X)$. Since $f$ is a monomorphism, $X \oplus 0 = 0 \oplus X$ and hence $X = 0$.

Proposition 3.8. If $M$ is a nonzero free co-multiplication $R$-module, then $M \cong R$.

Proof. since $M$ is a free $R$-module, then $M \cong \bigoplus_X R$ for some index set $X$. If $|X| \geq 2$, then there exists a epimorphism $f \in \text{Hom}_R(M, R \oplus R)$. Since $M$ is co-multiplication $R$-module, by 3.6, $R = 0$. It is a contradiction. Thus $|X| = 1$ and hence $M \cong R$.

On the other hand, a free module is a co-multiplication module if and only if it has rank 1.

Theorem 3.9. Let $M$ be a co-multiplication module satisfying ascending chain condition on submodules $N$ such that $M/N$ is co-multiplication $R$-module. Then $M$ satisfies Fitting’s Lemma.

Proof. This theorem proved in [1].

Theorem 3.10. Let $M$ be an indecomposable co-multiplication module satisfying ascending chain condition on submodules $N$ such that $M/N$ is a co-multiplication $R$-module. let $\varphi \in \text{End}_R(M)$. then the following are equivalent.
(1) $\varphi$ is an monomorphism.
(2) $\varphi$ is an epimorphism.
(3) $\varphi$ is an automorphism.
(4) $\varphi$ is not nilpotent.

Proof. This theorem proved in [1].

References


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Complex and hypercomplex structures on Lie superalgebras

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Abstract
Hypercomplex structures appeared in the physical theory of supersymmetry has been interested in physics and mathematics. Supersymmetry can be described by Lie superalgebras using commuting and anti-commuting variables. Recently, the complex structures on Lie superalgebras and invariant hypercomplex structures on Lie algebras have been interested in [2, 3, 4, 6, 8, 10, 11, 14]. The goal of this talk paper is to extend the hypercomplex structure notion to Lie superalgebras with emphasizing on solvability and the dimension of the derivation of even part of a Lie superalgebra.

Keywords: Supersymmetry, hypercomplex structure, Lie superalgebra.

Mathematics Subject Classification: Primary: 17B60; Secondary: 17B70, 17B30.

1 Introduction

Lie algebra theory describes the theory of symmetries using commuting variables. Similarly, Lie superalgebra theory describes super-symmetries using commuting and anti-commuting variables. There is a common physical motivation for theories of Lie algebras and Lie superalgebras as follows.

In quantum theory, any particle with spin number as $\frac{2k+1}{2}$ is called a fermion and any one with integer spin is called a boson. The Nucleus of an atom is a fermion or boson depending on whether the total number of its protons and neutrons is odd or even. Recently, physicists have discovered that this has caused some very strange behavior in certain atoms under unusual conditions, such as very cold helium. The physical theory of symmetries studies the transformations connecting bosons, but supersymmetry theory studies the transformations connecting bosons and fermions. So, Lie superalgebra theory is an algebraic model for supersymmetry. On the other hand, Hypercomplex structure (hcs) has been interested by physicians and mathematicians in the theory of supersymmetry. This paper defines and studies on the hypercomplex structures on Lie superalgebras, which is completely a new subject.

Definition 1.1. (a) A $\mathbb{Z}_2$-graded vector space or a super vector space is a direct sum of two vector spaces, $V = V_0 \oplus V_1$. Where, $V_0$ and $V_1$ are subspaces of $V$. Any member of $V_0$ is called even member and any one of $V_1$ is odd.

(b) A superalgebra is a $\mathbb{Z}_2$-graded algebra.

(c) A Lie superalgebra (ls) is a superalgebra $L = L_0 \oplus L_1$ (over $\mathbb{R}$ or $\mathbb{C}$) with a bilinear bracket $[\cdot,\cdot]$, such that for $x \in L_\alpha$, $y \in L_\beta$ and $z \in L_\gamma:

(i) $[x,y] = -(-1)^{\alpha\beta}[y,x],$
(ii) $(-1)^{\alpha\gamma}[x,y],z] + (-1)^{\alpha\beta}[[y,z],x] + (-1)^{\beta\gamma}[[z,x],y] = 0.$
In the super vector space representing physical particles, the even vectors represent bosons and the odd vectors represent fermions.

We say $L = L_0 \oplus L_1$ is $(m|n)$ dimensional if $L_0$ and $L_1$ be of dimensions $m$ and $n$, respectively.

**Definition 1.2.** A hypercomplex structure (hcs) on a Lie algebra $g$ is a pair $J_1, J_2$ of complex structures on $g$ satisfying $J_1 J_2 = - J_2 J_1$, $J_k^2 = -I$ and $N_k = 0$, for $k = 1, 2$, where $I$ is the identity and $N_k$ is the Nijenhuis tensor corresponding to $J_k$ as $N_k(X,Y) = [J_k X, J_k Y] - J_k ([X, J_k Y] + [J_k X, Y]) - [X, Y]$ (for $X, Y \in g$).

Note that $J_3 := J_1 J_2$ is another complex structure on $g$ and $J_1, J_2, J_3$ satisfy

$$J_1 J_2 = J_3, \quad J_2 J_3 = J_1, \quad J_3 J_1 = J_2.$$ 

In this talk, we extend the definition of hypercomplex structure to Lie superalgebras.

## 2 Main Result

Here is some results on Lie superalgebras equipped with hypercomplex structures.

**Theorem 2.1.** The restriction of an hcs on a Lie superalgebra $g = g_0 \oplus g_1$ to its even part $g_0$ gives an hcs on $g_0$ as a Lie algebra.

**Theorem 2.2.** If a Lie superalgebra $g = g_0 \oplus g_1$ admits an hcs and its even part is a non-solvable 4-dimensional Lie algebra, then $g_0 = R \oplus \mathfrak{so}(3)$. Moreover, the hcs on $R \oplus \mathfrak{so}(3)$ is unique up to equivalence.

In the case when $g_0$ is solvable we will analyze separately the cases $\dim g_0' = 0, 1, 2, 3$ (where $g_0'$ denotes the derivation of $g_0$). If $\dim g_0' = 0$ then $g_0$ is abelian. In this situation the integrability condition is automatically satisfied and there is a one to one correspondence between hypercomplex structures on $g$ and points in the space $GL(4n, \mathbb{R})/GL(n, \mathbb{H})$. The correspondence is established by fixing a hcs $\{J_0^0\}_{a=1,2}$ and sending $T \in GL(4n, \mathbb{R})$ to $T J_0^0 T^{-1}$. Moreover, it follows that every hcs is equivalent to $\{J_0^0\}_{a=1,2}$.

Now, we consider the cases $\dim g_0' = 1, 2, 3$

**Proposition 2.3.** Let $g = g_0 \oplus g_1$ be a Lie superalgebra with $\dim(g_0') = 1$, then $g$ does not admit any hcs.

**Theorem 2.4.** Let $g = g_0 \oplus g_1$ be a Lie superalgebra and $\dim g_0' = 2$.

(i) If $g$ admits a hcs, then $g_0 \cong a f f(C)$;

(ii) the equivalence classes of hcs on $g$ are parametrized by the space $R \mathbb{P}^2$.

When $\dim g_0' = 3$, we have

**Theorem 2.5.** Let $g = g_0 \oplus g_1$ be a Lie superalgebra and $\dim g_0' = 3$ and $g_0$ is solvable then one of the following holds:

(a) $g_0'$ is abelian, in this case if $g$ admits a hcs, then $g_0$ corresponds to the space $\mathbb{R} \mathbb{H}^4$;

(b) $g_0'$ is a Heisenberg algebra, in this case if $g$ admits a hcs, then $g_0$ corresponds to the space $\mathbb{C} \mathbb{P}^2$.

## References


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On the number of 5-ary algebraic operations of idempotent algebras

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Abstract
Let \((U; F)\) be an idempotent algebra and there is an \(r\)-ary essentially algebraic operation in \(F\) where there is not any \((r + 4)\)-ary algebraic operation depending on at least \(r + 1\) variables. In this work we prove that the set of all 5-ary algebraic operations of this algebras forms a finite De Morgan algebra with fixed element.

Keywords: Idempotent algebra, algebraic (or polynomial) operation, 5-ary algebraic operation, De Morgan algebra with fixed element

Mathematics Subject Classification: 03G10, 06E05, 62H05, 14H05, 62E10

1 Introduction
Let \((U; F)\) be an algebra. We call the \(n\)-ary operations
\[ e_k^{(n)}(x_1, x_2, ..., x_n) = x_k \quad (k = 1, 2, ..., n, \quad n = 1, 2, ...) \]
trivial. The smallest set containing trivial operations and closed under the composition with fundamental operations is called set of algebraic operations (or polynomials). The algebra \((U; F)\) is called idempotent if the subset of unary algebraic operations of \(U\) consists of trivial operations only. In other words, the algebra \((U; F)\) is idempotent iff the equation \(f(x, x, ..., x) = x\) holds for every algebraic operation \(f \in F\). We denote by \(\Phi_1(U)\) the set of all non-negative integers \(n\) for which there exists an algebraic non-trivial \(n\)-ary operation in \(U\) depending on every variable (such operation is said essentially \(n\)-ary algebraic operation) and by \(\Phi_2(U)\) the set of all non-negative integers \(n\) for which there exists an algebraic non-trivial \(n\)-ary operation in \(U\) depending on at least \(n - 2\) variables. Let \(P^{(n)}(U)\) be the set of all \(n\)-ary algebraic operations (or \(n\)-ary polynomials) in the operations of \(F\). In [10, 11] Gratzer and Plonka showed that under the condition that there is a commutative binary polynomial the sequence
\[ |P^{(2)}(U)|, |P^{(3)}(U)|, ..., |P^{(n)}(U)|, ... \]
(|\(P^{(n)}(U)\)| denote the number of \(n\)-ary polynomials in \(U\)) provided \(|P^{(n)}(U)| \neq 1\), is increasing and satisfies the inequality \(|P^{(n+1)}(U)| \geq |P^{(n)}(U)| + (n - 1)\). In [8], K.Urbanik is shown that if \(U\) is idempotent and \(2 \in \Phi_1(U)\) and there is an integer \(r \geq 3\) such that \(r \notin \Phi_1(U)\), then \((P^{(2)}(U); \lor, \land, -, 0, 1)\) is defined as follows
\[ (f \lor g)(x, y) = f(g(x, y), y) \]
\[ (f \land g)(x, y) = f(x, g(x, y)) \]
\[ \overline{f}(x, y) = f(y, x), \]
\[ 1(x, y) = x, \]
\[ 0(x, y) = y, \]
is a finite Boolean algebra satisfying some additional conditions, and \((U; P^{(3)}(U))\) is a diagonal algebra. It is natural to ask about the sets \(P^{(3)}(U), P^{(4)}(U), \ldots, P^{(n)}(U)\). The authors in [5] showed that if \(U\) is idempotent and \(2 < n \in \Phi_1(U)\) and there is no algebraic \(r + 2\)-ary operation in \(U\) depending on at least \(r + 1\) variables, then \((P^{(3)}(U), \lor, \land, -, 0, 1, 0)\) is defined as follows
\[
(f \lor g)(x, y, z) = f(g(x, y, z), g(y, y, z), z),
\]
\[
(f \land g)(x, y, z) = f(x, g(x, y, y), g(x, y, z)),
\]
\[
\overline{f}(x, y, z) = f(z, y, x),
\]
\[
0(x, y, z) = x,
\]
\[
1(x, y, z) = z,
\]
\[
\phi(x, y, z) = y,
\]
is a finite De Morgan algebra with fixed element satisfying some additional conditions. In this paper we show that the set \(P^{(5)}(U)\) of all 5-ary algebraic operations of an idempotent algebra \(U\) which satisfies the conditions of theorem 2.1 is a finite De Morgan algebra with fixed element.

2 Main Result

**Theorem 2.1.** Let \(U\) be an idempotent algebra and \(r \geq 2, r \in \Phi_1(U)\) and \(r + 4 \notin \Phi_2(U)\). Let \(q\) be an \(r\)-ary algebraic operation in the algebra \(U\) depending on every variable. For each 5-ary algebraic operation \(f\) there exist five pair-wise disjoint subsets \(Q_{1f}, Q_{2f}, Q_{3f}, Q_{4f}\) and \(Q_{5f}\) of the set \(S = \{1, 2, \ldots, r\}\) such that \(Q_{1f} \cup Q_{2f} \cup Q_{3f} \cup Q_{4f} \cup Q_{5f} = S\) and
\[
q(f(x_1, y_1, z_1, t_1, d_1), f(x_2, y_2, z_2, t_2, d_2), \ldots, f(x_r, y_r, z_r, t_r, d_r)) = q(u_1, u_2, \ldots, u_r),
\]
where
\[
u_j = \begin{cases} 
x_j & j \in Q_{1f}, 
y_j & j \in Q_{2f}, 
z_j & j \in Q_{3f}, 
t_j & j \in Q_{4f}, 
d_j & j \in Q_{5f} \end{cases}
\]
The correspondence between 5-ary algebraic operations \(f\) and pair-wise disjoint ordered 5-tuple \((Q_{1f}, Q_{2f}, Q_{3f}, Q_{4f}, Q_{5f})\) of subsets of the set \(S\) is one-to-one. Moreover, for each pair \(f, g\) of 5-ary algebraic operations we have the equation
\[
q(f(g(x_1, y_1, z_1, t_1, d_1), g(y_1, y_1, z_1, t_1, d_1), g(z_1, z_1, z_1, t_1, d_1), g(t_1, t_1, t_1, t_1, d_1), e_1), 
q(g(x_2, y_2, z_2, t_2, d_2), g(y_2, y_2, z_2, t_2, d_2), g(z_2, z_2, z_2, t_2, d_2), g(t_2, t_2, t_2, t_2, d_2), e_2), \ldots, 
q(g(x_r, y_r, z_r, t_r, d_r), g(y_r, y_r, z_r, t_r, d_r), g(z_r, z_r, z_r, t_r, d_r), g(t_r, t_r, t_r, t_r, d_r), e_r)) = q(v_1, v_2, \ldots, v_r),
\]
where
\[
v_j = \begin{cases} 
x_j & j \in Q_{1f} \cap Q_{1g}, 
y_j & j \in (Q_{1f} \cup Q_{2g}) \cap (Q_{2g} \cap (Q_{1f} \cup Q_{2f})), 
z_j & j \in (Q_{1g} \cup Q_{2g}) \cap (Q_{3f} \cup (Q_{1f} \cup Q_{2f} \cup Q_{3f})), 
t_j & j \in (Q_{1g} \cup Q_{2g} \cup Q_{3g}) \cap (Q_{1f} \cup (Q_{2f} \cup Q_{3f} \cup Q_{4f})), 
d_j & j \in (Q_{3g} \cap (Q_{1f} \cup Q_{2f} \cup Q_{3f} \cup Q_{4f})), 
e_j & j \in Q_{5f} \cap Q_{5f} \end{cases}
\]
Corollary 2.2. Let $U$ be an idempotent algebra and $r \geq 2$, $r \in \Phi_1(U)$ and $r + 4 \not\in \Phi_2(U)$, then the set of all 5-ary algebraic operations in $U$ is finite.

Proof. Since $r \in \Phi_1(U)$, let $q$ be an $r$-ary algebraic operation depending on each variable. According to the theorem 2.1, the operation $q$ induces a one-to-one correspondence between 5-ary algebraic operations $f$ and disjoint ordered 5-tuples $(Q_1f, Q_2f, Q_3f, Q_4f, Q_5f)$ of subsets of the set $S = \{1, 2, ..., r\}$. Since the number of such ordered 5-tuple in a finite set is finite, so the set of 5-ary algebraic operations of algebra $U$ is also finite. \qed

References


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The $n$-th commutativity degree of some classes of 2-generator groups with nilpotency class 2

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Abstract

Let $G$ be a finite group and $n$ be a positive integer. The $n$-th commutativity degree, denoted as $P_n(G)$, is defined to be the probability of commuting the $n$-th power of a random element of $G$ with another element. Here, we compute the explicit formula for $n$-th commutativity degree of some classes of 2-generator groups with nilpotency class two.

Keywords: commutativity degree, $n$-th commutativity degree.

Mathematics Subject Classification: 20D15, 20P05.

1 Introduction

In this paper, we consider the following finitely presented groups,

$$G_{mk} = \langle a, b | a^m = b^k = 1, [a, b]^m = [a, b], [a, b]^k = [a, b] \rangle, \ m, k \geq 2;$$

$$H_m = \langle a, b | a^{m^2} = b^m = 1, b^{-1}ab = a^{1+m} \rangle, m \geq 2;$$

$$K(s, l) = \langle a, b | ab^s = b^l a, ba^s = a^l b \rangle, \ where \ (s, l) = 1.$$ and find explicit formulas for their $n$-th commutativity degree. First, we state a lemma without proof that establishes some properties of groups of nilpotency class two.

Lemma 1.1. If $G$ is a group and $G' \subseteq Z(G)$, then the following hold for every integer $k$ and $u, v, w \in G$:

(i) $[uv, w] = [u, w][v, w]$ and $[u, vw] = [u, v][u, w]$;

(ii) $[u^k, v] = [u, v^k] = [u, v]^k$;

(iii) $[uv]^k = u^k v^k [v, u]^{k(k-1)/2}$.

The following lemmas can be seen in [3].

Lemma 1.2. Let $d = \text{gcd}(m, k)$, then we have

(i) every element of $G_{mk}$ may be uniquely represented by $a^i b^j [a, b]^t$, where $0 \leq i \leq m - 1$, $0 \leq j \leq k - 1$ and $0 \leq t \leq d - 1$.

(ii) $|G_{mk}| = mld$.

Lemma 1.3. Let $H_m = \langle a, b | a^{m^2} = b^m = 1, b^{-1}ab = a^{1+m} \rangle$, where $m \geq 2$; then

(i) every element of $H_m$ may be uniquely represented by $b^j a^i$, where $0 \leq i \leq m^2 - 1$ and $0 \leq j \leq m - 1$.

(ii) $|H_m| = m^3$. 

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Lemma 1.4. The groups $K(s,l) = \langle a, b | ab^s = b^l a, ba^s = a^l b \rangle$, where $(s,l) = 1$, have the following properties:

(i) $|K(s,l)| = |l - s|^3$, if $(s,l) = 1$ and is infinite otherwise;
(ii) if $(s,l) = 1$, then $|a| = |b| = (l - s)^2$;
(iii) if $(s,l) = 1$, then $a^{l-s} = b^{s-l}$.

Note: If $(s,l) = 1$, then $K(s,l) \cong K(1, l - s + 1)$, which we may write as $K_{l-s+1}$. Hence we only calculate $P_n(K_m)$, where $m = l - s + 1$.

Lemma 1.5. Every element of $K_m$ may be uniquely represented by $x = a^\beta b^\gamma a^{(m-1)\delta}$, where $1 \leq \beta, \gamma, \delta \leq m - 1$.

Also we have the following lemma[3].

Lemma 1.6. For the integer $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ and variables $x, y, i$ and $j$, the number of solutions of the equation $xy \equiv ij \mod n$ is

$$\prod_{i=1}^{s} p_i 2^{\alpha_i - 1} \left( p_i^{\alpha_i + 1} + p_i^{\alpha_i} - 1 \right).$$

2 Main Result

In this section, we give explicit formulas for n-th commutativity degree of $G_{mk}$, $H_m$ and $K_m$.

Proposition 2.1.

(i) Let $G = G_{mk}$, where $m, k \geq 2$, $d = \gcd(m, k)$, $l = \gcd(n, d)$ and $d/l = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}$. Then we have:

$$P_n(G) = \prod_{i=1}^{s} p_i^{\alpha_i+1} + p_i^{\alpha_i} - 1.$$

(ii) Let $G = H_m$, where $m \geq 2$, $l = \gcd(n, m)$ and $m/l = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}$. Then we have:

$$P_n(G) = \prod_{i=1}^{s} p_i^{\alpha_i+1} + p_i^{\alpha_i} - 1.$$
so

\[ P_n(G) = \prod_{i=1}^{s} \frac{p_i^{\alpha_i+1} + p_i^{\alpha_i} - 1}{p_i^{2\alpha_i+1}}. \]

The proof of parts (ii) and (iii) are similar.

We can check the results by GAP [1]. For example for \( G_{mk} \) for \( m = 6, k = 12 \), and different integers \( n \), we get the following table

<table>
<thead>
<tr>
<th>( n )</th>
<th>( l )</th>
<th>( d/l )</th>
<th>The solutions</th>
<th>( P_n(G) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>6</td>
<td>47520</td>
<td>( \frac{25}{225} )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>76032</td>
<td>( \frac{27}{27} )</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>116640</td>
<td>( \frac{7}{7} )</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>3</td>
<td>76032</td>
<td>( \frac{31}{31} )</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>6</td>
<td>47520</td>
<td>( \frac{35}{35} )</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>1</td>
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<td>1</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>1</td>
<td>186624</td>
<td>1</td>
</tr>
</tbody>
</table>

References


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A note on power values of derivation in semiprime rings

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Abstract
Let $R$ be a 2-torsion free semiprime ring and $d$ a non-zero derivation of $R$. Further, let $A = O(R)$ be the orthogonal completion of $R$ and $B = B(C)$ where $C$ is the extended centroid of $R$. We study that if $(d[x, y][x, y] + [x, y]d[x, y])^n - (d(x, y))^n = 0$ for all $x, y \in R$, where $n \geq 1$ is a fixed integer, then there exists idempotent $e \in B$ such that $d$ vanishes identically on $eA$ and $d$ induces a zero derivation on $(1 - e)A$.

Keywords: derivation, semiprime ring, Martindale quotient ring.

Mathematics Subject Classification: 16W25, 16N60, 16R50.

1 Introduction
Let $R$ be an associative ring with center $Z(R)$. Recall that an additive map $d : R \rightarrow R$ is called derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Many results in literature indicate that global structure of a prime (semiprime) ring $R$ is often lightly connected to the behaviour of additive mappings defined on $R$. A well-known result of Herstein [5] stated that if $d$ is a nonzero derivation of a prime ring $R$ such that $d(x)^n \in Z(R)$ for all $x \in R$, then $R$ satisfies $S_4$, the standard identity in four variables.

Many articles have been studied derivations with annihilator conditions. In [6] Herstein proved that if $R$ is a prime ring and $d$ is an inner derivation of $R$ such that $d(x)^n = 0$ for all $x \in R$ and $n$ is a fixed integer, then $d = 0$. The number of authors extended this theorem in several ways. In [4] Giambruno and Herstein extended this result to arbitrary derivations in semiprime rings. In [2] Carini and Giambruno proved that if $R$ is a prime ring with derivation $d$ such that $d(x)^n(x) = 0$ for all $x \in L$, a Lie ideal of $R$, then $d(L) = 0$, when $R$ has no non-zero nil right ideal and $charR \neq 2$.

The same conclusion holds when $n(x) = n$ is fixed and $R$ is a 2-torsion free semiprime ring.

Now, we will generalize Herstein’s results [6, 5] when the condition are more widespread. Here we will examine what happens in case $(d[x, y][x, y] + [x, y]d[x, y])^n - (d(x, y))^n = 0$ and $(d[x, y][x, y] + [x, y]d[x, y])^n - (d[x, y])^n \in Z(R)$ for any $x, y \in R$, a 2-torsion free semiprime ring, where $n \geq 1$ is a fixed integer.

The following result is useful tool needed in the proof of main results.

**Proposition 1.1.** Let $R$ be a prime ring of $charR \neq 2$, $d$ a non-zero derivation and $L$ be a non-central Lie ideal of $R$. Suppose $(d[x, y])^n - (d(x))^n(d(y))^n = 0$ for all $x, y \in L$ and $n \geq 1$ is a fixed integer. Then $R$ is commutative or $d = 0$.

The following example shows the hypothesis of primeness is essential in Proposition 1.1.
Example 1.2. Let $S$ be any ring, and $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in S \right\}$. Define $d : R \to R$ as follows:

$$d \left( \begin{array}{ccc} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

Then $0 \neq d$ is a derivation of $R$ such that $(d(xy))^n = (d(x))^n (d(y))^n$ for all $x, y \in R$, where $n > 1$ is a fixed integer, however $R$ is not commutative.

2 Main Result

Let $R$ be a semiprime orthogonally complete ring with extended centroid $C$. The notations $B = B(C)$ and $\text{spec}(B)$ denotes Boolean ring of $C$ and the set of all maximal ideal of $B$, respectively. It is well known that if $M \in \text{spec}(B)$ then $R_M = R/RM$ is prime [1, Theorem 3.2.7Venus Rahmani]. We use the notations $\Omega$-$\Delta$-ring, Horn formulas and Hereditary formulas. We refer the reader to [1, pages 37, 38, 43, 120Venus Rahmani] for the definitions and the related properties of these objects.

We establish the following technical result required in the proof of main results.

Lemma 2.1. [1, Theorem 3.2.18Venus Rahmani]. Let $R$ be an orthogonally complete $\Omega$-$\Delta$-ring with extended centroid $C$, $\Psi_i(x_1, x_2, \ldots, x_n)$ Horn formulas of signature $\Omega \Delta$, $i = 1, 2, \ldots$ and $\Phi(y_1, y_2, \ldots, y_m)$ a Hereditary first order formula such that $\overline{\Phi}$ is a Horn formula. $\overline{\alpha} = (a_1, a_2, \ldots, a_n) \in R^n$, $\overline{c} = (c_1, c_2, \ldots, c_m) \in R^m$. Suppose $R \models \Phi(\overline{c})$ and for every $M \in \text{spec}(B)$ there exists a natural number $i = i(M) > 0$ such that

$$R_M \models \Phi(\phi_M(\overline{c})) \implies \Psi_i(\phi_M(\overline{a})),
$$

where $\phi_M : R \to R_M = R/RM$ is the canonical projection. Then there exists a natural number $k > 0$ and pairwise orthogonal idempotents $e_1, e_2, \ldots, e_k \in B$ such that $e_1 + e_2 + \ldots + e_k = 1$ and $e_i R \models \Psi_i(\overline{e_i \overline{a}})$ for all $e_i \neq 0$.

We denote $O(R)$ the orthogonal completion of $R$ which is defined as the intersection of all orthogonally complete subset of $Q$ containing $R$.

Theorem 2.2. Let $R$ be a 2-torsion free semiprime ring with non-zero derivation $d$. Consider $(d([x, y])([x, y]) + [x, y]d([x, y]))^n - (d([x, y]))^{2n} = 0$ for all $x, y \in R$ and $n \geq 1$ is a fixed integer. Further, let $A = O(R)$ be the orthogonal completion of $R$ and $B = B(C)$ where $C$ is the extended centroid of $R$. Then there exists idempotent $e \in B$ such that $d$ vanishes identically on $eA$ and the ring $(1 - e)A$ is commutative.

Proof. By assumption we have $R$ satisfies

$$(d([x, y])([x, y]) + [x, y]d([x, y]))^n = (d([x, y]))^{2n}.$$

According to [1, Theorem 3.1.16Venus Rahmani] $d(A) \subseteq A$ and $d(e) = 0$ for all $e \in B$. Therefore, $A$ is an orthogonally complete $\Omega$-$\Delta$-ring, where $\Omega = \{\circ, +,-, \cdot,d\}$. Consider formulas

$$\Phi = (\forall x)(\forall y)(||d([x, y])([x, y]) + [x, y]d([x, y]))^n = (d([x, y]))^{2n}||,$$

$$\Psi_1 = (\forall x)||d(x) = 0||,$$

$$\Psi_2 = (\forall x)(\forall y)||xy = yx||.$$
One can easily check that $\Phi$ is a hereditary first order formula and $\neg \Phi$, $\Psi_1$, $\Psi_2$ are Horn formulas. So using Proposition 1.1 shows that all conditions of lemma 2.1 are fulfilled. Hence there exist two orthogonal idempotent $e_1$ and $e_2$ such that $e_1 + e_2 = 1$ and if $e_i \neq 0$, then $e_i A \models \Psi_i$, $i = 1, 2$. Which means that $eA$ is a commutative ring and $d$ induce a zero derivation on $(1 - e)A$. The proof is completed.

In a similar vein, we also have the following proposition.

**Proposition 2.3.** Let $R$ be a 2-torsion free semiprime ring with non-zero derivation $d$. Consider $(d([x,y])[x,y] + [x,y]d([x,y]))^n - (d([x,y]))^{2n} \in Z(R)$ for all $x, y \in R$ and $n \geq 1$ is a fixed integer. Further, let $A = O(R)$ be the orthogonal completion of $R$ and $B = B(C)$ where $C$ is the extended centroid of $R$. Then there exists idempotent $e \in B$ such that $d$ vanishes identically on $eA$ and the ring $(1 - e)A$ satisfies $S_4$.

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A result on generalized derivations with Engel conditions

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Abstract
Let $H$ be a generalized derivation of a prime ring $R$ without nonzero nil two-sided ideals. Suppose that $[H(x^n), x^m]_k = 0$ for all $x \in R$, where $n \geq 1, k \geq 1$ are fixed integers. Then $H(x) = bx$ for some $b \in Z(R)$ or $R$ is commutative.

Keywords: prime ring, generalized derivation, extended centroid.

Mathematics Subject Classification: 16W25, 16N60, 16R50.

1 Introduction
Let $R$ be a prime ring with center $Z(R)$, extended centroid $C$ and $U$ be Utumi quotient ring of $R$. An additive mapping $H$ from $R$ to $R$ is called a generalized derivation if there exists a derivation $d$ from $R$ to $R$ such that $H(xy) = H(x)y + xd(y)$. In [1] Chung proved that if $d$ is a derivation of a prime ring $R$ without nonzero nil two-sided ideal such that $[d(x^n), x^m]_k = 0$ for all $x \in R$, where $n \geq 1, k \geq 1$ are fixed integers, then $d = 0$ or $R$ is commutative. The aim of this paper is to study the situation when $[H(x^n), x^m]_k = 0$ for all $x \in R$, where $H$ is a generalized derivation of $R$.

Remark 1.1. [7] Let $R$ be a prime ring and $U$ be a Utumi quotient ring of $R$. Then every generalized derivation on $R$ can be uniquely extended to a generalized derivation of $U$ and assume the form $H(x) = bx + d(x)$ for all $x \in U$, for some $b \in U$ and a derivation $d$ of $U$.

2 Main Result

Lemma 2.1. Let $R = M_k(F)$, the ring of $k \times k$ matrices over a field $F$ and $a, b \in R$ such that $[bx^n, x^m]_k + [a, x^n]_{k+1} = 0$ for all $x \in R$, where $n \geq 1, k \geq 1$ are fixed integers. Then $a$ and $b$ are central.

Proof. Let $a = (a_{ij})_{k \times k}$ and $b = (b_{ij})_{k \times k}$. Put $x = e_{11}$, we have

$$0 = [be_{11}, e_{11}]_k + [a, e_{11}]_{k+1} = \begin{cases} (b + a)e_{11} + e_{11}(-b - 2a)e_{11} + e_{11}a & \text{if } k \text{ is odd} \\ (b + a)e_{11} - e_{11}be_{11} - e_{11}a & \text{if } k \text{ is even} \end{cases}$$

implying $a_{ij} = 0$ and $b_{ij} = 0$ for any $i, j$ ($i \neq j$). This give $a$ and $b$ are diagonal. Let $a = \sum_{i=1}^k w_i e_{ii}$. For some $F$-automorphism $\theta$ of $R$, $a^\theta$ and $b^\theta$ enjoy the same property as $a$ and $b$ do, namely, $[b^\theta x^n, x^m] + [a^\theta, x^n]_{k+1} = 0$ for all $x \in R$. Hence $a^\theta$ and $b^\theta$ must be diagonal. For each $j \neq 1$, we have

$$(1 + e_{1j})a(1 - e_{1j}) = \sum_{i=1}^k w_i e_{ii} + (w_{jj} - w_{11})e_{1j},$$
diagonal. Therefore $w_{jj} = w_{11}$ and so $a$ is central. Similarly, $b$ is central too. □
Theorem 2.2. Assume that $H$ is a generalized derivation of a prime ring $R$ without nonzero nil two-sided ideal. If for each $x \in R$, $[H(x^n), x^n]_k = 0$, where $n \geq 1$ and $k \geq 1$ are fixed integers, then $H(x) = bx$ for some $b \in Z(R)$ or $R$ is commutative.

Proof. By Remark 1.1, we may assume that for all $x \in U$, $H(x) = bx + d(x)$ for some $b \in U$ and a derivation $d$ of $U$. Hence $U$ satisfies

$$0 = [bx^n + d(x^n), x^n]_k = [bx^n, x^n]_k + \left[ \sum_{i=0}^{n-1} x^i d(x)x^{n-i-1}, x^n \right]_k.$$

Assume first $d$ is not $U$-inner, then by Kharchenko’s theorem [5]

$$[bx^n, x^n]_k + \left[ \sum_{i=0}^{n-1} x^i y x^{n-i-1}, x^n \right]_k = 0,$$

for all $x, y \in U$. In particular $U$ satisfies $[bx^n, x^n]_k = 0$. This is a polynomial identity and hence there exists a field $F$ such that $U \subseteq M_k(F)$ with $k > 1$ and $U$ and $M_k(F)$ satisfy the same polynomial identity [6]. But by choosing $x = e_{11}$, we get

$$0 = [bx^n, x^n]_k = be_{11} - e_{11} be_{11}$$

implying $b_{ij} = 0$ for any $i, j (i \neq j)$. This gives $b$ is central and so $[d(x^n), x^n]_k = 0$ for all $x \in R$. Therefore $d = 0$ or $R$ is commutative by [1].

If $d$ is inner derivation of $U$, i.e., there exist $a \in U$ such that $d(x) = [a, x]$ for all $x \in U$. Then

$$f(x) = [bx^n, x^n]_k + [[a, x^n], x^n]_k = 0.$$

By Chuang [1, Theorem 2] Shervin Sahebi ], this GPI is also satisfied by $Q$. If $R$ is commutative, there is nothing to prove. Otherwise, $f(x) = 0$ is a nontrivial GPI for $Q$. In case the center $C$ of $Q$ is infinite, we have $f(x) = 0$ for all $x \in Q \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of $C$. Since both $Q$ and $Q \otimes_C \overline{C}$ are prime and centrally closed [2], we may replace $R$ by $Q$ or $Q \otimes_C \overline{C}$ according to $C$ finite or infinite. Thus we may assume that $C = Z(R)$ and $R$ is $C$-algebra centrally closed, which satisfies $f(x) = 0$. By Martindale’s theorem [8], $R$ is then a primitive ring and hence by Jacobson’s theorem [4] $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$.

If $V$ is a finite dimensional over $C$, then the density of $R$ on $V$ implies that $R \cong M_k(C)$, where $k = \text{dim}_CV$. Hence $a$ and $b$ are central by Lemma 2.1. Hence $d = 0$ and so $H(x) = bx$ for all $x \in R$ and some $b \in Z(R)$, as desired.

If $V$ is an infinite dimensional over $C$, then for any $e = e^2 \in H = \text{soc}(R)$ we have $eRe \cong M_k(C)$ with $k = \text{dim}_C V$. Assume that either $a \notin C$ or $b \notin C$. Then one of them does not centralize the nonzero ideal $H = \text{soc}(R)$. Hence there exist $h_1, h_2 \in H$ such that $[a, h_1] \neq 0$ or $[b, h_2] \neq 0$. By Litoff’s Theorem [3], there exists idempotent $e \in H$ such that $ah_1, h_1a, bh_2, h_2b, h_2 \in eRe$. Since $R$ satisfies generalized identity $f(e(x^n)) = [b(e(x^n))^n, (e(x^n))^n]_{k+1} + [a, (e(x^n))^n]_{k+1}$, the subring $eRe$ satisfies $f(x) = [ebe(x^n), (x^n)]_k + [eae, (x^n)]_k$. Then by the above finite dimensional case, $eae$ and $ebe$ are central elements of $eRe$. Thus $ah_1 = (eae)h_1 = h_1(eae) = h_1a$ and $bh_2 = (ebe)h_2 = h_2(ebe) = h_2b$, a contradiction.

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A study of important group theoretical concept in graph theory

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Abstract
A subgraph $H$ of a graph $G$ is called characteristic if $\varphi(H) = H$, for each automorphism $\varphi \in \text{Aut}(G)$. In this paper, we use the main properties of characteristic subgroups in a finite group to obtain the main properties of this new concept in algebraic graph theory.

Keywords: Characteristic subgraph, Boolean algebra, automorphism group.

Mathematics Subject Classification: 05C10, 20E45

1 Introduction

Throughout this paper graph means simple finite graph and we follow the terminology and notation of [1, 3] for graphs. An trivial graph on $n$ vertices consists of $n$ isolated vertices with no edges. This graph is denoted by $\emptyset_n$. We refer to [2] for general properties of lattices and Boolean algebras.

We assume that $G$ is a graph and $u$, $v$ are vertices of $G$. The edge connecting $u$ and $v$ is denoted by $uv$ and the distance $d_G(u,v)$ is defined as the length of a shortest path connecting $u$ and $v$ in $G$. The eccentricity $\varepsilon(u)$ is the largest distance between $u$ and any other vertex $v$ of $G$. The minimum eccentricity is said to be radius and denoted by $r(G)$. The center of $G$ is the set of all vertices $u$ such that $\varepsilon(u) = r(G)$. A self-centered graph is one that the center is the same as vertex set. Consider the graph $G$ whose vertices are the $n$–tuples $(b_1, b_2, ..., b_n)$ with $b_i \in \{0,1\}$ and two vertices are adjacent if the corresponding tuples differ in precisely one place. Such a graph is called a $n$–dimensional hypercube and denote it by $Q_n$.

Suppose $G$ and $H$ are graphs. $H$ is said to be a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and it is a spanning subgraph of $G$ if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. A subgraph $H$ of $G$ is called characteristic if for every automorphism $\beta \in \text{Aut}(G)$, $\beta(H) = H$. We use the notation $H \leq_{ch} G$ to show that $H$ is a characteristic subgraph of $G$. It is easily seen that the graph $G$ itself and its trivial spanning subgraph are characteristic in $G$.

It is easy to see that if $P_n$ is a path of length $n$ with vertex set $V(P_n) = \{v_0, v_1, \cdots, v_n\}$ then for each $i$, $1 \leq i \leq n/2$, $v_i v_{i+1} \cdots v_{n-i}$ is a characteristic subgraph of $P_n$.

2 Main Result

In this section some basic properties of characteristic subgraphs are investigated. We first introduce an important class of characteristic subgraphs of a given graph $G$. Define $G[i] = \{x \in V(G) | \deg_G(x) = i\}$. Then it is easy to see that for each $i$, $1 \leq i \leq \Delta(G)$, where $\Delta$ is maximum
degree of vertices, \(\langle G[i]\rangle\) is a characteristic subgraph of \(G\). We begin by considering the center of a graph.

**Theorem 2.1.** \(\langle C(G)\rangle \leq ch G\).

**Corollary 2.2.** If \(T\) is a tree and \(\varphi \in Aut(T)\) then \(\varphi\) has at least a fixed vertex or a fixed edge.

**Theorem 2.3.** The number of characteristic spanning subgraphs of a non-trivial graph is always even.

**Remark:** Notice that trivial graphs have exactly one characteristic spanning subgraph.

**Corollary 2.4.** Suppose \(G\) is not regular or self-centered then \(G\) has at least four characteristic subgraphs.

**Definition 2.5.** A subgraph \(K\) of \(G\) is minimal characteristic in \(G\) if no proper nontrivial subgraph of \(K\) is characteristic in \(G\).

It is easy to see that every characteristic subgraph of \(G\) contains a minimal characteristic subgraph of \(G\).

**Theorem 2.6.** Let \(G\) be a graph and \(H\) be a minimal characteristic subgraph of \(G\). Then there exists positive integer \(i\) such that \(H \leq \langle G[i]\rangle\).

**Corollary 2.7.** Let \(G\) be a graph and for a positive integer \(i\), \(G[i]\) is singleton, then \(\langle G[i]\rangle\) is a characteristic subgraph of \(G\).

The converse of Corollary 2.7 is not hold. To do this, we assume that \(G\) is a cycle of length \(n\). Then \(G[2] = V(G)\) and \(\langle G[2]\rangle\) is minimal in \(G\).

**Theorem 2.8.** A graph \(G\) is vertex transitive graph if and only if except \(G\) and its trivial spanning subgraph, it doesn’t have any spanning characteristic subgraph.

Suppose \(S(G)\) and \(CSS(G)\) denote the set of all spanning and characteristic spanning subgraphs of \(G\), respectively. In the following theorem, we prove that \(S(G)\) has a Boolean algebra structure.

**Theorem 2.9.** Let \(G\) be a graph on \(n\) vertices, then \(S(G)\) is closed under taking intersection and union and \((S(G), \cap, \cup, ')\) is a Boolean algebra in which for each element \(H \in S(G)\), \(H'\) is a spanning subgraph such that \(E(H') = E(G) - E(H)\). Moreover, \(|S(G)| = 2^{\dim(G)}\).

**Theorem 2.10.** \(CSS(G)\) is closed under taking intersection and union and \((CSS(G), \cap, \cup, ')\) is a sub-Boolean algebra of \((S(G), \cap, \cup, ')\).

**Corollary 2.11.** \(H \in CSS(G)\) is an atom of Boolean algebra \((CSS(G), \cap, \cup, ')\) if and only if \(H\) is a minimal characteristic subgraph of \(G\). In particular, if \(G\) is a graph with exactly \(m\) minimal characteristic subgraph then \(|CSS(G)| = 2^m\).

**Corollary 2.12.** Every characteristic spanning subgraph of \(G\) can be represented as the union of some minimal characteristic subgraphs of \(G\).

In the end of this paper we construct a graph \(S'(G)\) with \(V(S'(G)) = S(G)\) and two spanning subgraph \(H\) and \(K\) are adjacent if and only if \(E(H) \subseteq E(K)\) and \(|E(H)| = |E(K)| + 1\). In the following theorem, it is proved that \(S'(G)\) is a hypercube.

**Theorem 2.13.** \(S'(G) \cong Q_m\), where \(m = |E(G)|\) and \(Q_m\) is \(m-\)dimensional hypercube.

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Analysis
A review of a theorem of Johnson on Jordan derivations and a theorem of Goldstein on bilinear forms

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Abstract

Every C*-algebra is weakly amenable. This theorem first was proved in [Invent. math. 74 (1983), pp. 305–319], based on a work in [J. Funct. Anal. 37 (1980), pp. 235–247]. In this note, we present a simplified and unified proof of this fact taken from [Banach algebras’97 (Blaubeuren), pp. 223–243, Berlin, 1998]. Johnson proved that every bounded Jordan derivation on a C*-algebra is a derivation; a new proof of this result will be presented here. We also bring a short proof of a theorem of Goldstein on bilinear forms.

Keywords: C*-algebras, Von Neumann algebras, Derivations, Jordan derivations, Weakly amenable, Bilinear forms.

Mathematics Subject Classification: 46L05

1 Weak Amenability of C*-Algebras

Let A be a Banach algebra. A Banach space X is called a Banach A-bimodule if X is an A-bimodule and, for all a ∈ A, x ∈ X,

\[ \|a \cdot x\| \leq \|a\| \|x\|, \quad \|x \cdot a\| \leq \|a\| \|x\|. \]

If X is a Banach A-bimodule, then X* is also a Banach A-module in a natural way: for a ∈ A, \( \varphi \in X^* \) and x ∈ X,

\[ (a \cdot \varphi)(x) = \varphi(x \cdot a), \quad (\varphi \cdot a)(x) = \varphi(a \cdot x). \]

Let X be a Banach A-bimodule. A linear map \( \delta : A \to X \) is called a derivation if \( \delta(ab) = \delta(a) \cdot b + a \cdot \delta(b) \), for all \( a, b \in A \). If x ∈ X is fixed, the mapping \( \delta_x : a \mapsto x \cdot a - a \cdot x \) is a bounded derivation; these kind of derivations are known as inner derivations. A Banach algebra A is called weakly amenable if every bounded derivation \( \delta : A \to A^* \) is inner.

Every C*-algebra is weakly amenable. This fact was first shown by Haagerup in [3] based on a work in [1]. Then, in 1997, Haagerup and Jakob [4] unified and simplified these proofs and presented the following theorem as one of their main results.

**Theorem 1.1.** Let A be a C*-algebra. Then A is weakly amenable.

First, it is proved that every von Neumann algebra is weakly amenable.

**Theorem 1.2** ([4, Corollary 1.8M. Abtahi and M. Tajari]). Let \( \mathcal{M} \) be a von Neumann algebra and \( \delta : \mathcal{M} \to \mathcal{M}^* \) be a derivation. Then, there is a functional \( \omega \in \mathcal{M}^* \) for which \( \delta = \delta_\omega \).
Next, to prove Theorem 1.1, for a given derivation $\delta : A \to A^*$, define $V : A \times A \to \mathbb{C}$ by $V(a,b) = \delta(a)(b)$. This $V$ is a bilinear form and $V(ab,c) = V(a,bc) + V(b,ca)$ since $\delta$ is a derivation. By [5, Corollary 2.4M. Abtahi and M. Tajari], one can extend $V$ to a bilinear form $\tilde{V} : A^{**} \times A^{**} \to \mathbb{C}$ such that $\|V\| = \|\tilde{V}\|$ and $\tilde{V}$ is continuous. If $a, b, c \in A^{**}$, by Goldstein’s Theorem, [8, Theorem 5.3.5M. Abtahi and M. Tajari], there are norm-bounded nets $(a_\lambda), (b_\mu)$ and $(c_\nu)$ in $A$ such that $a_\lambda \to a$, $b_\mu \to b$ and $c_\nu \to c$ so that
\[
\tilde{V}(ab,c) = \lim_{\lambda,\mu,\nu} V(a_\lambda b_\mu c_\nu) = \lim_{\lambda,\mu,\nu} V(a_\lambda b_\mu) + \lim_{\lambda,\mu,\nu} V(b_\mu c_\nu a_\lambda) = \tilde{V}(a,bc) + \tilde{V}(b,ca).
\]

Define a linear map $\tilde{\delta} : A^{**} \to A^{***}$ by $\tilde{\delta}(a)(b) = \tilde{V}(a,b)$. Since $A^{**}$ is a von Neumann algebra and $\delta$ is a derivation, there is a functional $\tilde{\omega}$ such that $\tilde{\delta}(a) = \tilde{\omega} \cdot a - a \cdot \tilde{\omega}$. If $\omega = \tilde{\omega}|_A$ then $\omega \in A^*$ and $\delta(a) = \omega \cdot a - a \cdot \omega$.

### 2 Johnson’s Theorem

A Jordan derivation from a Banach algebra $A$ into a Banach $A$-bimodule $X$ is a linear map $\delta : A \to X$ with $\delta(a^2) = a \cdot \delta(a) + \delta(a) \cdot a$. Of course, any derivation is a Jordan derivation and Johnson proved that the converse is true for $C^*$-algebras:

**Theorem 2.1** ([6]). Let $A$ be a $C^*$-algebra. Then every bounded Jordan derivation form $A$ into a Banach $A$-bimodule is a derivation.

Johnson’s proof uses the concept of symmetric amenability for Banach algebras. Here, we bring a new proof, taken from [4], which does not rely on symmetric amenability. First, it is proved that the statement in Theorem 2.1 holds for any von Neumann algebra:

**Theorem 2.2.** If $\mathcal{M}$ is a von Neumann algebra, then every bounded Jordan derivation form $\mathcal{M}$ into a Banach $\mathcal{M}$-bimodule is a derivation.

Then, in order to prove Theorem 2.1, instead of dealing directly with derivations, work with trilinear forms using the fact that, for a Banach algebra $A$, the following assertion are equivalent:

(a) every bounded Jordan derivation is a derivation;

(b) every bounded trilinear form $V : A \times A \times A \to \mathbb{C}$ which satisfies
\[
V(a^2, b, c) = V(a, ab, c) + V(a, b, ca), \quad (a, b, c \in A)
\]
will also satisfy
\[
V(ad, b, c) = V(a, db, c) + V(d, b, ca), \quad (a, b, c, d \in A).
\]

Now, let $\delta : A \to X$ be a bounded Jordan derivation. Take arbitrary functional $\varphi \in X^*$ and define $V_\varphi : A \times A \times A \to \mathbb{C}$ by $V_\varphi(a, b, c) = \varphi(c \cdot \delta(a) \cdot b)$. This $V_\varphi$ is a bounded bilinear form with $\|V_\varphi\| = \|\delta\|$. By [7, Theorem 2.3M. Abtahi and M. Tajari], one can extend $V_\varphi$ to a bounded trilinear form $\tilde{V} : A^{**} \times A^{**} \times A^{**} \to \mathbb{C}$ which is $w^*$-continuous in each variable. Then $\tilde{V}$ satisfies (1). In fact, if $a \in A^{**}$ and $b, c \in A$ then $a = \lim a_\lambda$ for some norm bounded net $(a_\lambda)$ in $A$ and thus
\[
\tilde{V}(a^2, b, c) = \lim_\lambda V(a_\lambda^2, b, c) = \lim_\lambda V(a_\lambda, a_\lambda b, c) + \lim_\lambda V(a_\lambda, b, ca_\lambda)
\]
\[
= \tilde{V}(a, ab, c) + \tilde{V}(a, b, ca) + \tilde{V}(a, b, ca).
\]

Now, if $b, c \in A^{**}$, take norm-bounded nets $(b_\mu)$ and $(c_\nu)$ in $A$ such that $b_\mu \to b$ and $c_\nu \to c$ and proceed as above. This proves that $\tilde{V}$ satisfies (1). Since $A^{**}$ is a von Neumann algebra, Theorem 2.2 implies that $\tilde{V}$ must satisfy (2) which shows that $V$ satisfies (2).
3 Goldstein’s Theorem

In this section, using results obtained in the previous sections, we include a simplified proof of the following theorem of Goldstein.

**Theorem 3.1** (Goldstein, [2]). Let $A$ be a $C^*$-algebra and $V : A \times A \to \mathbb{C}$ a bounded bilinear form. The following statements are equivalent:

(a) $V(a, b) = 0$ whenever $a, b \in A$ are orthogonal and self-adjoint;

(b) there exist functionals $\varphi, \psi \in A^*$ such that

$$V(a, b) = \varphi(ab) + \psi(ba), \quad (a, b \in A).$$

Goldstein worked with sesquilinear forms rather than bilinear forms. The crucial idea in the proof presented here of the above theorem is to relate $V$ to a Jordan derivation form $A \to A^*$, and then apply the result of the previous sections.

**Proof sketch.** First, it is proved that the statement of theorem holds when $A$ is a von Neumann algebra, [4, Proposition 3.6M. Abtahi and M. Tajari]. In the general case, the implication $(b) \Rightarrow (a)$ is evident. Conversely, suppose that $V(a, b) = 0$ whenever $a, b$ are orthogonal and self-adjoint. By [5, Corollary 2.4M. Abtahi and M. Tajari], one can extend $V$ to a bilinear form $\tilde{V} : A^{**} \times A^{**} \to \mathbb{C}$.

Let $p, q$ be orthogonal projections in $A^{**}$, and define $a = p - q$. By Kaplansky’s Density Theorem, there is a norm-bounded net $(a_\lambda)$ of self-adjoint elements in $A$ such that $a_\lambda \to a$, so that $a_\lambda^+ \to p$ and $a_\lambda^- \to q$, where $a_\lambda^+ = (|a_\lambda| + a_\lambda)/2$ and $a_\lambda^- = (|a_\lambda| - a_\lambda)/2$. Since $V(a_\lambda^+, a_\lambda^-) = 0$ for each $\lambda$, we obtain that $\tilde{V}(p, q) = 0$. Since $A^{**}$ is a von Neumann algebra, there are functionals $\tilde{\varphi}$ and $\tilde{\psi}$ in $A^{***}$ such that $\tilde{V}(a, b) = \tilde{\varphi}(ab) + \tilde{\psi}(ba)$, for all $a, b \in A^{**}$. Now, take $\varphi = \tilde{\varphi}|_A$ and $\psi = \tilde{\psi}|_A$. \(\square\)

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The composition of \(*\)-frames in Hilbert \(C^*\)-modules

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Abstract
A special class of sequences in Hilbert spaces which is near to the class of bases is called frame. Frames have been generalized to \(g\)-frames in Hilbert spaces and in Hilbert \(C^*\)-modules. This paper shows that the composition of two \(g\)-Bessel sequences in Hilbert \(C^*\)-modules is also a \(g\)-Bessel sequence. And it considers this fact about \(*\)-frames in Hilbert \(C^*\)-modules by an example. At the end, the \(*\)-frames in Hilbert \(C^*\)-modules are characterized with respect to \(*\)-orthonormal basis.

Keywords: dual \(*\)-frame, \(*\)-Bessel sequence, \(-\)frames, \(*\)-frame operator, Hilbert \(C^*\)-module.

Mathematics Subject Classification: Primary: 42C15; Secondary: 47C15, 46L99.

1 Introduction
Frames were first introduced in 1952 by Duffin and Schaeffer [2] in the study of nonharmonic fourier series. Frames possess many nice properties which make them very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields. The theory of frames has been generalized rapidly and various generalizations of frames in Hilbert spaces and Hilbert \(C^*\)-modules have been proposed recently. In 2005, Sun [5] introduced the notion of \(g\)-frames as a generalization of frames for bounded operators on Hilbert spaces. In this paper, we show that the composition of two \(g\)-Bessel sequences is a \(g\)-Bessel sequence but this fact does not hold for \(g\)-frames and by the composition of \(*\)frames we characterize \(*\)-frames.

Let us recall some definitions and basic properties of \(C^*\)-algebras and Hilbert \(C^*\)-modules that we need in the rest of the parer. For more details, we refer the interested reader to [4, 6].

Let \(\mathcal{A}\) be a unital \(C^*\)-algebra and \(a \in \mathcal{A}\). The nonzero element \(a\) is called strictly nonzero if zero doesn’t belong to \(\sigma(a)\) and \(a\) is also strictly positive if it is strictly nonzero and positive. We use this to define the absolute value \(|a| := (a^*a)^{\frac{1}{2}}\) and if \(a\) is positive, then \(\sqrt{a} = |a|\). A pre-Hilbert \(C^*\)-module \(\mathcal{H}\) is a Hilbert \(C^*\)-module or, simply, a Hilbert \(\mathcal{A}\)-module if it is complete with respect to the norm \(\|f\| = \|\langle f, f \rangle\|^{\frac{1}{2}}_{\mathcal{A}}\). Let \((\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})\) and \((\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})\) be Hilbert \(\mathcal{A}\)-modules. A map \(T : \mathcal{H} \rightarrow \mathcal{K}\) is said to be adjointable if there exists a map \(T^* : \mathcal{K} \rightarrow \mathcal{H}\) satisfying \(\langle Tf, g \rangle_{\mathcal{K}} = (f, T^*g)_{\mathcal{H}}\) whenever \(f \in \mathcal{H}\) and \(g \in \mathcal{K}\). The map \(T^*\) is called the adjoint of \(T\). The class of all adjointable maps from \(\mathcal{H}\) into \(\mathcal{K}\) is denoted by \(B_*(\mathcal{H}, \mathcal{K})\) and the class of all bounded \(\mathcal{A}\)-module maps from \(\mathcal{H}\) into \(\mathcal{K}\) is denoted by \(B_0(\mathcal{H}, \mathcal{K})\) where a \(\mathcal{A}\)-module map is a \(\mathbb{C}\)- and \(\mathcal{A}\)-linear map. It is known that \(B_*(\mathcal{H}, \mathcal{K}) \subseteq B_0(\mathcal{H}, \mathcal{K})\).

Let \(\{\langle K_j, \langle \cdot, \cdot \rangle_j \rangle\}_{j \in J}\) be a finite or countably infinite family of finitely or countably generated Hilbert \(\mathcal{A}\)-modules. Set \(K_j = \{(f_j)_{j \in J} : f_j \in K_j, \sum_{j \in J} \langle f_j, f_j \rangle_j < \infty\}\)
and define
\[(f_j)_{j \in J}, (g_j)_{j \in J} = \sum_{j \in J} (f_j, g_j)_J.\]

It is known that \((K_J, \langle \cdot, \cdot \rangle_J)\) is a Hilbert \(A\)-module.

Throughout the paper, we fix the notations \(A\) and \(J\) for a given unital \(C^*\)-algebra and a finite or countably infinite index set, respectively. Also, the Hilbert \(A\)-modules \(H\) and \(K_j\), for \(j \in J\), are assumed to be finitely or countably generated and ordered pairs \(\{\Lambda_j, K_j\} : j \in J\) consisting of Hilbert \(A\)-modules \(K_j\) and operators \(\Lambda_j \in B_0(H, K_j)\).

Before, authors \([3]\) extended the concept of \(g\)-frames from Hilbert spaces to Hilbert \(C^*\)-modules. By a \(g\)-frame for \(H\) we mean a family of ordered pairs \(\{\Lambda_j, K_j\} : j \in J\) satisfying
\[A(f, f) \leq \sum_{j \in J} (\Lambda_j f, \Lambda_j f) \leq B(f, f)\]
for all \(f \in H\) and some positive constants \(A, B\) independent of \(f\).

In [1], the authors introduced \(*-g\)-frames and \(*-g\)-Bessel sequences that are \(C^*\)-algebraic version of \(g\)-frames and \(g\)-Bessel sequences. Actually, we need strictly positive elements of \(C^\ast\)-algebra \(A\) instead of positive real numbers.

\(A \ast - g\)-frame for \(H\) is a collection of ordered pairs \(\{\Lambda_j, K_j\} : j \in J\) such that
\[A(f, f)A^* \leq \sum_{j \in J} (\Lambda_j f, \Lambda_j f) \leq B(f, f)B^*,\]
for all \(f \in H\) and strictly nonzero elements \(A\) and \(B\) in \(A\). (Throughout the paper, series like \((2)\) are assumed to be convergent in the norm sense.)

The numbers \(A\) and \(B\) are called lower and upper \(\ast - g\)-frame bounds, respectively. The sequence \(\{\{\Lambda_j, K_j\} : j \in J\}\) is called to be a \(\ast - g\)-Bessel sequence for \(H\) if it has the upper bound condition in \((2)\). If \(\{\{\Lambda_j, K_j\} : j \in J\}\) is a \(\ast - g\)-frame for \(H\) with an upper bound \(B\), then \(\{\Lambda_j\}_{j \in J}\) is uniformly bounded by \(\|B\|\).

Also, the sequence \(\{\{\Lambda_j, K_j\} : j \in J\}\) is called to be a \(\ast - g\)-orthonormal basis if it is a \(\ast - g\)-frame for \(H\) and satisfies
\[i.\ \Lambda_i\Lambda_j^\ast g_j = \delta_{ij} g_j, \text{ for any } i, j \in J, \text{ and} \]
\[ii.\ \sum_{j \in J} \Lambda_j^\ast \Lambda_j f = f, \text{ for all } j \in J.\]

Operators corresponding to a given \(g\)-frame \(\{\{\Lambda_j, K_j\} : j \in J\}\) are following. The pre-\(\ast - g\)-frame operator is the operator \(\theta\) such that
\[\theta : H \to \oplus_{j \in J} K_j, \ \theta f = (\Lambda_j f)_{j \in J}.\]

The adjoin of \(\theta\) is said the synthesis operator of \(\{\Lambda_j\}_{j \in J}\) and defined by
\[\theta^* : \oplus_{j \in J} K_j \to H, \ \theta^*(k_j)_{j \in J} = \sum_{j \in J} \Lambda_j^\ast k_j.\]

The \(g\)-frame operator \(S\) is given by composition of two the above operators; means,
\[S = \theta^* \theta, \ S f = \sum_{j \in J} \Lambda_j^\ast \Lambda_j f, \ \forall f \in H.\]

At this time, we consider the composition of two \(\ast - g\)-Bessel sequences.

**Theorem 1.1.** Assume that \(\Lambda = \{(\Lambda_j, K_j) : j \in J\}\) and \(\Gamma = \{(\Gamma_j, K_j) : j \in J\}\) are \(\ast - g\)-Bessel sequences for \(H_1\) and \(H_2\) with \(\ast - g\)-Bessel bounds \(B_\Lambda\) and \(B_\Gamma\), respectively. Then \(\Omega = \{(\Lambda_j \Gamma_j, K_j) : j \in J\}\) is a \(\ast - g\)-Bessel sequence for \(H_2\) with \(\ast - g\)-Bessel bound \(\|B_\Lambda\| B_\Gamma\) and the pre-\(\ast - g\)-frame operator of \(\Omega\) is a bounded operator \(\Theta_\Omega\) from \(H_2\) into \(\oplus_{j \in J} H_1\) by
\[\Theta_\Omega f = (\Lambda_j \Gamma_j f)_{j \in J}.\]
Proof. By the properties of adjointable operators and the definition of \( g \)-Bessel sequence \( \Gamma \), we obtain for \( f \in \mathcal{H}_2 \),
\[
\sum_{j \in J} < \Lambda_j^* \Gamma_j f, \Lambda_j^* \Gamma_j f > \leq \sum_{j \in J} \| \Lambda_j^* \|^2 < \Gamma_j f, \Gamma_j f >^2 \\
\leq \| B_\Lambda \|^2 \sum_{j \in J} < \Gamma_j f, \Gamma_j f > \leq \| B_\Lambda \| B_\Gamma < f, f > \| B_\Lambda \| B_\Gamma^*.
\]

Then \( \{ \Lambda_j^* \Gamma_j \}_{j \in J} \) is a \( g \)-Bessel sequence with bound \( \| B_\Lambda \| B_\Gamma \). The definition of pre-\( g \)-frame operator shows that the pre-\( g \)-frame operator of \( \Omega \) is \( \Theta_\Omega f = (\Lambda_j^* \Gamma_j f)_{j \in J} \) for all \( f \in \mathcal{H}_2 \).

The following example illustrates that Theorem 1.1 is not valid for \( g \)-frames.

Example 1.2. Let \( T \in B_\ell(\ell^2(A)) \) be the right shift operator and let \( \alpha \) be an element in the center of \( A \). Set \( \Lambda = \alpha T \). Since
\[
< \Lambda(a_i)_{i \in \mathbb{N}}, \Lambda(a_i)_{i \in \mathbb{N}} > = |\alpha|^2 \sum_{i \in \mathbb{N}} a_i a_i^* = \alpha < (a_i)_{i \in \mathbb{N}}, (a_i)_{i \in \mathbb{N}} > \alpha^*, \quad \forall (a_i)_{i \in \mathbb{N}} \in \ell^2(A).
\]
The sequence \( \{ \Lambda \} \) is an \( \alpha \)-tight \( g \)-frame for \( \ell^2(A) \). The operator \( \Lambda^* \), the adjoint of \( \Lambda \), is \( \Lambda^*(a_i)_{i \in \mathbb{N}} = (\alpha^* a_{i+1})_{i \in \mathbb{N}} \) on \( \ell^2(A) \) and the set \( \{ \Lambda \} \) is not a \( g \)-frame. To see this, choose the subsequence \( \{(n,1,0,0,...) : n \in \mathbb{N} \} \) in \( \ell^2(A) \). There do not exist \( \lambda > 0 \) such that,
\[
A < (n,1,0,0,...), (n,1,0,0,...) > A^* \leq < \Lambda^*(n,1,0,0,...), \Lambda^*(n,1,0,0,...) >, \\
\| A(n^2+1)A^* \|^2 \leq \| \alpha \|^2, \quad \forall n \in \mathbb{N}.
\]
Then \( \{ \Lambda^* \} \) has not lower bound condition and is not a \( g \)-frame.

On the other hand, the identity operator \( I \) on \( \ell^2(A) \) is a normalized \( g \)-frame. Therefore, the composition of two \( g \)-frames \( \{ \Lambda \} \) and \( \{ I \} \), the sequence \( \{ \Lambda^* I \} \), is not a \( g \)-frame.

Now, we characterize the class of \( g \)-frames by \( g \)-orthonormal basis and the composition of \( g \)-frames. The following theorem illustrates that the lower bound condition is preserved in the composition of some \( g \)-frames.

Theorem 1.3. Let \( \mathcal{H}_1, \mathcal{H}_2 \) and \( K_j \), for \( j \in J \), be Hilbert \( C^* \)-modules and \( \Lambda_j \in B_\mathcal{H}_j(\mathcal{H}_1, K_j) \) and \( \Gamma_j \in B_\mathcal{H}_j(\mathcal{H}_2, K_j) \). Let \( \Lambda = \{ (\Lambda_j, K_j) : j \in J \} \) be a \( g \)-orthonormal basis for \( \mathcal{H}_1 \) and \( \Gamma = \{ (\Gamma_j, K_j) : j \in J \} \). Then \( \Omega = \{ (\Lambda_j^* \Gamma_j, \mathcal{H}_1) : j \in J \} \) is \( g \)-frame for \( \mathcal{H}_2 \) if and only if \( \Gamma \) is a \( g \)-frame for \( \mathcal{H}_2 \) Moreover, \( S_\Omega = S_\Gamma \) where \( S_\Omega \) and \( S_\Gamma \) are \( g \)-frame operators for \( \Omega \) and \( \Gamma \), respectively.

Proof. By the definition of \( g \)-orthonormal basis \( \Lambda \), we have
\[
\sum_{j \in J} < \Lambda_j^* \Gamma_j f, \Lambda_j^* \Gamma_j f > = \sum_{j \in J} < \Gamma_j f, \Gamma_j f >
\]
So \( \{ \Lambda_j^* \Gamma_j \}_{j \in J} \) is a \( g \)-frame if and only if the sequence \( \{ \Gamma_j \}_{j \in J} \) is a \( g \)-frame and their \( g \)-frame bounds are the same. By (3), we have for all \( f \in \mathcal{H}_2 \),
\[
< S_\Omega f, f > = < \sum_{j \in J} \Gamma_j^* \Lambda_j \Lambda_j^* \Gamma_j f, f > \\
= < \sum_{j \in J} \Lambda_j^* \Gamma_j f, \Lambda_j^* \Gamma_j f > \\
= < \sum_{j \in J} \Gamma_j f, \Gamma_j f > \\
= < \sum_{j \in J} \Gamma_j^* \Gamma_j f, f > = < S_\Gamma f, f >,
\]
then it concludes that \( S_\Omega = S_\Gamma \) on \( \mathcal{H}_2 \).
The composition of \( \ast\-g\)-frames in Hilbert \( C^\ast\)-modules

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Fixed points of \((G, \psi)\)-contractions in metric spaces endowed with a graph

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Abstract

The purpose of this paper is to prove Boyd and Wong’s fixed point theorem type in a metric space endowed with a graph. We obtain sufficient conditions for the existence of fixed points for mappings defined on a complete metric space \(X\) endowed with a graph.

Keywords: Fixed point, \((G, \psi)\)-contraction, Graph.

Mathematics Subject Classification: 47H10, 47H09.

1 Introduction

Let \((X, d)\) be a metric space. Let \(\Delta\) denote the diagonal of the Cartesian product \(X \times X\). Consider a directed graph \(G\) such that \(V(G)\), the set of all vertices coincides with \(X\), and \(E(G)\), the set of all edges contains all loops, i.e., \(E(G) \supseteq \Delta\). An edge between two vertices \(x\) and \(y\) of \(G\) is denoted \((x, y) \in E(G)\) \((x, y \in X)\).

If \(f : X \to X\) is a mapping, then by \(\text{Fix}(f)\) we denote the set of all fixed points for \(f\) and we put \(X_f = \{x \in X : (x, fx) \in E(G)\}\).

Throughout this note, we assume that \(X\) is a metric space, \(G\) is a directed graph such that \(V(G) = X\) and \(E(G) \supseteq \Delta\), and we will use the following property:

\((P)\): For any sequence \((x_n)_{n \in \mathbb{N}} \subseteq X\), if \(x_n \to x\) as \(n \to \infty\) and \((x_n, x_{n+1}) \in E(G)\) for all \(n \in \mathbb{N}\), then there exists a subsequence \((x_{n_k})_{k \in \mathbb{N}}\) of \((x_n)_{n \in \mathbb{N}}\) such that \((x_{n_k}, x) \in E(G)\) for all \(k \in \mathbb{N}\).

Definition 1.1 ([2]). A mapping \(f : X \to X\) is said to be a \(G\)-contraction if \(f\) preserves the edges of \(G\), i.e.,

\[(x, y) \in E(G) \Rightarrow (fx, fy) \in E(G) \quad (x, y \in X),\]

and there exists \(\alpha \in (0, 1)\) such that for all \(x, y \in X\) with \((x, y) \in E(G)\) we have

\[d(fx, fy) \leq \alpha d(x, y).\]

In [1], Bojor introduced the notion of \((G, \varphi)\)-contraction in metric spaces endowed with a graph.

Definition 1.2 ([1]). A mapping \(f : X \to X\) is said to be a \((G, \varphi)\)-contraction if:

1. \((x, y) \in E(G) \Rightarrow (fx, fy) \in E(G) \quad (x, y \in X).\)
2. there exists a function \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) such that:

\[d(fx, fy) \leq \varphi(d(x, y)) \quad \text{for all}\ (x, y) \in E(G),\]

where \(\varphi\) is nondecreasing and \(\varphi^n(t) \to 0\) for all \(t > 0\).
2 Main Results

Definition 2.1. A mapping $f : X \to X$ is called a $(G, \psi)$-contraction if:

1. $f$ preserves the edges of $G$;
2. the contractive condition
   \[ d(fx, fy) \leq \psi(d(x, y)) \]
   holds for all $x, y \in X$ with $(x, y) \in E(G)$, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is upper semicontinuous from the right (i.e., $\lambda_n \downarrow \lambda \geq 0 \Rightarrow \limsup_{n \to \infty} \psi(\lambda_n) \leq \psi(\lambda)$) and satisfies $\psi(t) < t$ for each $t > 0$.

Definition 2.2 ([2]). A mapping $f : X \to X$ is called orbitally $G$-continuous if for all $x, y \in X$ and any sequence $(k_n)_{n \in \mathbb{N}}$ of positive integers,
\[
 f^{k_n}x \to y \text{ and } (f^{k_n}x, f^{k_n+1}x) \in E(G) \text{ for } n \in \mathbb{N} \text{ imply } f(f^{k_n}x) \to fy.
\]

Theorem 2.3. Let $X$ be a complete metric space endowed with a transitive graph $G$ and $f : X \to X$ be a mapping. We suppose that

(i) $X_f \neq \emptyset$;

(ii) there exists a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f$ is a $(G, \psi)$-contraction;

(iii) $f$ is orbitally $G$-continuous

or

(iv) $(X,d,G)$ has the property (P);

Then $f$ has a fixed point.

Proof. Suppose that $x_0 \in X_f$; so $(x_0, fx_0) \in E(G)$, and by Condition (ii) and easy induction we obtain $(f^{n}x_0, f^{n+1}x_0) \in E(G)$. Define a sequence $(x_n)$ in $X$ by $x_n = f^n x_0$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Set $d_n := d(x_n, x_{n+1})$, for $n \geq 0$. We shall show that $d_n \to 0$. Note
\[
 d_{n+1} = d(x_{n+1}, x_{n+2}) = d(f^{n+1}x_0, f^{n+2}x_0) \leq \psi(d_n) \leq d_n, \quad n \in \mathbb{N}_0.
\]
Hence $(d_n)$ is nonincreasing and bounded below. Let $\lim_{n \to \infty} d_n = a \geq 0$. Assume that $a > 0$. By the right continuity of $\psi$,
\[
 a = \lim_{n \to \infty} d_{n+1} \leq \lim_{n \to \infty} \psi(d_n) \leq \psi(a) < a,
\]
so $a = 0$.

Now we prove that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Assume that $(x_n)_{n \in \mathbb{N}}$ is not Cauchy. Then there exist $\varepsilon > 0$ and integers $m_k, n_k \in \mathbb{N}_0$ such that $m_k > n_k \geq k$ and
\[
 d(x_{m_k}, x_{m_k}) \geq \varepsilon \quad \text{for} \quad k = 0, 1, 2, \ldots.
\]
Also, choosing $m_k$ as small as possible, it may be assumed that
\[
 d(x_{m_k-1}, x_{m_k}) < \varepsilon.
\]
Hence for each $k \in \mathbb{N}_0$, we have,
\[
 \varepsilon \leq d(x_{m_k}, x_{m_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \leq d(x_{m_k}, x_{m_k-1}) + \varepsilon = d_{m_k-1} + \varepsilon,
\]
and therefore \( \lim_{k \to \infty} d(x_{m_k}, x_{n_k}) = \varepsilon \). By the transitivity of \( G \) observe that
\[
d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \leq d_{m_k} + \psi(d(x_{m_k}, x_{n_k})) + d_{n_k}.
\]
Letting \( k \to \infty \) and using the upper semicontinuity of \( \psi \) from the right, we obtain
\[
\varepsilon = \lim_{k \to \infty} d(x_{m_k}, x_{n_k}) \leq \lim_{k \to \infty} \psi(d(x_{m_k}, x_{n_k})) \leq \psi(\varepsilon),
\]
which is a contradiction. Hence \( (x_n) \) is a Cauchy sequence in \( X \). So by the completeness of \( X \), there exists \( x^* \in X \) such that \( x_n \to x^* \).

Now we will prove that \( x^* \in \text{Fix}(f) \). If (iii) holds, then clearly \( x^* \in \text{Fix}(f) \). If (iv) holds, then we have
\[
d(x^*, fx^*) \leq d(x^*, x_{n_k+1}) + d(x_{n_k+1}, fx^*) \\
\leq d(x^*, x_{n_k+1}) + \psi(d(x_{n_k}, x^*)) \\
\leq d(x^*, x_{n_k+1}) + d(x_{n_k}, x^*).
\]
Letting \( k \to \infty \), we obtain \( x^* \in \text{Fix}(f) \).

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Relations between representations, ternary representations and medial representations of a ternary group

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Abstract
Let \((G,[\ ],O)\) be a topological ternary group. In this article we define representation, ternary and medial representation of a ternary group and study its relations between.

Keywords: Medial representation, Representation, Ternary Representation, Topological Ternary Group.

Mathematics Subject Classification: 43A20

1 Introduction
Ternary and n-ary generalizations of algebraic structures are the most natural ways for further development and deeper understanding of their fundamental properties. Firstly, ternary algebraic operations were introduced already in the nineteenth century by A. Cayley [1]. As the development of Cayley's ideas it were considered n-ary generalization of matrices and their determinants [6].

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their possible applications in physics. Some significant physical applications are described in [8] and [9].

The notion of an n-ary group was introduced by Dörnte [3] and is a natural generalization of the notion of a group and a ternary group considered by Certaine [2] and Kasner [7]. In 1940 E. L. Post [10] published an extensive study of n-groups in which the well-known Post's Coset Theorem appeared.

In this article we introduce concepts representation, ternary representation and medial representation of a ternary group.

Definition 1.1. A nonempty set \(G\) with one ternary operation \([\ ]:\ G \times G \times G \to G\) is called a ternary groupoid and denoted by \((G,[\ ])\). The ternary groupoid \((G,[\ ])\) is called a ternary semigroup if the operation \([\ ]\) is associative, i.e., if \([xyz]uv = [x]yzu[v] = [xy]zu[v]\) hold for all \(x, y, z, u, v \in G\).

Definition 1.2. A ternary semigroup \((G,[\ ])\) is a ternary group if for all \(a, b, c \in G\), there are \(x, y, z \in G\) such that

\([xab] = [ayb] = [abz] = c\).
The subset $H$ of $G$ is called ternary subgroup of $G$ if itself ternary group under the ternary operation of $G$.

If

$$[x_1x_2x_3] = [x_3x_2x_1]$$

holds for all $x_1, x_2, x_3 \in G$. Then $(G, [\ ])$ is called semicommutative.

In a ternary group, the equation $[xxz] = x$ has a unique solution which is denoted by $z = \bar{x}$ and called the skew element to $x$ [3]. Other properties of skew elements are described in [4] and [5].

**Definition 1.3.** A ternary group $(G, [\ ])$ is medial if it satisfies the identity

$$[[x_1x_2x_3][y_1y_2y_3][z_1z_2z_3]] = [[x_1y_1z_1][x_2y_2z_2][x_3y_3z_3]],$$

hold for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in G$.

**Definition 1.4.** [11] Let $(G, [\ ])$ be a ternary group, $^{-1}$ its inverse operation, and $G$ be equipped with a topology $O$. Then, we say that $(G, [\ ], O)$ is a topological ternary group iff
(i) ternary operation $[\ ]$ is continuous in $O$; and
(ii) The $x$-operation $^{-1}$ is continuous in $O$.

## 2 Main Result

Let $E$ be a linear space and $\text{End}(E)$ denote a set of endomorphisms of $E$.

**Definition 2.1.** A representation of a ternary group $(G, [\ ])\ldots$ in a linear space $E$ is a map $\Pi : G \to \text{End}(E)$ such that

$$\Pi_x \Pi_y \Pi_z = \Pi_{[xyz]} \quad \forall x, y, z \in G.$$ 

The linear space $E$ is called the representation space of $\Pi$. A subspace $E_1$ of $E$ is said to be invariant under the representation $\Pi$ if $\Pi_x(E_1) \subseteq E_1$ for all $x \in G$.

For any locally compact ternary group $G$, we denote by $\text{Rep}(G, [\ ])$ the collection of all of representations of $G$.

A first obvious specialization is appropriate for representation of ternary groups.

**Theorem 2.2.** Let $\Pi$ be a representation of a ternary group $G$ as in (2.1). Then $E$ is the direct sum of invariant subspaces $E_0$ and $E_1$ such that:

(i) $\Pi_x(E_0) = \{0\}$ for all $x \in G$;  
(ii) $\Pi_x\Pi_x(\xi) = \Pi_x\Pi_x(\xi) = \xi$ for all $\xi \in E_1$ and $x \in G$.

If $E$ is a topological linear space and $\Pi_x\Pi_x$ is a continuous operator, then $E_0$ and $E_1$ are closed in $E$.

**2.3.** Theorem (3) shows that in studying representations of ternary groups, we lose nothing in supposing that $\Pi_x\Pi_x$ is the identity operator $I$ on $E$. From this point on, therefore, we shall mean by “representation $\Pi$ of a ternary group” a representation such that $\Pi_x\Pi_x = I$.

**Theorem 2.4.** Let $G$ be a ternary group and $\Pi$ a representation of $G$ by unitary operators on a Hilbert space $H$. If $H_1$ is a linear subspace of $H$ invariant under $\Pi$, then $H_1$ is also invariant under $\Pi$.

**Corollary 2.5.** Let $G$ be a ternary group and $\Pi$ a representation of $G$ by (bounded) operators on a Hilbert space $H$. Then $H$ is the direct sum of closed orthogonal invariant subspaces $N$ and $H_1$ such that $\Pi_x(N) = 0$ for all $x \in G$, and for all nonzero $\xi \in H_1$ there is an $x \in G$ such that $\Pi_x\xi \neq 0$.  

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Theorem 2.6. Let $\Pi$ be a representation of a ternary group $G$ by unitary operators on a Hilbert space $H$. Let $L$ be a closed linear subspace of $H$, and let $P$ be the projection operator mapping $H$ onto $L$. Then $L$ reduces $\Pi$ if and only if 
(i) $\Pi|_{L} \{ x \} = \Pi|_{L} P$ for all $x \in G$.

Definition 2.7. A ternary representation of a ternary group $(G, [\,])$ in a linear space $E$ is a map $\Pi^3 : G \times G \times G \to \text{End}(E)$ such that 
$\Pi^3(x_1, x_2, x_3)\Pi^3(y_1, y_2, y_3)\Pi^3(z_1, z_2, z_3) = \Pi^3([x_1y_1z_1], [x_2y_2z_2], [x_3y_3z_3])$
for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in G$ and
$\Pi^3(x, y, z)\Pi^3(\bar{x}, \bar{y}, \bar{z}) = \text{id}_E$ \quad $\forall x, y, z \in G$.

For any locally compact group $G$, we denote by $\text{Rep}^3(G, [\,])$ the collection of all of ternary representations of $G$.

Theorem 2.8. Let $G$ be a semicommutative ternary group. Then we have
$\Pi^3(x_1, x_2, x_3)\Pi^3(y_1, y_2, y_3)\Pi^3(z_1, z_2, z_3) = \Pi^3(z_1, z_2, z_3)\Pi^3(y_1, y_2, y_3)\Pi^3(x_1, x_2, x_3)$

Theorem 2.9. Let $(G, [\,])$ be a semicommutative ternary group. There is one-to-one correspondence between representations of $G$ and ternary representations of $G$.

Definition 2.10. A medial ternary representation of a ternary group $(G, [\,])$ in a linear space $E$ is a map $\Pi^m : G \to \text{End}(E)$ such that 
$\Pi^m([x_1y_1z_1])\Pi^m([y_1z_1z_3]) = \Pi^m([x_1y_1z_1])\Pi^m([x_2y_2z_2], [x_3y_3z_3])$
for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in G$ and
$\Pi^m([xyz])\Pi^m([xyz]) = \Pi^m([xyz])\Pi^m([xyz]) = \Pi^m([xyz]) = \text{id}_E$ \quad $\forall x, y, z \in G$.

Theorem 2.11. Let $(G, [\,])$ be a ternary group, and $\Pi$ be a representation of $G$. Then $\Pi$ is a ternary representation of a ternary group.

Theorem 2.12. Let $(G, [\,])$ be a ternary group, and $\Pi^m : G \to \text{End}(E)$ be a medial representation ternary group. Then $\Pi^m : G \to \text{End}(E)$ is a representation.

References


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On properties of some classes of bounded linear operators on Hilbert spaces

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Abstract

In this paper we consider the properties of operators satisfying $|T^*|^2 \leq |T|^2$. We show that if $T \in B(H)$ is a $(A, \ast)$-class operator, then $r(T) = \|T\|$. 

Keywords: Operator Inequality, Spectral Radius, Spectrum of an Operator

Mathematics Subject Classification: Primary: 47A12; Secondary: 47A30, 47B20

1 Introduction

In this paper we denote the set of bounded linear operators on $H$ by $B(H)$, where $H$ is a complex Hilbert space. We say $T \in B(H)$ is a $A_\ast$-class operator if $|T|^2 \geq |T^2|$. In [1],[2] considered case of properties and that shown this operators satisfy in Weyl theorem. Our main purpose in this paper to consider properties of $(A, \ast)$-class operators. This operators satisfy $|T^*|^2 \leq |T|^2$. In main section we discuss if $T$ be $(A, \ast)$-class operator, then $r(T) = \|T\|$. Also we answer this question that, if there are operators that belong to $(A, \ast)$-class operators but do not belong to $A_\ast$-class operators. Let $T \in B(H)$. We denote the spectrum of $T \in B(H)$ by $\sigma(T)$, and also $r(T) = \sup\{|\lambda|, \lambda \in \sigma(T)\}$. 

For $T \in B(H)$, we write $N(T)$ and $R(T)$ for the null space and the range of $T$. An Operator $T \in B(H)$ is called a Fredholm if it has closed range and, 

$$dimN(T) < \infty, \quad dimR(T)^\perp < \infty.$$ 

If $T \in B(H)$ is a Fredholm then the index of $T$, denoted by $Ind(T)$, is given by 

$$ind(T) = dim(N(T)) - dim(R(T)^\perp).$$

An operator $T \in B(H)$ is called a Weyl if it is Fredholm of index zero. Also we let 

$$\pi_{00}(T) = \{\lambda \in C : \lambda \in isos(T) \quad dimN(T - \lambda I) < \infty\}$$

denote the set of isolated eigenvalues of finite multiplicity.
2 Main Result

We give some lemmas that we use in the following.

Lemma 2.1. [2] Let $T$ is a self-adjoin operator Therefore

$$\| T \| = \sup \{ \langle Tx, x \rangle, \| x \| = 1 \}$$

Lemma 2.2. [1] (Hansen inequality). If $A, B \in B(H)$ satisfy $A \geq 0$ and $\| B \| \leq 1$, then

$$(B^* AB)^k \geq B^* A^k B$$

for all $0 < \delta \leq 1$.

Lemma 2.3. [3] (Holder-MCCarty inequality). If $A \in B(H)$ and $x \in H$, then

i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r \| x \|^{2(1-r)}$ for $r > 1$, 
ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r \| x \|^{2(1-r)}$ for $0 \leq r \leq 1$.

Definition 2.4. We say that a bounded linear operator $T \in B(H)$ is a $(A, * )$-class operator if $|T^*|^2 \leq |T|^2$.

Theorem 2.5. Let $T$ be a $(A, * )$-class operator, then $r(T) = \| T \|$.

Theorem 3 implies that the zero operator is the only nilpotent operator in the $(A, * )$-class operators.

Corollary 2.6. If $T$ is a nilpotent $(A, * )$-class operator and operator, then $T = 0$.

Assume that $x \in H$ and $\| x \| = 1$. Then we have

$$\| (T - \lambda)^* x \|^2 = \langle (T - \lambda)^* x, (T - \lambda)^* x \rangle$$
$$= \langle (T - \lambda)(T^* - \lambda^*) x, x \rangle$$
$$= \langle TT^* x, x \rangle - \lambda \langle T^* x, x \rangle - \langle \lambda T x, x \rangle + \lambda \| x \|^2$$
$$= \langle \| T^* x \|^2, x \rangle - \lambda \langle x, T x \rangle - \lambda \langle T x, x \rangle$$
$$+ \lambda \| x \|^2 \leq \langle \| T^2 \| x, x \rangle - \lambda \| x \|^2 + \lambda \| x \|^2$$
$$= \langle (T^2 T^2)^{1/2} x, x \rangle - \lambda \| x \|^2 \leq (T^2 x, T^2 x)^{1/2} - \lambda \| x \|^2 - \lambda \| x \|^2.$$

Hence we have the following theorem.

Theorem 2.7. Let $T \in B(H)$ be an operator belongs to the set of $(A, * )$-class operators. Suppose that $\lambda \in C$ and $(T - \lambda) (x) = 0$ then $(T - \lambda)^* (x) = 0$.

Theorem 2.8. Let $T$ is an $(A, * )$-class operator and $0 \in \delta(T) \setminus \omega(T)$, then $0 \in \pi_{00}(T)$.

Proof. Let $0 \in \delta(T) \setminus \omega(T)$. We show that zero is an isolated eigenvalue of $T$. If zero is not eigenvalue of $T$, then $\dim N(T) = 0$. therefore

$$\dim R(T)^\perp = 0, R(T) = 0$$

and $T$ is invertible, hence $0 \notin \sigma(T)$ this is a contradiction. Now if $0 \notin iso(\sigma(T))$, we have

$$N(T) \subseteq N(T^* ) = R(T)^\perp$$
Since $\dim N(T) = \dim R(T)^\perp$, therefore $N(T) = R(T)^\perp$. Now we consider the following decomposition of $T$,

$$T = 0 \bigoplus B$$

where $B$ is an invertible, and since $0 \notin \text{iso}(\sigma(T))$, we get $0 \notin \text{iso}(\sigma(B))$. Then there exists a sequence of $\sigma(B), \{\lambda_n\}$, such that $\lambda_n \to 0$, which is a contradiction.

Example 2.9. There exists this question that, if there are operators such that satisfy in $(A, \ast)_-$-class operator but donot satisfy in $A_-$ class operator. For this mean we consider the infinite matrix operator $T$ as follows,

$$T = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \ldots \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \ldots \\
0 & \frac{1}{2} & 0 & 0 & 0 & \ldots \\
0 & 0 & \frac{1}{2} & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}$$

Hence we get $TT^* \leq (T^{*2}T^2)^{\frac{1}{2}}$ and $T^*T \nleq (T^{*2}T^2)^{\frac{1}{2}}$. Thus $T$ is an $(A, \ast)_-$ class operator and is not an $A_-$ class operator.

Remark 2.10. If $T \in B(H)$ is a $(A, \ast)_-$ class operator and $T$ is invertible, then

$$|T^{*-1}|^2 \geq |T^{-2}|.$$  

Now suppose that $T$ and $T^{-1}$ are $(A, \ast)_-$ class operators. Then we have

$$|T^*|^2 = |T^2|.$$  

Remark 2.11. If $T$ is an $(A, \ast)_-$ class operator, then by Theorem 2.7 we have $N(T) \subset N(T^*)$. Now if $T^*$ is $(A, \ast)_-$ class operator then $N(T^*) \subset N(T)$. Therefore if $T$ and $T^*$ are $(A, \ast)_-$ class operators then we have $N(T) = N(T^*)$ and we get

$$R(T) = R(T^*).$$

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Best approximation and new results in fixed point

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Abstract
In this paper we introduce the concept of asymptotic best approximation of maps and consider relation between it and fixed point theory in the Banach space.

Keywords: Best approximation, fixed point, Hausdorff metric, nonexpansive maps.

Mathematics Subject Classification: 46A32, 46M05, 41A17

1 Introduction and Preliminaries

Let $X$ be a normed linear space and $M$ be a subset of $X$. A point $y_0 \in M$ is said to be a best approximation for $x \in X$, if for every $y \in W$,

$$
\|x - y_0\| \leq \|x - y\|.
$$

The set of all the best approximation of $x$ in $M$ is denoted by $P_M(x)$, for more results see [7]. Some results about relation between fixed point and best approximation can be see in [1-7]. Now, we introduce a generalization of the best approximation concept with sequences that is important in different aspect of mathematics.

Definition 1.1. Let $X$ be a normed space, $M$ be a closed subset of $X$ and a sequence $\{x_n\} \subseteq X \setminus M$. Put

$$
d^a(\{x_n\}, M) = \liminf_{n \to \infty} \inf_{y \in M} \|x_n - y\|.
$$

A point $y_0 \in M$ is said to be a asymptotic best approximation for $\{x_n\} \subseteq X \setminus M$, if

$$
\liminf_{n \to \infty} \|x_n - y_0\| = d^a(\{x_n\}, M).
$$

If every bounded sequence $\{x_n\} \subseteq X$ has at least one asymptotic best approximation in $M$, then $M$ is called a asymptotic proximinal subspace of $X$. If each sequence $\{x_n\} \subseteq X$ has an unique asymptotic best approximation in $M$, then $M$ is called a asymptotic Chebyshev subspace of $X$.

For $x \in X$ put,

$$
P^a_M(\{x_n\}) = \{y_0 \in M : \liminf_{n \to \infty} \|x_n - y_0\| = d^a(\{x_n\}, M)\}.
$$

This set may be empty, a singleton, or certain infinitely many points. It is clear that

$$
P^a_M(\{x_n\}) = \{\cap_{n=1}^{\infty} \cup_{i=n}^{\infty} B_{d^a(\{x_n\}, M)}(x_i)\} \cap M \quad (*)
$$

Also, if $\{x_n\}$ converges strongly to $x \in C$, then $P^a_M(\{x_n\}) = \{x\}$. 

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Definition 1.2. Let $C$ be a bounded closed convex subset of $X$. A sequence $\{x_n\} \subseteq X$ is said to be an asymptotic best approximation for a mapping $T : C \to C$ if, for each $x \in C$,

$$\liminf_{n \to \infty} \|Tx - x_n\| \leq \liminf_{n \to \infty} \|x_n - x\|.$$ 

Definition 1.3. Let $C$ be a nonempty subset of $X$. We say that $C$ has the fixed point property for continuous mappings of $C$ with asymptotic best approximation if every continuous mapping $T : C \to C$ admitting an asymptotic best approximation has a fixed point.

Definition 1.4. Let $C$ be a nonempty subset of $X$. We say that $C$ has Property $(U)$ if for every bounded sequence $\{x_n\} \subset X \setminus C$, the set $P_C(\{x_n\})$ is a nonempty and compact subset of $C$.

Example 1.5. Let $X$ be a normed space and $C$ a nonempty subset of $X$. It is clear that:

I) If $C$ is a compact set, then $P_C(\{x_n\})$ is a nonempty and compact set and so has Property $(U)$.

II) If $C$ is an open set, since $P_C(\{x_n\}) \subset \partial C$, therefore $P_C(\{x_n\})$ is empty and so fail to have Property $(U)$.

2 Main results

Our new results are presented in this section.

Proposition 2.1. Let $X$ be a Banach space and let $C$ be a nonempty closed bounded and convex subset of $X$. If $C$ satisfies Property $(U)$, then every continuous mapping $T : C \to C$ that admitting an asymptotic best approximation in $C$ has a fixed point.

Theorem 2.2. Let $X$ be a Banach space and let $C$ be a nonempty closed bounded and convex subset of $X$. If $C$ has the fixed point property for continuous mappings admitting an asymptotic best approximation, then $C$ has Property $(U)$.

Corollary 2.3. Let $C$ be a nonempty closed bounded and convex subset of a Banach space $X$. The following conditions are equivalent.

1) $C$ has the fixed point property for continuous mappings admitting asymptotic best approximation in $C$.

2) $C$ has Property $(U)$.

Let $C$ be a nonempty closed convex bounded subset of a Banach space $X$. By $KC(C)$ we denote the family of all nonempty compact convex subsets of $C$. On $KC(C)$ we consider the well-known Hausdorff metric $H$. Recall that a mapping $T : C \to KC(C)$ is said to be nonexpansive whenever

$$H(Tx, Ty) \leq d(x, y), \quad x, y \in C.$$ 

Theorem 2.4. Let $X$ be a Banach space and let $C$ be a nonempty closed convex and bounded subset of $X$ satisfying Property $(U)$. If $T : C \to KC(C)$ is a nonexpansive mapping, then $T$ has a fixed point.

Let $C$ be a bounded closed and convex subset of a Banach space $X$, $T : C \to C$ a nonexpansive mapping and $\alpha \in (0, 1)$. Then a mappings $T_\alpha : C \to C$ defined by $T_\alpha(x) = \alpha x + (1 - \alpha)Tx$ is always asymptotically regular, that is, for every $x \in C$, $\lim_{n \to \infty} \|T_\alpha^{n+1}x - T_\alpha^nx\| = 0$. 

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Proposition 2.5. Let $X$ be a Banach space and $C$ a closed bounded convex subset of $X$, $x_0 \in C$ and $\alpha \in (0, 1)$. If $T : C \rightarrow C$ is a nonexpansive mapping, then the sequence $\{T^n_\alpha x_0\}$ is an asymptotic best approximation for $T$.

Theorem 2.6. Let $X$ be a normed space, $T : X \rightarrow X$ a nonexpansive mapping with an approximating fixed point sequence $\{x_n\} \subseteq X$ and $C$ be a nonempty subset of $X$ such that $P_C(\{x_n\})$ is a nonempty starshaped subset of $X$. Then $T$ has an approximating fixed point sequence in $P_C(\{x_n\})$.

References

Weak amenability of fourth dual of a Banach algebra $A$

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Abstract

Let $A$ be a Banach algebra and $((A'', \Box)'', \Box)$ be its forth dual with first Arens product $\Box$. We consider three $((A', \Box)'', \Box)-$bimodule structure on fifth dual $A^{(5)}$ of $A$. This paper determines the conditions that make these structures equal. Among other results we show that if $A^{(4)}$ is weakly amenable with some conditions, then $A$ is weakly amenable.

Keywords: Banach algebra, Arens products, Arens regularity, Derivation, weak amenability.

Mathematics Subject Classification: 46H20, 46H25, 46G05.

1 Introduction

Throughout this paper $A$ is a Banach algebra, and $A', A'', A''', A^{(4)}$ and $A^{(5)}$ denote the first, second, third, forth and fifth duals of $A$, respectively. Let $X$, $Y$ and $Z$ be Banach space and let $m : X \times Y \rightarrow Z$ be a bounded bilinear map. The first extension $m_1$ of $m$ can be done by

\[ m_1' : Z' \times X \rightarrow Y', \quad \langle m_1'(f, x), y \rangle = \langle f, m(x, y) \rangle; \]

\[ m_1'' : Y' \times Z' \rightarrow X', \quad \langle m_1''(G, f), x \rangle = \langle G, m_1'(f, x) \rangle; \]

\[ m_1 : X' \times Y'' \rightarrow Z'', \quad \langle m_1(F, G), f \rangle = \langle F, m_1''(G, f) \rangle. \]

Similarly, the second extension $m_2$ of $m$ can be done by

\[ m_2' : Y \times Z' \rightarrow X', \quad \langle m_2'(y, f), x \rangle = \langle f, m(x, y) \rangle; \]

\[ m_2'' : Z' \times X'' \rightarrow Y', \quad \langle m_2''(f, G), y \rangle = \langle F, m_2'(y, f) \rangle; \]

\[ m_2 : X'' \times Y'' \rightarrow Z'', \quad \langle m_2(F, G), f \rangle = \langle G, m_2''(f, F) \rangle. \]

The bilinear map $m$ is called Arens regular if $m_1 = m_2$. This can now be specialized to a product map $m = \pi : A \times A \rightarrow A$ of a Banach algebra $A$. For $F$ and $G$ in $A''$ we denote $m_1(F, G)$ and $m_2(F, G)$ by symbols $F \square G$ and $F \Diamond G$, respectively. These are called the first and second Arens product on $A''$. These products are defined in stages as follows. For every $F, G \in A'$, $f \in A'$ and $a, b \in A$, we define $f.a, a.f, G.f$ and $F, f$ in $A'; F \square G$ and $F \Diamond G$ in $A''$ by

\[ \langle f.a, b \rangle = \langle f, ab \rangle, \quad \langle a.f, b \rangle = \langle f, ba \rangle, \]

\[ \langle G.f, a \rangle = \langle G, fa \rangle, \quad \langle F, f.a \rangle = \langle F, af \rangle, \]

\[ \langle F \square G, f \rangle = \langle F, G.f \rangle, \quad \langle F \Diamond G, f \rangle = \langle G, F.f \rangle. \]
Throughout this paper we consider \( A \) as a Banach algebra. Let \( A^\prime \) be the dual space of \( A \). We define \( A^\prime \) as the space of all continuous linear functionals on \( A \). The dual space of \( A^\prime \) is denoted by \( A^{\prime\prime} \).

The Banach algebra \( A \) is said to be Arens regular if \( F \square G = F \circ G \) for each \( F, G \in A^\prime \). When \( A^\prime \) is given the \( w^\prime \)-topology, the extensions \( m_1 \) and \( m_2 \) of \( m \), in the topological sense, can be written by

\[
m_1(F, G) = w^\prime - \lim_{\alpha} w^\prime - \lim_{\beta} (m(x_\alpha, y_\beta)),
\]
\[
m_2(F, G) = w^\prime - \lim_{\beta} w^\prime - \lim_{\alpha} (m(x_\alpha, y_\beta)).
\]

For \( F = w^\prime - \lim_{\alpha} x_\alpha \) and \( G = w^\prime - \lim_{\beta} y_\beta \) in \( A^\prime \), such that \( (x_\alpha), (y_\beta) \) are nets in \( A \). So we have

\[
F \square G = w^\prime - \lim_{\alpha} x_\alpha - \lim_{\beta} y_\beta,
\]
\[
F \circ G = w^\prime - \lim_{\beta} y_\beta - \lim_{\alpha} x_\alpha.
\]

The topological dual space \( A^\prime \) of \( A \) becomes a Banach \( A \)-bimodule from

\[
(\langle f, a, b \rangle) = \langle f, ab \rangle, \quad (\langle a, f, b \rangle) = \langle fa, b \rangle.
\]

Now let \( E \) be a Banach \( A \)-bimodule then \( E^\prime \) becomes a Banach \( A^\prime \)-bimodule by actions as follows (see also [7] and Theorem 2.6.15 in [3])

\[
(\langle f, a \rangle) = \langle f, a \rangle = \langle f, a \rangle
\]
 \[ (\langle a, f \rangle) = \langle f, a \rangle = \langle f, a \rangle \]

for \( F = w^\prime - \lim_{\alpha} a, \beta \) in \( A^\prime \) and \( A = w^\prime - \lim_{\beta} y_\beta \) in \( E^\prime \), such that \( (a_\alpha), (x_\beta) \) are nets in \( A \) and \( E \), respectively.

Let \( X \) be a Banach \( A \)-bimodule. Then a continuous linear map \( D : A \rightarrow X \) is called a derivation if \( D(ab) = aD(b) + b(b, a \in A) \). For \( x \in X \) we define \( \delta_x : A \rightarrow X \) as follows \( \delta_x(a) = ax - ax(a \in A) \), it is easy to show that \( \delta_x \) is a derivation. Such derivations are called inner derivations. A is called amenable, if every derivations \( D : A \rightarrow X \) is inner, for each Banach \( A \)-bimodule \( X \). If every derivations from \( A \) into \( A^\prime \) is inner, then \( A \) is called weakly amenable. We regard \( A \) as a subspace of \( A^\prime \) by canonical embedding \( i : A \rightarrow A^\prime (a \mapsto i) \). We write \( A \) as the image of \( A \) under this mapping.

**Theorem 1.1.** Let \( A \) be Banach algebra and \( X \) be a Banach \( A \)-bimodule and \( D : A \rightarrow X \) is a derivation, then \( D^\prime : A^\prime \rightarrow X^\prime \) is a derivation. [3, theorem 2.7.17]

**Remark 1.2.** In above Theorem \( X^\prime \) is an \((A^\prime, \square)\)-bimodule by module structure as in formula (2).

In section 2 of this paper by using formula (1) and (2) we investigate three different \( A^{(4)} \)-bimodule structures on \( A^{(5)} = (A^\prime)^\prime \), \( A^{(5)} = (A^\prime)^\prime \) and \( A^{(5)} = (A^\prime)^\prime \). We find conditions that make these structures equal. By using of these conditions, we will discuss the forth transpose \( D^{(4)} : A^{(4)} \rightarrow A^{(5)} \) of a derivation \( D : A \rightarrow A^\prime \), in section 3. Finally we will show that when weak amenability of \( A^{(4)} \) implies weak amenability of \( A \).

### 2 \( A^{(4)} \)-bimodule structure on fifth dual of a Banach algebra

Throughout this paper we consider \( A^{(2n)}(n \geq 1) \) with the first Arens product. Consider the third dual \( A^{(3)} \) of \( A \). First we regard \( A^{(3)} \), as the dual space of \( A^\prime \), \( A^{(3)} = (A^\prime)^\prime \) and so \( A^{(3)} \) can be made into an \( A^\prime \)-bimodule by the following actions

\[
\langle \lambda, F, G \rangle = \langle \lambda, F \square G \rangle, \quad \langle F, \lambda, G \rangle = \langle \lambda, G \square F \rangle, \quad (\lambda \in A^{(3)}; F, G \in A^{(3)}). \]

In the second way, \( A^{(3)} \) as the second dual of \( A \), \( A^{(3)} = (A^\prime)^\prime \), can be an \( A^\prime \)-bimodule by the following formulæ. For \( \lambda \in A^{(3)} \) and \( F \in A^{\prime} \), we have
\[ \lambda \circ F = w^* - \lim_i w^* - \lim_{\alpha} j_{\alpha} a_{\alpha}, \]
\[ F \circ \lambda = w^* - \lim_{\alpha} w^* - \lim_i a_{\alpha} f_i, \]
where \( F = w^* - \lim_{\alpha} a_{\alpha} \) in \( A' \) and \( \lambda = w^* - \lim_i f_i \) in \( A'' \), such that \((a_{\alpha})\) and \((f_i)\) are nets in \( A \) and \( A' \) respectively. These two \( A'' \)-bimodule structures on \( A'' \) are considered in [4].

**Lemma 2.1.** Let \( A \) be Arens regular, then for bounded nets \((F_{\alpha})\) and \((G_{\beta})\) in \( A' \), we have
\[ (w^* - \lim_{\alpha} F_{\alpha}) \Box (w^* - \lim_{\beta} G_{\beta}) = w^* - \lim_{\alpha} (F_{\alpha} \Box G_{\beta}) = w^* - \lim_{\beta} (w^* - \lim_{\alpha} (F_{\alpha} \Box G_{\beta})). \]

**Proof.** This follows from Lemma 2.2 part (iii) of [2]. \( \square \)

**Lemma 2.2.** If for every \( F \in A'' \) the map \( \phi : A'' \to A''(G \mapsto G \Box F) \) is \( w^* - w \)-continuous, then for every bounded net \((a_{\alpha})\) in \( A \)
\[ \left\langle \lambda, (w^* - \lim_{\beta} a_{\beta}) \Box F \right\rangle = \lim_{\alpha} \left\langle \lambda, a_{\beta} \Box F \right\rangle, \quad (\lambda \in A'''). \]

**Proof.** This follows from Lemma 2.3 of [2]. \( \square \)

**Lemma 2.3.** If for every \( G \in A'' \) the map \( \rho : A'' \to A''(F \mapsto G \Box F) \) is \( w^* - w \)-continuous, then for every bounded net \((F_j)\) in \( A' \)
\[ \left\langle \lambda, G \Box (w^* - \lim_j F_j) \right\rangle = \lim_j \left\langle \lambda, G \Box F_j \right\rangle, \quad (\lambda \in A'''). \]

**Proof.** This is obvious from Lemmas 16, 17 and 10. \( \square \)

Now we consider \( A^{(4)} = (A')'' \) as the second dual of \( A'' \) with first Arens product. Take \( \Gamma = w^* - \lim_{\alpha} F_{\alpha} \) and \( \Lambda = w^* - \lim_{\beta} G_{\beta} \) in \( A^{(4)} \), such that \((F_{\alpha})\) and \((G_{\beta})\) are nets in \( A'' \), so for first Arens product in \( A^{(4)} \) we have
\[ \Gamma \Box \Lambda = w^* - \lim_{\alpha} w^* - \lim_{\beta} F_{\alpha} \Box G_{\beta}. \]

Now the Banach algebra \( A^{(5)} \) has three \( A^{(4)} \)-bimodule structures.

(a) We consider \( A^{(5)} = (A^{(4)})' \) in which \( A^{(4)} = (A'')'' \), by using formula (1) we have following \( A^{(4)} \)-bimodule structures on \( A^{(5)} \)
\[ \left\langle \Psi \bullet \Lambda, \Gamma \right\rangle = \left\langle \Psi, \Lambda \Box \Gamma \right\rangle, \quad \left\langle \Lambda \bullet \Psi, \Gamma \right\rangle = \left\langle \Psi, \Gamma \Box \Lambda \right\rangle \quad (\Psi \in A^{(5)} \text{ and } \Gamma, \Lambda \in A^{(4)}). \]

(b) Let \( A^{(5)} = ((A'')')'' \) as the second dual of \((A'')'' = (A')'' \). Take \( \Psi \in A^{(5)} \), \( \Lambda \in A^{(4)} \) and bounded nets \((\lambda_{\gamma}) \subset A''', (G_{\beta}) \subset A'' \) with \( \Psi = w^* - \lim_{\gamma} \lambda_{\gamma} \) and \( \Lambda = w^* - \lim_{\beta} G_{\beta} \). By using formula (2) two \( A^{(4)} \)-bimodule actions are defined by
\[ \Psi \circ \Lambda = w^* - \lim_{\gamma} w^* - \lim_{\beta} \lambda_{\gamma} \circ G_{\beta}, \]
\[ \Lambda \circ \Psi = w^* - \lim_{\beta} w^* - \lim_{\gamma} G_{\beta} \circ \lambda_{\gamma} . \]
(c) Let \( A^{(5)} = ((A^{''})^{'}) \) as the second dual of \( A^{''} = (A^{''})' \). Take \( \Psi \in A^{(5)}, \Lambda \in A^{(4)} \) and bounded nets \( (\lambda_\gamma) \subset A^{''}, (G_\beta) \subset A' \) with \( \Psi = w^* - \lim_\gamma \lambda_\gamma \) and \( \Lambda = w^* - \lim_\beta G_\beta \). By using formula (2) two \( A^{(4)} \)-bimodule actions are defined by

\[
\Psi.\Lambda = w^* - \lim_\gamma w^* - \lim_\beta \overline{\lambda_\gamma} G_\beta , \\
\Lambda.\Psi = w^* - \lim_\beta w^* - \lim_\gamma G_\beta \lambda_\gamma .
\]

We show that these three \( (A^{''}, \square, \square) \)-bimodule structures on \( A^{(5)} \) are not always equal. Suppose that \( \Psi = w^* - \lim_\gamma \lambda_\gamma \) in \( A^{(5)} \), \( \Lambda = w^* - \lim_\beta G_\beta \) and \( \Gamma = w^* - \lim_\alpha F_\alpha \) in \( A'' \) such that \( (\lambda_\alpha), (G_\beta) \) and \( (F_\alpha) \) are nets in \( A^{''} \) and \( A' \) respectively. For right actions, then

\[
\langle \Psi \bullet \Lambda, \Gamma \rangle = \langle \Psi, \Lambda \square \Gamma \rangle = \lim_\gamma \langle \Lambda \square \Gamma, \lambda_\gamma \rangle = \lim_\gamma \left( \langle w^* - \lim_\beta \lambda_\gamma G_\beta, \lambda_\gamma \rangle \right) = \lim_\gamma \lim_\beta \lim_\alpha \langle \lambda_\gamma, G_\beta \square F_\alpha \rangle .
\]

and

\[
\langle \Psi \circ \Lambda, \Gamma \rangle = \left\langle w^* - \lim_\gamma w^* - \lim_\beta \overline{\lambda_\gamma} G_\beta, \Gamma \right\rangle = \lim_\gamma \lim_\beta \langle \Gamma, \lambda_\gamma \square G_\beta \rangle = \lim_\gamma \lim_\beta \lim_\alpha \langle \lambda_\gamma, G_\beta \square F_\alpha \rangle .
\]

For structure (c), we have

\[
\langle \Psi.\Lambda, \Gamma \rangle = \left\langle w^* - \lim_\gamma w^* - \lim_\beta \overline{\lambda_\gamma} G_\beta, \Gamma \right\rangle = \lim_\gamma \lim_\beta \langle \Gamma, \lambda_\gamma \square G_\beta \rangle = \lim_\gamma \lim_\beta \lim_\alpha \langle \lambda_\gamma, G_\beta \square F_\alpha \rangle .
\]

We see all right actions in parts (a), (b) and (c) are equal. For left actions, then

\[
\langle \Lambda \bullet \Psi, \Gamma \rangle = \langle \Psi, \Gamma \square \Lambda \rangle = \lim_\gamma \langle \Gamma \square \Lambda, \lambda_\gamma \rangle = \lim_\gamma \left( \langle w^* - \lim_\alpha w^* - \lim_\beta F_\alpha \square G_\beta, \lambda_\gamma \rangle \right) = \lim_\gamma \lim_\alpha \lim_\beta \langle \lambda_\gamma, F_\alpha \square G_\beta \rangle .
\]

and

\[
\langle \Lambda \circ \Psi, \Gamma \rangle = \left\langle w^* - \lim_\beta w^* - \lim_\gamma G_\beta \square \lambda_\gamma, \Gamma \right\rangle = \lim_\beta \lim_\gamma \langle \Gamma, \lambda_\gamma \square G_\beta \rangle = \lim_\beta \lim_\gamma \lim_\alpha \langle G_\beta \lambda_\gamma, F_\alpha \rangle = \lim_\beta \lim_\gamma \lim_\alpha \langle G_\beta, \lambda_\gamma, F_\alpha \rangle .
\]

For structure (c), we have

\[
\langle \Lambda.\Psi, \Gamma \rangle = \left\langle w^* - \lim_\beta w^* - \lim_\gamma G_\beta \lambda_\gamma, \Gamma \right\rangle = \lim_\beta \langle \Gamma, G_\beta \lambda_\gamma \rangle = \lim_\beta \lim_\gamma \lim_\alpha \langle G_\beta \lambda_\gamma, F_\alpha \rangle = \lim_\beta \lim_\gamma \lim_\alpha \langle \lambda_\gamma, F_\alpha \square G_\beta \rangle .
\]
We see that left actions in part (a), (b) and (c) are not equal. We put some conditions on A and show that with these conditions all $A^{(4)}$-bimodule structure on $A^{(5)}$ are equal.

**Proposition 2.5.** Let $A$ be an Arens regular Banach algebra such that the map $\omega: A'' \times A''' \to A'''((F,\lambda) \mapsto F,\lambda)$ is Arens regular, the map $\varphi : A'' \to A''(F \mapsto G\square F)$ and $\rho : A'' \to A''(F \mapsto G\square F)$ for every $G \in A''$ are $w^* - w$-continuous. Then the two $A^{(4)}$-bimodule actions on $A^{(5)}$ in (a) and (b) are equal.

**Proof.** We know that the two right $((A'',\Box)'',\Box)$-bimodule actions on $A^{(5)}$ in (a) , (b) are equal to

$$\lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle$$

in which $(\lambda), (G_{\beta})$ and $(F_{\alpha})$ are bounded nets in $A'''$ and $A''$, respectively. It is enough show left $((A'',\Box)'',\Box)$-bimodule actions on $A^{(5)}$ is equal.

By Arens regularity of the map $\omega$ we have

$$\lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle = \lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle.$$  

By Lemma 2.4 and Lemma 17 we have

$$(\Lambda \bullet \Psi, \Gamma) = \lim_{\beta} \lim_{\alpha} \langle \lambda, F_{\alpha} \Box G_{\beta} \rangle$$

$$= \lim_{\beta} \lim_{\alpha} \langle \lambda, F_{\alpha} \Box G_{\beta} \rangle$$

$$= \lim_{\beta} \lim_{\alpha} \langle \lambda, (w^* - \lim_{\alpha} F_{\alpha}) \Box G_{\beta} \rangle$$

$$= \lim_{\beta} \lim_{\alpha} \langle \lambda, (w^* - \lim_{\alpha} F_{\alpha}) \Box G_{\beta} \rangle$$

$$= \lim_{\beta} \lim_{\alpha} \langle \lambda, (w^* - \lim_{\alpha} F_{\alpha}) \Box G_{\beta} \rangle$$

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$$= \langle \lambda, (w^* - \lim_{\alpha} F_{\alpha}) \Box G_{\beta} \rangle$$

By Arens regularity of the map $\omega$ we have

$$\lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle = \lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle.$$  

**Remark 2.6.** Similarly by using conditions in proposition 2.5, two $A^{(4)}$-bimodule structure on $A^{(5)}$ in (a) and (c) are equal.

**Theorem 2.7.** Let $A$ be an Arens regular Banach algebra. Then the two $A^{(4)}$-bimodule actions on $A^{(5)}$ in (b) and (c) are equal.

**Proof.** We know that the two right $((A'',\Box)'',\Box)$-bimodule actions on $A^{(5)}$ in (b) and (c) are equal to

$$\lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle$$

in which $(\lambda), (G_{\beta})$ and $(F_{\alpha})$ are bounded nets in $A''$ and $A''$, respectively. For left $((A'',\Box)'',\Box)$-bimodule actions on $A^{(5)}$ it is enough show the following equality

$$\lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle = \lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle$$

Since $A$ is Arens regular we can write

$$\lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle = \lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle$$

$$= \lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle$$

$$= \lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle$$

$$= \lim_{\beta} \lim_{\alpha} \langle \lambda, G_{\beta} \Box F_{\alpha} \rangle.$$  

$$\square$$
3 Derivations and weak amenability of $A^{(4)}$ and $A$

Let $D : A \rightarrow A'$ be a derivation, then by theorem 13, $D^{(4)} : ((A''', \Box)''') \rightarrow (((A')''')$ is also a derivation, that means that for every $\Gamma, \Lambda \in A^{(4)}$

$$D^{(4)}(\Gamma \Box \Box \Lambda) = D^{(4)}(w^* - \lim_{\alpha} w^* - \lim_{\beta} F_{\alpha} \Box G_{\beta})$$
$$= D''(F \Box G)$$
$$= D''(F) \circ G + F \circ D''(G)$$
$$= D^{(4)}(\Gamma) \circ \Lambda + \Gamma \circ D^{(4)}(A).$$

But $D^{(4)} : ((A', \Box)', \Box) \rightarrow (((A'')'')$ is not always a derivation, that means that for every $\Gamma, \Lambda \in A^{(4)}$ and $F = w^* - \lim_i a_i$,

$$D^{(4)}(\Gamma \Box \Box \Lambda) = D^{(4)}(w^* - \lim_{\alpha} w^* - \lim_{\beta} F_{\alpha} \Box G_{\beta})$$
$$= D''(F \Box G)$$
$$= D''(F).G + w^* - \lim_i a_i.D''(G)$$
$$= D^{(4)}(\Gamma).\Lambda + w^* - \lim_i a_i.D''(G).$$

**Theorem 3.1.** Let $A$ be a Banach algebra such that $((A''', \Box)''')$ is weakly amenable. If conditions of theorem 2.5 hold, then $A$ is weakly amenable.

**Proof.** Let $D : A \rightarrow A'$ be a derivation, then by Theorem 13 and Theorem 2.5

$$D^{(4)} : ((A''', \Box)''') \rightarrow (((A')''')$ is also a derivation. Since $A^{(4)}$ is weakly amenable, there exists $\Lambda \in A^{(4)}$ such that

$$D^{(4)}(\Lambda) = \Lambda.\Psi - \Psi.\Lambda,$$

$(\Psi \in A^{(5)})$.

Let $f = \iota^* \circ \iota^{***} (\Psi)$, where $\iota : A \rightarrow A''$ is the canonical injection. Then

$$D(a) = a.f + f.a,$$

$(a \in A)$,

and so $D$ is inner. Thus $A$ is weakly amenable. \hfill $\Box$

**References**


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Majorization and Euclidean distance matrices

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Abstract
Let $D_1$ and $D_2$ be two Euclidean distance matrices (EDM) with corresponding positive semidefinite matrices $B_1$ and $B_2$ respectively. Suppose that $\lambda(A)$ is the vector of eigenvalues of a matrix $A$. In this note we investigate the conditions under which $\lambda(D_1) \prec \lambda(D_2)$ whenever $\lambda(B_1) \prec \lambda(B_2)$. Also for $n = 2$ we show that $\lambda(D_1) \prec_t \lambda(D_2)$ if and only if $\lambda(B_1) \prec_t \lambda(B_2)$.

Keywords: Left matrix majorization, Majorization, Euclidean distance matrix, Positive semidefinite matrix.

Mathematics Subject Classification: 15A03, 15A45

1 Introduction
An $n \times n$ nonnegative and symmetric matrix $D = (d_{ij}^2)$ with zero diagonal elements is called a predistance matrix. A predistance matrix $D$ is called Euclidean or a Euclidean distance matrix (EDM) if there exist a positive integer $r$ and a set of $n$ points $\{p_1, \ldots, p_n\}$ such that $p_1, \ldots, p_n \in \mathbb{R}^r$ and $d_{ij}^2 = \|p_i - p_j\|^2$ ($i, j = 1, \ldots, n$), where $\|\cdot\|$ denotes the usual Euclidean norm. The smallest value of $r$ that satisfies the above condition is called the embedding dimension.

Let $\Lambda_n$ be the set of $n \times n$ EDMs, and $\Omega_n(e)$ be the set of $n \times n$ positive semidefinite matrices $B$ such that $Be = 0$, where $e$ is the vector of all ones. Then the linear mapping $\tau : \Lambda_n \to \Omega_n(e)$ defined by $\tau(D) = -\frac{1}{2} PDP$ with $P = I_n - \frac{1}{n} ee^t$, where $I_n$ is the $n \times n$ identity matrix, is invertible, and its inverse mapping, say $\kappa : \Omega_n(e) \to \Lambda_n$ is given by $\kappa(B) = be^t + eb^t - 2B$ with $b = diag(B)$, where $diag(B)$ is the vector consisting of the diagonal elements of $B$.

As is well-known, a necessary and sufficient condition for a predistance matrix $D$ to be an EDM is that $\tau(D)$ is positive semidefinite matrix.

Majorization is one of the vital topics in mathematics and statistics. It plays a basic role in matrix theory. For instance, majorization relation among eigenvalues and singular values of matrices produce a lot of norm inequalities. For more details on this subject see [3, 5].

For two vectors $x = (x_1, \ldots, x_n)^t$ and $y = (y_1, \ldots, y_n)^t$, we say that $x$ majorizes $y$ and denote it by $y \prec x$, if $\sum_{i=1}^k y_i \leq \sum_{i=1}^k x_i$ ($k = 1, \ldots, n - 1$) and $\sum_{i=1}^n y_i = \sum_{i=1}^n x_i$, where $x_i$ and $y_i$ are the $i$th largest elements of $x$ and $y$, respectively. It is equivalent to existence of an $n \times n$ doubly stochastic matrix $D$ such that $y = Dx$.

Some properties of the eigenvalues of EDMs are studied by Alfakih in [1]. Recently a connection between majorization and Euclidean matrices has been found by Hayden and Tarazaga in [2].

Let $M_n$ be the set of all $n \times n$ real matrices. A matrix $R \in M_n$ with nonnegative entries is called row stochastic if $Re = e$. Let $x, y \in \mathbb{R}^n$. It is said that $x$ is left matrix majorized by $y$, and is denoted as $x \prec_l y$, if $x = Ry$ for some row stochastic matrix $R$.

First, we state some results from [4].
Proposition 1.1. Let $B$ and $\tilde{B}$ be two arbitrary positive semidefinite matrix in $\Omega_n(e)$, and let $\beta = \lambda(B)$ and $\tilde{\beta} = \lambda(\tilde{B})$. Then a necessary and sufficient condition for the majorization $\tilde{\beta} < \beta$ to hold is that the equality $\tilde{B} = \sum_{i=1}^{m} c_i \Gamma_i B \Gamma_i^T$ for some $\Gamma_i \in O(n; e)$ ($i = 1, \ldots, m$), where $m$ is a positive integer, $c_i$s are positive real satisfying $\sum_{i=1}^{m} c_i = 1$, and $O(n; e)$ is the set of orthogonal matrices $\Gamma$ satisfying $\Gamma e = e$.

Till the end of this section, the relation between the eigenvalues of EDMs and those of the corresponding positive semidefinite matrices will be characterized.

Proposition 1.2. Let $B$ and $\tilde{B}$ be two arbitrary positive semidefinite matrix in $\Omega_n(e)$, and let $D = \kappa(B)$ and $\tilde{D} = \kappa(\tilde{B})$. Suppose that $\tilde{B} = \sum_{i=1}^{m} c_i \Pi_i B \Pi_i^T$ for some $\Pi_i \in P(n)$ ($i = 1, \ldots, m$), where $m$ is a positive integer, $c_i$s are positive real satisfying $\sum_{i=1}^{m} c_i = 1$, and $P(n)$ is the set of $n \times n$ permutation matrices. Then the eigenvalues of $D$ majorizes those of $\tilde{D}$: $\lambda(\tilde{D}) < \lambda(D)$.

Proposition 1.3. Let $B, \tilde{B} \in \Omega_n(e)$, $D = \kappa(B)$ and $\tilde{D} = \kappa(\tilde{B})$. Suppose that eigenvalues of $B$ majorizes those of $\tilde{B}$: $\lambda(B) < \lambda(\tilde{B})$, that is, $\tilde{B} = \sum_{i=1}^{m} c_i \Gamma_i B \Gamma_i^T$ for some $\Gamma_i \in O(n; e)$, where $m$ is a positive integer, $c_i$s are positive real satisfying $\sum_{i=1}^{m} c_i = 1$. Then $D$ and $\tilde{D}$ admit the following expression $D = \sum_{i=1}^{m} c_i \Gamma_i D \Gamma_i^T + ye^T + ey^T$ with $y = b - \sum_{i=1}^{m} c_i \Gamma_i b$, where $b = \text{diag}(B)$ and $\tilde{b} = \text{diag}(\tilde{B})$. And hence, $\lambda(\tilde{D} - (ye^T + ey^T)) < \lambda(D)$.

Proposition 1.4. Suppose $B, \tilde{B} \in \Omega_n(e)$ with equal diagonal elements, and $\lambda(\tilde{B}) < \lambda(B)$. Then $\lambda(\tilde{D}) < \lambda(D)$.

Proposition 1.5. Let $B, \tilde{B} \in \Omega_n(e)$, $D = \kappa(B)$ and $\tilde{D} = \kappa(\tilde{B})$. Suppose that $\tilde{B} = \sum_{i=1}^{m} c_i \Pi_i B \Pi_i^T$ for some $\Pi_i \in P(n)$, where $m$ is a positive integer, $c_i$s are positive real satisfying $\sum_{i=1}^{m} c_i = 1$. Then the eigenvalues of $B$ majorizes those of $\tilde{B}$: $\lambda(B) < \lambda(\tilde{B})$.

2 Main Result

Remark 2.1. [3] It is well-known, $x \prec_1 y$ if and only if $\min y \leq \min x \leq \max x \leq \max y$, where $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, $\min x = \min \{x_1, \ldots, x_n\}$, and $\max x = \max \{x_1, \ldots, x_n\}$.

The following theorem will specify the relation between the eigenvalues of an EDM and those of the corresponding positive semidefinite matrix with respect to $\prec_1$ on $\mathbb{R}^2$.

Theorem 2.2. Let $B, \tilde{B} \in \Omega_2(e)$, and let $D = \kappa(B)$ and $\tilde{D} = \kappa(\tilde{B})$. Then $\lambda(\tilde{B}) \prec_1 \lambda(B)$ if and only if $\lambda(\tilde{D}) \prec_1 \lambda(D)$.

Proof. Since $B, \tilde{B} \in \Omega_2(e)$, there exist $\alpha, \beta \geq 0$ such that $B = \begin{pmatrix} \alpha & -\beta \\ -\beta & \alpha \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \beta \end{pmatrix}$, and $\{0, 2\alpha\}$ and $\{0, 2\beta\}$ are the set of eigenvalues of $B$ and $\tilde{B}$, respectively. Now by Remark 2.1, $\lambda(\tilde{B}) \prec_1 \lambda(B)$ if and only if $0 \leq \beta \leq \alpha$. By the definition of $\kappa$, $D = \begin{pmatrix} \alpha & -\beta \\ -\beta & \alpha \end{pmatrix}$ and $\tilde{D} = \begin{pmatrix} \alpha & -\beta \\ -\beta & \alpha \end{pmatrix}$. So $\{-4\alpha, 4\alpha\}$ and $\{-4\beta, 4\beta\}$ are the set of eigenvalues of $D$ and $\tilde{D}$, respectively. By applying Remark 2.1, $\lambda(\tilde{D}) \prec_1 \lambda(D)$ if and only if $0 \leq \beta \leq \alpha$. The previous discussion shows $\lambda(\tilde{B}) \prec_1 \lambda(B)$ if and only if $\lambda(\tilde{D}) \prec_1 \lambda(D)$.

Theorem 2.3. Let $B = (b_{ij}), \tilde{B} = (\tilde{b}_{ij}) \in \Omega_3(e)$ such that $b_{11} = \tilde{b}_{11} = 0$ and let $D = \kappa(B)$ and $\tilde{D} = \kappa(\tilde{B})$. Then $\lambda(\tilde{B}) \prec_1 \lambda(B)$ if and only if $\lambda(\tilde{D}) \prec_1 \lambda(D)$.

Proof. Since $B \in \Omega_3(e)$, there exist $\alpha, \beta \in \mathbb{R}$ such that $B = \begin{pmatrix} \alpha & -\beta & -\alpha \\ -\beta & \alpha & -\beta \\ -\alpha & -\beta & \alpha \end{pmatrix}$. Since $B$ is positive semidefinite matrix, we have $\alpha = 0$ and $\beta \geq 0$. So $B = \begin{pmatrix} \alpha & -\beta & -\alpha \\ -\beta & \alpha & -\beta \\ -\alpha & -\beta & \alpha \end{pmatrix}$ and hence $\{0, 0, 2\beta\}$ and $\{-4\beta, (2 + \sqrt{6})\beta, (2 - \sqrt{6})\beta\}$ are the set of eigenvalues of $B$ and $D$.
respectively. Similarly, $\tilde{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\beta \\ 0 & -\beta & 0 \end{pmatrix}$, $\tilde{D} = \begin{pmatrix} 0 & \beta & 0 \\ -\beta & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix}$, $\tilde{\beta} \geq 0$ and hence $\{0, 0, 2\tilde{\beta}\}$ and 
$\{-4\tilde{\beta}, (2 + \sqrt{6})\tilde{\beta}, (2 - \sqrt{6})\tilde{\beta}\}$ are the set of eigenvalues of $\tilde{B}$ and $\tilde{D}$ respectively. By Remark 2.1, $\lambda(\tilde{B}) \prec_{l} \lambda(B)$ if and only if $0 \leq \tilde{\beta} \leq \beta$. Also, $\lambda(\tilde{D}) \prec_{l} \lambda(D)$ if and only if $-4\beta \leq -4\tilde{\beta} \leq (2 + \sqrt{6})\tilde{\beta} \leq (2 + \sqrt{6})\beta$ if and only if $0 \leq \tilde{\beta} \leq \beta$. This complete the proof. 

References


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Induced Hilbert $C^*$-modules by $*$-isomorphisms

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Abstract

If $A$, $B$ are two $C^*$-algebras and $M$ is a Hilbert $A$-module one can induce, by means of a $*$-isomorphism $\varphi : A \to B$, a Hilbert $B$-module structure over $M$. In this paper we show that if $M$ is a full Hilbert $A$-module, then the induced Hilbert $B$-module is also full and their inner products will be equivalent.

Keywords: Hilbert $C^*$-Module, Morphism.

Mathematics Subject Classification: 46L08

1 Introduction

Suppose $A$ is a $C^*$-algebra and $M$ is a right $A$-module, then $M$ is called a pre-Hilbert $A$-module if there exists a sesquilinear form $\langle \cdot, \cdot \rangle : M \times M \to A$ with the following properties:

i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

ii) $\langle y, x \rangle = \langle x, y \rangle^*$;

iii) $\langle x, ya \rangle = \langle x, y \rangle a$;

for each $x, y \in M$ and $a \in A$. Let $M$ be a pre-Hilbert $A$-module and $x \in M$. $M$ is called a Hilbert $C^*$-module or a Hilbert $A$-module if it is complete with respect to the norm $||\cdot||_M$, where $||x||_M = ||\langle x, x \rangle||^{1/2}$.

Definition 1.1. Let $A$ be a $C^*$-algebra. An increasing net $\{u_\alpha\}_{\alpha \in I}$ in $A^+$ with $||u_\alpha|| \leq 1$ for all $\alpha$ is called an approximate unit for $A$ if $\lim_{\alpha} ||a(1 - u_\alpha)|| = 0$ for each $a$ in $A$. A $C^*$-algebra is called $\sigma$-unital if it has a countable approximate unit. If $M$ is a Hilbert $A$-module then we denote by $\langle M, M \rangle$ the closure of the linear span of all $\langle x, x \rangle$, $x \in M$. In fact

$$\langle M, M \rangle = \text{span}\{\langle x, x \rangle; \ x \in M\}.$$  

The Hilbert $A$-module $M$ is called full if $\langle M, M \rangle = A$.

Suppose $A$ and $B$ are $C^*$-algebras and $\varphi : A \to B$ is a $*$-isomorphism and $M$ is a Hilbert $A$-module, then one can make $M$ a Hilbert $B$-module. Define $\cdot : M \times B \to M$ by $(m, b) \mapsto m.b = m\varphi^{-1}(b)$.

Then $M$ is a right $B$-module, since

i) $(am_1 + m_2).b = (am_1 + m_2)\varphi^{-1}(b) = am_1\varphi^{-1}(b) + m_2\varphi^{-1}(b) = am_1.b + m_2.b$;

ii) $m.(ab_1 + b_2) = m\varphi^{-1}(ab_1 + b_2) = m(ab_1 + b_2) = m(\alpha\varphi^{-1}(b_1) + \varphi^{-1}(b_2)) = \alpha m\varphi^{-1}(b_1) + m\varphi^{-1}(b_2) = \alpha m.b_1 + m.b_2$;

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iii) \((m.b_1).b_2 = (m\varphi^{-1}(b_1)).\varphi^{-1}(b_2) = m\varphi^{-1}(b_1)\varphi^{-1}(b_2) = m\varphi^{-1}(b_1b_2) = m.b_1b_2\)

Moreover if the \(A\)-valued inner product of \(M\) is denoted by \(\langle \cdot, \cdot \rangle\), then one can define a \(B\)-valued inner product by \(\langle \cdot, \cdot \rangle'M \times M \rightarrow B\) by \(\langle x, y \rangle' = \varphi((x, y))\). In fact

1) \(\langle ax + y, z \rangle' = \varphi(\langle ax + y, z \rangle) = \varphi(\alpha(x, y) + \langle y, z \rangle) = \alpha \varphi(x, z) + \varphi(y, z) = \alpha \langle x, z \rangle' + \langle y, z \rangle'\);

2) \(\langle x, ay + z \rangle' = \varphi(\langle x, ay + z \rangle) = \varphi(\alpha(x, y) + \langle y, z \rangle) = \alpha \varphi(x, y) + \varphi(x, z) = \alpha \langle x, y \rangle' + \langle x, z \rangle';\)

3) \(\langle x, x \rangle' = \varphi((x, x)) \geq 0;\)

4) \(\langle x, x \rangle' = 0 \iff \varphi((x, x)) = 0 \iff \langle x, x \rangle = 0 \iff x = 0;\)

5) \(\langle x, y \rangle' = \varphi((x, y)) = \varphi((x, y)^*) = [\varphi((x, y))]^* = [(x, y')]^*;\)

6) \(\langle x, y, b \rangle' = \varphi((x, y, a)) = \varphi((x, y)) = \varphi((x, y)a) = \varphi((x, y))' = \varphi((x, y)^*);\)

So \((\cdot, \cdot)'\) is a \(B\)-valued inner product on \(M\). Moreover \(M\) with \(\| \cdot \|_M\) is complete and if we define \(\| x \|_M^2 = \| \varphi((x, x)) \| = \| (x, x) \| = \| x \|_M^2\) then \(\| x \|_M = \| x \|_M\). So \(M\) is complete with respect to \(\| \cdot \|_M\). So \(M\) is a Hilbert \(B\)-module which is called the induced Hilbert \(B\)-module by \(\varphi: A \rightarrow B\).

If \(M\) is a Hilbert \(A\)-module with \(A\)-valued inner product \(\langle \cdot, \cdot \rangle\), we denote it by \((M, A, \langle \cdot, \cdot \rangle)\).

**Definition 1.3.** Two inner products \(\langle \cdot, \cdot \rangle_1\) and \(\langle \cdot, \cdot \rangle_2\) are called equivalent if the norms defined by these inner products are equivalent.

**Theorem 1.3.** Inner products \(\langle \cdot, \cdot \rangle'\) and \(\langle \cdot, \cdot \rangle\) are equivalent.

**Proof.** Since \(\| x \|_M = \| x \|_M\), two norms \(\| \cdot \|_M\) and \(\| \cdot \|_M\) are equivalent. \(\square\)

**Definition 1.4.** Let \(V\) and \(W\) be Hilbert \(C^*\)-modules over \(C^\ast\)\(-\)algebras \(A\) and \(B\), respectively. Let \(\varphi: A \rightarrow B\) be a morphism of \(C^\ast\)\(-\)algebras. A map \(\Phi: V \rightarrow W\) is said to be a \(\varphi\)-morphism of Hilbert \(C^\ast\)-modules if \(\Phi(x) \Phi(y) = \varphi(\langle x, y \rangle)\) is satisfied for all \(x, y\) in \(V\).

**Theorem 1.5.** Suppose \(A\) and \(B\) are \(C^\ast\)-algebras and \(\varphi: A \rightarrow B\) is a \(\ast\)-isomorphism, then if \((M, A, \langle \cdot, \cdot \rangle)\) is a Hilbert \(A\)-module and \((M, B, \langle \cdot, \cdot \rangle')\) be the induced Hilbert \(B\)-module, then \(\Phi: (M, A, \langle \cdot, \cdot \rangle) \rightarrow (M, B, \langle \cdot, \cdot \rangle')\) defined by \(\Phi(x) = x\) is a \(\varphi\)-morphism.

**Proof.** \(\Phi(x) \Phi(y)' = \langle x, y \rangle' = \varphi((x, y)).\) \(\square\)

### 2 Main Result

**Theorem 2.1.** If \(A\) and \(B\) are \(C^\ast\)-algebras and \(M\) is a full Hilbert \(A\)-module and \(\varphi: A \rightarrow B\) is a \(\ast\)-isomorphism, then the Hilbert \(B\)-module \(M\) is full.

**Proof.** Since the Hilbert \(A\)-module \(M\) is full we have

\[
\langle M, M \rangle = \text{span}\{\langle x, x \rangle : x \in M\} = A.
\]

So for \(\epsilon > 0\) and \(a \in A\) there exists \(\sum_{i=1}^{n} \lambda_i\langle x_i, x_i \rangle \in \text{span}\{\langle x, x \rangle : x \in M\}\) such that \(\| \sum_{i=1}^{n} \lambda_i\langle x_i, x_i \rangle - a \| < \epsilon\). Now if \(b \in B\) is arbitrary then there exists \(a \in A\) such that \(b = \varphi(a)\), so

\[
\| \sum_{i=1}^{n} \lambda_i\langle x_i, x_i \rangle' - b \| = \| \sum_{i=1}^{n} \lambda_i\varphi(\langle x_i, x_i \rangle) - b \| = \| \varphi(\sum_{i=1}^{n} \lambda_i\langle x_i, x_i \rangle) - \varphi(a) \| \leq \| \varphi(\sum_{i=1}^{n} \lambda_i\langle x_i, x_i \rangle - a) \| = \| \sum_{i=1}^{n} \lambda_i\langle x_i, x_i \rangle - a \| < \epsilon.
\]

So \(\langle M, M \rangle' = \text{span}\{\langle x, x \rangle' : x \in M\} = B\), so \(M\) is a full Hilbert \(B\)-module. \(\square\)
Lemma 2.2. Suppose A and B are $C^*$-algebras and M is a Hilbert A-module and $\varphi : A \rightarrow B$ is a $*-$isomorphism, then $a \perp b$ in Hilbert A-module M if and only if $a \perp b$ in Hilbert B-module M.

Proof. $a \perp b$ in Hilbert A-module M $\iff \langle a, b \rangle = 0 \iff \varphi(\langle a, b \rangle) = 0 \iff \langle a, b' \rangle = 0 \iff a \perp b$ in Hilbert B-module M. □

Theorem 2.3. Let A and B be $C^*$-algebras and $\varphi : A \rightarrow B$ be a $*-$isomorphism and M is a Hilbert A-module and $C, D \subset M$, then $C \perp D$ in Hilbert A-module M if and only if $C \perp D$ in Hilbert B-module M.

Proof. $C \perp D$ in Hilbert A-module M $\iff \forall c \in C, d \in D : \langle c, d \rangle = 0 \iff \forall c \in C, d \in D : \varphi(\langle c, d \rangle) = 0 \iff \forall c \in C, d \in D : \varphi(\langle c, d \rangle) = 0 \iff C \perp D$ in Hilbert B-module M. □

Theorem 2.4. If A and B are $C^*$-algebras and $\varphi : A \rightarrow B$ be a $*-$isomorphism and M is a Hilbert A-module and $M = R \oplus S$ in $(M, A, \langle, \rangle)$, then $M = R \oplus S$ in $(M, B, \langle, \rangle')$.

Proof. $M = R \oplus S$ in $(M, A, \langle, \rangle) \iff \forall m \in M : m = n + n'$ where $n \in R, n' \in S$ and $R \perp S$ in $(M, A, \langle, \rangle) \iff \forall m \in M : m = n + n'$ where $n \in R, n' \in S$ and $R \perp S$ in $(M, B, \langle, \rangle') \iff \forall m \in M : m = n + n'$ where $n \in R, n' \in S$ and $R \oplus S$ in $(M, B, \langle, \rangle') \iff M = R \oplus S$ in $(M, B, \langle, \rangle')$. □

Lemma 2.5. If A and B are $C^*$-algebras and $\varphi : A \rightarrow B$ be a $*-$isomorphism and if $\{u_\alpha\}_{\alpha \in I}$ is an approximate unit of A, then $\{\varphi(u_\alpha)\}_{\alpha \in I}$ is an approximate unit of B.

Proof. Suppose $b \in B$ is arbitrary. So there exists a in A such that $\varphi(a) = b$. Hence $\|b\varphi(u_\alpha) - b\| = \|\varphi(a)\varphi(u_\alpha) - \varphi(a)\| = \|\varphi(au_\alpha) - \varphi(a)\| = ||\varphi(au_\alpha - a)|| = ||au_\alpha - a|| \rightarrow 0$. □

Theorem 2.6. Let A and B be $C^*$-algebras and $\varphi : A \rightarrow B$ be a $*-$isomorphism and $(M, A, \langle, \rangle)$ is a Hilbert A-module and $(M, B, \langle, \rangle')$ is the induced Hilbert B-module, then if $(x_i)_{i=1}^\infty$ is a sequence in M such that $(\sum_{i=1}^k \langle x_i, x_i \rangle)_{k=1}^\infty$ is an approximate unit of A, then $(\sum_{i=1}^k \langle x_i, x_i \rangle')_{k=1}^\infty$ is an approximate unit of B.

Proof. By Lemma 2.5, if $(\sum_{i=1}^k \langle x_i, x_i \rangle)_{k=1}^\infty$ is an approximate unit of A, then $(\varphi(\sum_{i=1}^k \langle x_i, x_i \rangle))_{k=1}^\infty$ is an approximate unit of B. But

$$\varphi(\sum_{i=1}^k \langle x_i, x_i \rangle) = \sum_{i=1}^k \varphi(\langle x_i, x_i \rangle) = \sum_{i=1}^k \langle x_i, x_i \rangle'.$$

By Lemma 2.4.3 of [3] we know that if A is unital, then there exists a finite number $k$ and elements $x_1, x_2, \cdots, x_k \in M$ such that $\sum_{i=1}^k \langle x_i, x_i \rangle = 1$. Now if $\varphi : A \rightarrow B$ is a $*-$isomorphism and $B$ is unital, then

$$\sum_{i=1}^k \langle x_i, x_i \rangle' = \sum_{i=1}^k \varphi(\langle x_i, x_i \rangle) = \varphi(\sum_{i=1}^k \langle x_i, x_i \rangle) = \varphi(1_A) = 1_B.$$

References


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Some characterizations of EP matrices

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Abstract
The EP elements in an algebra can be considered as a generalization of invertible elements. In the case of matrix spaces it contains a wide range of elements such as normal elements. In this note we review some of the characterizations of this class.

Keywords: Moore-Penrose inverse, Drazin inverse, EP elements

Mathematics Subject Classification: 15A09; 15A57.

1 Introduction
A Matrix $A \in M_n(\mathbb{C})$ is said to have a Moore-Penrose inverse if there exists $X \in M_n(\mathbb{C})$ such that

$$(AX)^* = AX, \quad (XA)^* = XA, \quad AXA = A \quad \text{and} \quad XAX = X.$$  

It can be proved that if $A$ has a Moore-Penrose inverse, then its Moore-Penrose inverse is unique and under this situation, we shall write $X = A^\dagger$.

A matrix $A$ is called EP when $R(A) = R(A^*)$. It is a well known result that $A$ is an EP matrix if and only if $AA^\dagger = A^\dagger A$. [1]

The range equality is a fairly weaker restriction to $A$, so that EP matrices are a wide class of objects that include many matrices as their special cases, such as, Hermitian and skew-Hermitian matrices (i.e., $A^* = \pm A$), normal matrices (i.e., $AA^* = A^*A$), as well as all nonsingular matrices. EP matrices also have close links with other types of matrix, such as, orthogonal projector (i.e., $A^2 = A = A^*$), GP matrix (i.e., $R(A) = R(A^2)$), bi-normal matrix (i.e., $(AA^*)(A^*A) = (A^*A)(AA^*)$), star-dagger matrix (i.e., $A^*A^\dagger = A^\dagger A^*$), bi-dagger matrix (i.e., $(A^2)^\dagger = (A^\dagger)^2$). In addition, EP matrices were also extended to some other types of matrix, such as, bi-EP matrix (i.e., $(AA^\dagger)(A^\dagger A)(AA^\dagger)$), conjugate EP matrix (i.e., $AA^\dagger = \overline{A^\dagger A}$), $K$-EP matrix (i.e., $R(A) = R(KA^*)$) and power EP matrix (i.e., $R(A^k) \subseteq R(A^*)$ and $R((A^k)^*) \subseteq R(A)$).

This paper aims at collecting various known characterizations of EP matrix in the literature.

2 Main Result
In [1] the following characterizations of EP matrices is stated.

Theorem 2.1. For $A \in M_n(\mathbb{C})$ the following statements are equivalent:

- $A$ is $EP$;
- $N(A) = N(A^*)$;
- $\mathbb{C}^n = N(A) \oplus ^\perp R(A)$.
• $A = U(K \oplus 0)U^*$ for some unitary $U$ and nonsingular matrix $K$;

• there exists a matrix $Y$ such that $A = YA^*$;

• $\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} P_{R(A)} = A^1$.

Some other properties is listed in [2, 4].

**Theorem 2.2.** A is an EP matrix if and only if either $\dim[R(A) \cap R(A^*)] = \text{rank}(A)$ or $\dim[R(A) + R(A^*)] = \text{rank}(A)$.

In addition, using the concept of Drazin inverse, we can discuss about EP matrices. For this purpose, we need the following definitions.

**Definition 2.3.** The index of $A \in M_n(\mathbb{C})$ is the least nonnegative integer $k = \text{ind}(A)$ such that $\mathcal{N}(A^k) = \mathcal{N}(A^{k+1})$ and the eigenprojection $A^\pi$ of $A$ (corresponding to the eigenvalue 0) is the uniquely determined idempotent matrix with $\mathcal{R}(A^\pi) = \mathcal{N}(A^k)$ and $\mathcal{N}(A^\pi) = \mathcal{R}(A^k)$.

It can be proved that $A$ is EP if and only if $AA^\pi = 0$ and $(A^\pi)^* = A^\pi$. We note that $AA^\pi = 0$ is equivalent to $\text{ind}(A) \leq 1$.

In [3], the authors find the following characterization of EP matrices.

**Theorem 2.4.** Let $A \in M_n(\mathbb{C})$. Then the following conditions are equivalent:

• $A$ is EP;

• $(A^\pi)D A^* = A^D A$ and $A^D AA^* = A^*$;

• $A^D AA^* = A^* = A^* A^D A$ and $I + A^* - A^D A$ is nonsingular;

• $A^D AA^* = A^* = A^* A^D A$ and $I + A^D A^* - A^D A$ is nonsingular.

Although, in [5] Zhang and Chen prove the following theorem.

**Theorem 2.5.** A matrix $A$ is EP if and only if $A^\pi A^* = A^* A^\pi = 0$.

In their proof they also show that the conditions "$I + A^* - A^D A$ is invertible" in (iii) and "$I + A^D A^* - A^D A$ is invertible" in (iv) in Theorem 2.4, can be deleted. In fact both $I + A^* - A^D A$ and $I + A^D A^* - A^D A$ are invertible if $A^D AA^* = A^* = A^* A^D A$.

Another approach is as follows:

Recall that for $A \in M_n(\mathbb{C})$, the null space ideals of $A$ is defined as $A^0 = \{X \in M_n(\mathbb{C}) : AX = 0\}$ and $^0 A = \{X \in M_n(\mathbb{C}) :XA = 0\}$.

**Theorem 2.6.** A matrix $A$ is EP if and only if one of the following statements holds.

• There exists a unique projection $P$ such that $A + P$ is invertible and $AP = PA = 0$.

• $A^0 = (A^*)^0$.

• $^0 A = (A^*)^0$

Note that the projection $P$ in this theorem is exact the matrix $A^\pi$. 

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σ−Biflatness of Banach algebras

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Abstract

Let $A$ be a Banach algebra and $\sigma$ be continuous homomorphism on $A$. Suppose that $X$ and $Y$ be Banach $A$-bimodules. A linear bounded mapping $T : X \to Y$ is a Anti $\sigma - A$-bimodule homomorphism if $T(\sigma(a) \cdot x) = a \cdot T(x)$, $T(x \cdot \sigma(a)) = T(x) \cdot a$ $(a \in A, x \in X)$. The Banach algebra $A$ is called $\sigma$-biflat if there exists a bounded Anti $\sigma - A$-bimodule homomorphism $\rho : (A \otimes A)^* \to A^*$ such that $\rho \circ \pi^* = \sigma^*$.

In this paper, we investigate the relation between $\sigma$-biprojectivity and $\sigma$-biflatness of Banach algebras. We also introduce the concept of $\sigma$-splitness of short exact sequences $\Pi$ and $\Pi^*$ of Banach $A$-bimodules and $A$-bimodule homomorphisms. Finally, we investigate the relation between $\sigma$-amenability and $\sigma$-biflatness where $\sigma$ is an idempotent epimorphism of $A$.

Keywords: Banach algebras, $\sigma$-contractible, $\sigma$-Biflatness, $\sigma$-derivation.

Mathematics Subject Classification: Primary: 46H20; Secondary: 46H35, 46H25

1 Introduction and preliminaries

The concept of amenability for Banach algebras was introduced by B.E. Johnson in 1972 [5]. In fact, he defined the amenability of a Banach algebra $A$ through vanishing of first cohomology groups of $A$ with coefficients in $X^*$ for each Banach $A$-bimodule $X$, where $X^*$ denotes the first dual space of $X$, which is a Banach $A$-bimodule in the usual way. He showed that the concept of amenability is of great importance in the theory of Banach algebras. Other notions of amenability such as contractibility (or super amenability) and weak amenability were introduced by other authors (c.f. [1]). All of above notions of amenability are generalized in some ways during recent decades. For example, Ghahramani and Loy defined approximate versions of these notions ([2]) and extensively studied them on different classes of Banach algebras through several papers. Another extension of the mentioned notions of amenability, that we are interested, were done by using and endomorphism of a Banach algebra (see, for example, [6], [9], [10]). There are some other concepts related to the notions of amenability such as biflatness and biprojectivity, introduced by Hellemskii in [3] (see also [4] and [11]).

The notions of $\sigma$-contractibility and $\sigma$-biprojectivity in the context of Banach algebras as an extension of the ordinary notions of contractibility and biprojectivity was introduced in [10], where $\sigma$ is a dense range or an idempotent bounded endomorphism of the corresponding Banach algebra. Also it was investigated relations between $\sigma$-contractibility and $\sigma$-biprojectivity of Banach algebras.

In this paper, we develop definition of biflatness of a Banach algebra to $\sigma$-biflatness. We investigate the relation between $\sigma$-biprojectivity and $\sigma$-biflatness of Banach algebras, too. And also the relation between $\sigma$-amenability
and \( \sigma \)-biflatness where \( \sigma \) is an idempotent epimorphism of \( A \).

Let \( A \) be a Banach algebra and \( A \hat{\otimes} A \) be the projective tensor product of \( A \) and \( A \). The product map on \( A \) extends to a map \( \pi : A \hat{\otimes} A \to A \) determined by
\[
\pi(a \otimes b) = ab \quad (a, b \in A).
\]
The projective tensor product \( A \hat{\otimes} A \) becomes a Banach \( A \)-bimodule with the following usual module actions:
\[
a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).
\]
Obviously, by above actions, \( \pi \) becomes an \( A \)-bimodule homomorphism.

\section{\( \sigma \)-Biflatness}

\textbf{Definition 2.1.} Let \( A \) be a Banach algebra, \( \sigma \in \text{Hom} \ A \) and let \( X \) and \( Y \) be left Banach \( A \)-modules. The map \( T \in B(X,Y) \) is called a left \( \sigma \)-\( A \)-module homomorphism if
\[
T(a \cdot x) = \sigma(a) \cdot T(x) \quad (a \in A, \ x \in X).
\]
Also \( T \in B(X,Y) \) is called a Anti left \( \sigma \)-\( A \)-module homomorphism if
\[
T(\sigma(a) \cdot x) = a \cdot T(x) \quad (a \in A, \ x \in X).
\]
Similarly, right and two-sided \( \sigma \)-\( A \)-module homomorphism, Anti right and two-sided \( \sigma \)-\( A \)-module homomorphism are defined.

\textbf{Lemma 2.2.} Let \( A \) be a Banach algebra, \( \sigma \in \text{Hom} \ A \) and let \( X \) and \( Y \) be left Banach \( A \)-modules. If \( T \in B(X,Y) \) be a left \( \sigma \)-\( A \)-module homomorphism, then \( T^* \in B(Y^*,X^*) \) is a Anti right \( \sigma \)-\( A \)-module homomorphism.

\textbf{Proof.} For each \( x \in X \), we have
\[
\langle x , T^*(y^* \cdot \sigma(a)) \rangle = \langle T(x) , y^* \cdot \sigma(a) \rangle = \langle \sigma(a) \cdot T(x) , y^* \rangle = \langle x , T(y^*) \rangle \quad (a \in A, \ y^* \in Y^*)
\]
Hence, \( T^* \) is a Anti right \( \sigma \)-\( A \)-module homomorphism.

\textbf{Lemma 2.3.} Let \( A \) be a Banach algebra, \( \sigma \in \text{Hom} \ A \) and let \( X \) and \( Y \) be left Banach \( A \)-modules. If \( T \in B(X,Y) \) be a Anti left \( \sigma \)-\( A \)-module homomorphism, then \( T^* \in B(Y^*,X^*) \) is a right \( \sigma \)-\( A \)-module homomorphism.

\textbf{Proof.} For each \( x \in X \), we have
\[
\langle x , T^*(y^* \cdot a) \rangle = \langle T(x) , y^* \cdot a \rangle = \langle a \cdot T(x) , y^* \rangle = \langle T(\sigma(a) \cdot x) , y^* \rangle = \langle \sigma(a) \cdot x , T^*(y^*) \rangle = \langle x , T^*(y^*) \cdot \sigma(a) \rangle \quad (a \in A, \ y^* \in Y^*)
\]
Hence, \( T^* \) is a right \( \sigma \)-\( A \)-module homomorphism.
Corollary 2.4. Let $A$ be a Banach algebra, $\sigma \in \text{Hom}A$ and let $X$ and $Y$ be Banach left $A$-modules. If $T \in B(X,Y)$ be a left $\sigma$-$A$-module homomorphism, then $T^{**} \in B(X^{**},Y^{**})$ is a left $\sigma$-$A$-module homomorphism, too.

The same statement also established about the Anti left $\sigma$-$A$-module homomorphisms.

Proposition 2.5. Let $A$ be a Banach algebra, $\sigma \in \text{Hom} A$ and let $X$, $Y$ and $Z$ be Banach left $A$-modules. Let also that $T \in B(X,Y)$ and $S \in B(Y,Z)$. If one of the operators $T$ or $S$ is left $A$-module homomorphism and other one is (Anti) left $\sigma$-$A$-module homomorphism, then $SOT$ is (Anti) left $\sigma$-$A$-module homomorphism.

**Proof.** Suppose that $S$ is left $A$-module homomorphism and $T$ is left $\sigma$-$A$-module homomorphism. We have

$$SOT(a \cdot x) = S(T(a \cdot x)) = S(\sigma(a) \cdot T(x)) = \sigma(a) \cdot SOT(x) \quad (a \in A, x \in X)$$

Similarly, if $T$ is left $A$-module homomorphism and $S$ left $\sigma$-$A$-module homomorphic, then $SOT$ is left $\sigma$-$A$-module homomorphism.

Proposition 2.6. Let $A$ be a Banach algebra, $\sigma \in \text{Hom} A$ and let $X$, $Y$ and $Z$ be Banach left $A$-modules. Let also that $T \in B(X,Y)$ and $S \in B(Y,Z)$.

(i) If $T$ is left $\sigma$-$A$-module homomorphism and $S$ is Anti left $\sigma$-$A$-module homomorphism then, $SOT$ is left $A$-module homomorphism.

(ii) Let $\sigma(A)$ be dense in $A$. If $T$ is Anti left $\sigma$-$A$-module homomorphism and $S$ is left $\sigma$-$A$-module homomorphism then, $SOT$ is left $A$-module homomorphism.

**Proof.** (i) The proof is same as proposition 5.

(ii) Since $T$ is Anti left $\sigma$-$A$-module homomorphism and $S$ be a left $\sigma$-$A$-module homomorphism so, we have

$$SOT(\sigma(a) \cdot x) = S(a \cdot T(x)) = \sigma(a) \cdot SOT(x) \quad (a \in A)$$

Because $\sigma(A)$ is dense in $A$ and $SOT$ is continuous, thus for each $b \in A$ we have

$$SOT(b \cdot x) = b \cdot SOT(x)$$

Hence, $SOT$ is left $A$-module homomorphism.

Proposition 2.7. Let $A$ be a Banach algebra, $\sigma \in \text{Hom} A$ and let $X$, $Y$ and $Z$ be Banach left $A$-modules. Let also that $T \in B(X,Y)$ and $S \in B(Y,Z)$ be left $\sigma$-$A$-module homomorphisms. then, $SOT$ is left $\sigma^2A$-module homomorphism. In particular, if $\sigma$ be an idempotent (i.e., $\sigma^2 = \sigma \sigma = \sigma$) then, $SOT$ is left $\sigma$-$A$-module homomorphism.

**Proof.** It is clear.

Definition 2.8. Let $A$ be a Banach algebra. $A$ is called $\sigma$-$biflat$ if there exists a bounded Anti $\sigma$-$A$-bimodule homomorphism $\rho : (A \hat{\otimes} A)^* \to A^*$ such that $\rho \circ \pi^* = \sigma^*$.

Theorem 2.9. Let $A$ be a Banach algebra, $\sigma \in \text{Hom} A$. The following phrases are equivalent:

(i) $A$ is $\sigma$-$biflat$.

(ii) There is a bounded $\sigma$-$A$-bimodule homomorphism $\theta : A \to (A \hat{\otimes} A)^{**}$ such that for each $a \in A$, $\pi^{**} \theta(a) = \sigma(a)$. 
\textbf{Proof.} (i) $\Rightarrow$ (ii) Let $A$ be $\sigma$-biflat. Then there exists a bounded Anti $\sigma$-bimodule homomorphism $\rho : (A \hat{\otimes} A)^* \to A^*$ such that $\rho \circ \pi^* = \sigma^*$. Define $\theta : A \to (A \hat{\otimes} A)^*$ by
\[
\theta(a) = \rho^* \circ o \wedge (a) \quad (a \in A)
\]
where $\wedge$ is a canonical mapping. Clearly, $\theta$ is an bounded $\sigma$-A-bimodule homomorphism. We have also
\[
< a^* , \pi^* o \theta(a) > = < \pi^*(a^*) , \theta(a) >
\]
\[
= < \pi^*(a^*) , \rho^*(\tilde{a}) >
\]
\[
= < \rho o \pi^*(a^*) , \tilde{a} >
\]
\[
= < \sigma^*(a^*) , \tilde{a} >
\]
\[
= < a , \sigma^*(a^*) >
\]
\[
= < \sigma(a) , a^* >
\]
\[
= < a^* , \sigma(a) > \quad (a \in A , a^* \in A^*)
\]
Hence, for $a \in A$ we have $\pi^* o \theta(a) = \tilde{\sigma}(a)$.

(ii) $\Rightarrow$ (i) Let there is a bounded $\sigma$-A-bimodule homomorphism $\theta : A \to (A \hat{\otimes} A)^*$ such that for each $a \in A$, $\pi^* o \theta(a) = \tilde{\sigma}(a)$. Define $\rho : (A \hat{\otimes} A)^* \to A^*$ by
\[
\rho(\varphi) = \theta^* \circ o \wedge (\varphi) \quad (\varphi \in (A \hat{\otimes} A)^*)
\]
where $\wedge$ is a canonical mapping. Clearly, $\rho$ is an bounded Anti $\sigma$-A-bimodule homomorphism. We have also
\[
< a , \rho o \pi^*(a^*) > = < a , \theta^*(\pi^*(a^*)) >
\]
\[
= < \theta(a) , \pi^*(a^*) >
\]
\[
= < \pi^*(a^*) , \theta(a) >
\]
\[
= < a^* , \pi^* o \theta(a) >
\]
\[
= < a^* , \sigma(a) >
\]
\[
= < \sigma(a) , a^* >
\]
\[
= < a , \sigma^*(a^*) > \quad (a \in A , a^* \in A^*)
\]
Hence, we have $\rho o \pi^* = \sigma^*$. So $A$ is $\sigma$-biflat. \qed

\textbf{Theorem 2.10.} Every $\sigma$-biprojective Banach algebra, is $\sigma$-biflat.

\textbf{Proof.} Let $A$ be $\sigma$-biprojective Banach algebra. Then there exists a bounded $\sigma$-A-bimodule homomorphism $\rho : A \to A \hat{\otimes} A$ such that $\pi o \rho = \sigma$. Define
\[
\theta := \wedge o \rho : A \to (A \hat{\otimes} A)^*
\]
where $\wedge$ is a canonical mapping. Clearly, $\theta$ is a bounded $\sigma$-A-bimodule homomorphism. We have also
\[
\pi^* o \theta(a) = \pi^* (\rho(\tilde{a})) = \tilde{\pi o \rho}(\tilde{a}) = \tilde{\sigma}(a) \quad (a \in A)
\]
Hence, according to the theorem 9, $A$ is $\sigma$-biflat. \qed
Let $A$ be a Banach algebra that $A^2 = A$ and $K = \ker \pi$. Then

$$\Pi : 0 \to K \xrightarrow{\iota} A \hat{\otimes} A \xrightarrow{\pi} A \to 0$$

is a short exact sequence of Banach $A$-bimodules and $A$-bimodule homomorphisms. So

$$\Pi^* : 0 \to A^* \xrightarrow{\sigma} (A \hat{\otimes} A)^* \xrightarrow{\pi^*} K^* \to 0$$

is a short exact sequence of Banach $A$-bimodules and $A$-bimodule homomorphisms, too.

**Definition 2.11.** The short exact sequence $\Pi$ of Banach $A$-bimodules and $A$-bimodule homomorphisms is called to be $\sigma$-split if there exists a bounded $\sigma$-$A$-bimodule homomorphism $\rho : A \to A \hat{\otimes} A$ such that $\pi \rho = \sigma$.

**Definition 2.12.** The short exact sequence $\Pi^*$ of Banach $A$-bimodules and $A$-bimodule homomorphisms is called to be $\sigma$-split if there exists a bounded Anti $\sigma$-$A$-bimodule homomorphism $\Gamma : (A \hat{\otimes} A)^* \to A^*$ such that $\Gamma \pi^* = \sigma^*$.

Definition 11 is equivalent to that $A$ be $\sigma$-biprojective; i.e., the short exact sequence $\Pi$ of Banach $A$-bimodules and $A$-bimodule homomorphisms is $\sigma$-split if and only if $A$ is $\sigma$-biprojective.

Definition 12 is equivalent to that $A$ be $\sigma$-biflat; i.e., the short exact sequence $\Pi^*$ of Banach $A$-bimodules and $A$-bimodule homomorphisms is $\sigma$-split if and only if $A$ is $\sigma$-biflat.

**Theorem 2.13.** Let $A$ be a unital Banach algebra with unit $e$ and $\sigma \in \text{Hom} A$. Then, the short exact sequence $\Pi$ of Banach left $A$-modules and left $A$-module homomorphisms be $\sigma$-split.

**Proof.** Define $\rho : A \to A \hat{\otimes} A$ by $\rho(a) = \sigma(a) \otimes e$ for all $a \in A$. Clearly, $\rho$ is linear and bounded. We have also

$$\rho(ab) = \sigma(ab) \otimes e = (\sigma(a) \sigma(b)) \otimes e = \sigma(a) \cdot (\sigma(b) \otimes e) = \sigma(a) \cdot \rho(b) \quad (a, b \in A)$$

So, $\rho$ is a bounded left $\sigma$-$A$-module homomorphism. In addition

$$\pi \rho(a) = \pi(\sigma(a) \otimes e) = \sigma(a)e = \sigma(a) \quad (a \in A)$$

Hence, $\pi \rho = \sigma$. Thus, the short exact sequence $\Pi$ of Banach left $A$-modules and left $A$-module homomorphisms, be $\sigma$-split. \qed

**Theorem 2.14.** Let $A$ be a unital Banach algebra with unit $e$ and $\sigma \in \text{Hom} A$. Then, the short exact sequence $\Pi$ of Banach right $A$-modules and right $A$-module homomorphisms be $\sigma$-split.

**Proof.** It is enough to define $\rho : A \to A \hat{\otimes} A$ by $\rho(a) = e \otimes \sigma(a)$ for all $a \in A$. The rest of the proof is similar to the proof of the theorem 13. \qed

Let $A$ be a Banach algebra, $\sigma \in \text{Hom} A$ and $\sigma(A)$ be dense in $A$. Recall that $A$ is $\sigma$-contractible if and only if it is unital and $\sigma$-biprojective [12]. So, we have the following result.

**Corollary 2.15.** Let $A$ be a unital Banach algebra with unit $e$, $\sigma \in \text{Hom} A$ and $\sigma(A)$ be dense in $A$. Then, the short exact sequence $\Pi$ of Banach $A$-bimodules and $A$-bimodule homomorphisms be $\sigma$-split if and only if $A$ is $\sigma$-contractible.
Theorem 2.16. Let $A$ be a Banach algebra and $\sigma$ be an idempotent epimorphism of $A$. Then, the following phrases are equivalent:

(i) $A$ is $\sigma$-amenable.

(ii) $A$ has a bounded approximate identity and the short exact sequence $\Pi^*$ of Banach $A$-bimodules and $A$-bimodule homomorphisms be $\sigma$-split.

Proof. (i) $\Rightarrow$ (ii) Let $A$ is $\sigma$-amenable. So $A$ has a bounded approximate identity and has a $\sigma$-virtual diagonal such as $M \in (A\hat{\otimes}A)^{**}$. Define $\Gamma : (A\hat{\otimes}A)^* \rightarrow A^*$ by

$$<a, \Gamma(\varphi)> = <\sigma(a) \cdot \varphi, M>, \quad (a \in A, \varphi \in (A\hat{\otimes}A)^*)$$

Clearly, $\Gamma(\varphi)$ is linear and bounded so $\Gamma$ is linear and bounded too. Also

$$<b, \Gamma(\sigma(a) \cdot \varphi)> = <\sigma(b) \cdot (\sigma(a) \cdot \varphi), M> = <\sigma(b) \cdot \sigma(a) \cdot \varphi, M> = <b \cdot a, \Gamma(\varphi)> = <b, a \cdot \Gamma(\varphi)>, \quad (b \in A)$$

Thus, $\Gamma(\sigma(a) \cdot \varphi) = a \cdot \Gamma(\varphi)$ for all $a \in A$. In addition

$$<b, \Gamma(\varphi \cdot \sigma(a))> = <\sigma(b) \cdot (\varphi \cdot \sigma(a)), M> = <\sigma(b) \cdot \varphi \cdot \sigma(a), M> = <\sigma(b) \cdot \varphi, \sigma(a) \cdot M> = <\sigma(b) \cdot \varphi, M \cdot \sigma(a)> = <\sigma(a) \cdot (\sigma(b) \cdot \varphi), M> = <\sigma(a) \cdot b \cdot \varphi, M> = <a \cdot b, \Gamma(\varphi)> = <b, \Gamma(\varphi) \cdot a>, \quad (b \in A)$$

Thus, $\Gamma(\varphi \cdot \sigma(a)) = \Gamma(\varphi) \cdot a$ for all $a \in A$. So $\Gamma$ is bounded Anti $\sigma$-$A$-bimodule homomorphism. For each $a \in A$, we have

$$<a, \Gamma_{o\pi^*}(a^*)> = <\sigma(a) \cdot \pi^*(a^*), M> = \pi^*(\sigma(a) \cdot a^*), M> = <\sigma(a) \cdot a^*, \pi^*(M)> = <a^*, \pi^*(M) \cdot \sigma(a)> = <a^*, \sigma(a)> = <\sigma(a), a^*>$$

Hence, $\Gamma_{o\pi^*} = \sigma^*$. So, the short exact sequence $\Pi^*$ of Banach $A$-bimodules and $A$-bimodule homomorphisms be $\sigma$-split.

(ii) $\Rightarrow$ (i) Let $A$ has a bounded approximate identity such as $\{e_\alpha\}$ and there exists a bounded Anti $\sigma$-$A$-bimodule homomorphism $\Gamma : (A\hat{\otimes}A)^* \rightarrow A^*$ such that $\Gamma_{o\pi^*} = \sigma^*$. Since $\{e_\alpha \hat{\otimes} e_\alpha\}$ is a bounded net in $(A\hat{\otimes}A)^{**}$, thus, by Banach Alaoglu’s theorem there exists a subnet $\{e_\beta \hat{\otimes} e_\beta\}$ of $\{e_\alpha \hat{\otimes} e_\alpha\}$ and $M_0 \in (A\hat{\otimes}A)^{**}$ such that

$$e_\beta \hat{\otimes} e_\beta \xrightarrow{w^*} M_0$$
Set $M := \Gamma^*o\pi^{**}(M_0)$. Clearly, $M \in (A \hat{\otimes} A)^{**}$ and
\[
\sigma(a) \cdot M = \sigma(a) \cdot \Gamma^*\left(\pi^{**}(M_0)\right) = \Gamma^*\left(a \cdot \pi^{**}(M_0)\right) = \Gamma^*\sigma\pi^{**}(a \cdot M_0)
\]
So, we have
\[
< \varphi \cdot \sigma(a) \cdot M > = < \varphi \cdot \Gamma^*o\pi^{**}(a \cdot M_0) > \\
= < \pi^*o\Gamma(\varphi) \cdot a \cdot M_0 > \\
= < \pi^*o\Gamma(\varphi) \cdot a > \\
= \lim_{\beta} < \pi^*o\Gamma(\varphi) \cdot a, e_\beta \otimes e_\beta > \\
= \lim_{\beta} < e_\beta \otimes e_\beta, \pi^*o\Gamma(\varphi) \cdot a > \\
= \lim_{\beta} < ae_\beta^2, \Gamma(\varphi) > \\
= < a, \Gamma(\varphi) > \\
= \lim_{\beta} < e_\beta^2a, \Gamma(\varphi) > \\
= < \varphi \cdot M \cdot \sigma(a) > \quad (\varphi \in (A \hat{\otimes} A)^*)
\]
Hence, $\sigma(a) \cdot M = M \cdot \sigma(a)$ for all $a \in A$. Also
\[
< a^* \cdot \pi^{**}(M) \cdot \sigma(a) > = < \sigma(a) \cdot a^* , \pi^{**}(M) > \\
= < \pi^*(\sigma(a) \cdot a^* ) , M > \\
= < \pi^*(\sigma(a) \cdot a^* ) , \Gamma^*\sigma\pi^{**}(M_0) > \\
= < \pi^*o\Gamma\pi^*(\sigma(a) \cdot a^* ) , M_0 > \\
= \lim_{\beta} < e_\beta^2 \otimes e_\beta, \pi^*o\Gamma\pi^*(\sigma(a) \cdot a^* ) > \\
= \lim_{\beta} < e_\beta^2, \Gamma\pi^*(\sigma(a) \cdot a^* ) > \\
= \lim_{\beta} < e_\beta^2, \Gamma(\sigma(a) \cdot \pi^*(a^*)) > \\
= \lim_{\beta} < e_\beta^2, a \cdot \Gamma\pi^*(a^* ) > \\
= \lim_{\beta} < e_\beta^2, a \cdot \Gamma\pi^*(a^* ) > \\
= < a, \Gamma\pi^*(a^* ) > \\
= < \Gamma\pi^*(a^* ) \cdot \widehat{a} > \\
= < \sigma^*(a^* ) \cdot \widehat{a} > \\
= < a^* , \sigma^*(\widehat{a}) > \\
= < a^* , \widehat{\sigma(a)} > \quad (a^* \in A^*)
\]
Hence, $\pi^{**}(M) \cdot \sigma(a) = \widehat{\sigma(a)}$ for all $a \in A$. Thus, $M$ is a $\sigma$–virtual diagonal for $A$ and so $A$ is $\sigma$–amenable. \hfill \Box

**Corollary 2.17.** Let $A$ be a Banach algebra, and $\sigma$ be an idempotent epimorphism of $A$. Then, $A$ is $\sigma$–amenable if and only if $A$ has a bounded approximate identity and it is $\sigma$–biflat.

**References**


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On subspace hypercyclicity criterion

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Abstract
In this paper we prove an equivalent conditions to subspace- hypercyclicity criterion. The results improve earlier work.

Keywords: Hypercyclicity Criterion, hereditarily subspace-transitive, subspace-hypercyclicity criterion.

Mathematics Subject Classification: Primary: 47A16; Secondary: 47B37, 37B99.

1 Introduction
Let $X$ be a Banach space. An operator $T$ on $X$ is hypercyclic if there exists a vector $x \in X$ whose orbit under $T$, $\text{orb}(T, x) = \{x, Tx, T^2x, \ldots\}$, is dense in $X$. Such a vector $x$ is called hypercyclic vector for $T$.

A nice criterion, namely the Hypercyclicity Criterion, was developed independently by Kitai[7], Gethner and Shapiro [3]. This criterion has been used to show that hypercyclic operators arise within the special classes of operators, like composition operators [2] and weighted shifts [9].

The formulation of the Hypercyclicity Criterion in the following theorem, was given by[6].

**Theorem 1.1.** (The Hypercyclicity Criterion) Let $T : X \to X$ be an operator. If there are dense subsets $Y$ and $Z$ in $X$ and an increasing sequence sequence $\{n_k\}$ of positive integers such that:

(i) for each $y \in Y$, $T^{n_k}y \to 0$,

(ii) for each $z \in Z$, there is a sequence $\{x_k\}$ in $X$ such that

$$x_k \to 0 \quad \text{and} \quad T^{n_k}x_k \to z$$

Then $T$ is hypercyclic.

Recently, B. F. Madore and R. A. Martínez-Avendaño in [8] introduced the concept of subspace- hypercyclicity for an operator.

Let $X$ be a Banach space and $M$ a non-zero subspace of $X$ and $T$ be a bounded linear operator on $X$. We say that $T$ is subspace-hypercyclic for $M$ if there exists $x \in X$ such that $\text{orb}(T, x) \cap M$ is dense in $M$. Such a vector $x$ is called a subspace-hypercyclic vector for $T$.

Madore and R. A. Martínez-Avendaño state subspace- hypercyclicity criterion as follows:

**Theorem 1.2.** ([8]) Let $T \in B(X)$ and $M$ be a non-zero subspace of $X$. Assume there exist $Y$ and $Z$, dense subsets of $M$ and an increasing sequence of positive integers $\{n_k\}$ such that:

(i) $T^{n_k}y \to 0$ for all $y \in Y$,

(ii) for each $z \in Z$, there exists a sequence $\{x_k\}$ in $M$ such that

$$x_k \to 0 \quad \text{and} \quad T^{n_k}x_k \to z$$
(iii) $M$ is an invariant subspace for $T^{n_k}$ for all $k \in N$. Then $T$ is subspace transitive with respect to $M$ and hence $T$ is subspace-hypercyclic for $M$.

In this paper, we define hereditarily subspace-transitive and hereditarily subspace-hypercyclic operators and prove an equivalent conditions to subspace-hypercyclicity criterion. The results improve earlier work.

In this paper, $N$ will refer to positive numbers and $\{n_k\} \subseteq N$ will always refer to an increasing sequence of positive integers. Whenever we talk about a subspace $M$ of $X$, we will assume that $M$ is non-zero, infinite dimensional and topologically closed.

**Definition 1.3.** Let $T \in B(X)$, $M$ be a subspace of $X$ and $\{n_k\}_{k=1}^{\infty}$ be a sequence. We say $\{T^{n_k}\}_{k=1}^{\infty}$ is subspace-hypercyclic with respect to $M$ or $M$-hypercyclic, if there exists $x \in X$ such that $\{(T^{n_k}(x)) \cap M : k \in N\}$ is dense in $M$. In this case we say $x$ is a subspace-hypercyclic vector for $\{T^{n_k}\}$.

**Definition 1.4.** Let $T \in B(X)$, $M$ be a subspace of $X$ and $\{n_k\}_{k=1}^{\infty}$ be a sequence. We say $\{T^{n_k}\}_{k=1}^{\infty}$ is subspace-transitive with respect to $M$ or $M$-transitive, if for all non-empty sets $U \subseteq M$, $V \subseteq M$ both relatively open, there exists $k \in N$ such that $T^{-n_k}(U) \cap V$ contains a relatively open non-empty subset of $M$.

**Theorem 1.5.** Let $T \in B(X)$ and $M$ be a subspace of $X$ and $\{n_k\}_{k=1}^{\infty}$ be a sequence. Then the following conditions are equivalent:

(i) $\{T^{n_k}\}_{k=1}^{\infty}$ is subspace-transitive with respect to $M$.

(ii) for all non-empty sets $U \subseteq M$, $V \subseteq M$ both relatively open, there exists $k \in N$ such that $T^{-n_k}(U) \cap V$ is a relatively open non-empty subset of $M$.

(iii) for all non-empty sets $U \subseteq M$, $V \subseteq M$ both relatively open, there exists $k \in N$ such that $T^{-n_k}(U) \cap V$ is non-empty and $T^{n_k}(M) \subseteq M$.

2 Main Result

**Definition 2.1.** Let $T \in B(X)$ and $M$ be a subspace of $X$ and $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of non-negative integers. We say $T$ is hereditarily $M$-transitive with respect to $\{n_k\}$, if for every subsequence $\{n_{k_j}\}$ of $\{n_k\}$, $\{T^{n_{k_j}}\}$ is subspace-transitive with respect to $M$ or $M$-transitive.

**Theorem 2.2.** Let $T \in B(X)$, $M$ be a closed non-zero subspace of $X$ and $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of non-negative integers such that $T^{n_k}(M) \subseteq M$ for every $k$. Then the following conditions are equivalent:

(i) $T$ is hereditarily $M$-transitive with respect to $\{n_k\}$.

(ii) for every non-empty sets $U \subseteq M$, $V \subseteq M$ both relatively open, there exists a positive integer $N$ such that $T^{n_k}(U) \cap V$ is non-empty for all $k > N$.

**Definition 2.3.** Let $T \in B(X)$ and $M$ be a subspace of $X$ and $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of non-negative integers. We say $T$ is hereditarily $M$-hypercyclic with respect to $\{n_k\}$, if for every subsequence $\{n_{k_j}\}$ of $\{n_k\}$, $\{T^{n_{k_j}}\}$ is subspace-hypercyclic with respect to $M$ or $M$-hypercyclic.

**Theorem 2.4.** Let $T \in B(X)$ and $M$ be a subspace of $X$ and $\{n_k\}_{k=1}^{\infty}$ be a sequence such that $T^{n_k}(M) \subseteq M$. Then conditions (A) and (B) are equivalent:
(A) There exist $X_0$ and $Y_0$ dense subsets of $M$, and mappings $S_{n_k}: Y_0 \to M$ such that:

(i) $T^{n_k} \to 0$ pointwise on $X_0$.

(ii) $S_{n_k} \to 0$ pointwise on $Y_0$.

(iii) $T^{n_k}S_{n_k} \to \text{Id}_{Y_0}$ pointwise on $Y_0$.

(B) (subspace-hypercyclicity criterion) There exist dense subsets $X_0$ and $Y_0$ of $M$ such that:

(iv) $T^{n_k} \to 0$ pointwise on $X_0$.

(v) for every $y \in Y_0$ there exists sequence $\{u_k\}$ in $M$ such that $u_k \to 0$ and $T^{n_k}u_k \to y$ pointwise on $Y_0$.

(vi) $T^{n_k}(M) \subseteq M$ for all $k$.

**Corollary 2.5.** Let $T \in B(X)$ and $M$ be a subspace of $X$ and $\{n_k\}_{k=1}^{\infty}$ be a sequence such that $T^{n_k}(M) \subseteq M$. Then if there exist $X_0$ and $Y_0$ dense subsets of $M$, and mappings $S_{n_k}: Y_0 \to M$ such that:

(i) $T^{n_k} \to 0$ pointwise on $X_0$.

(ii) $S_{n_k} \to 0$ pointwise on $Y_0$.

(iii) $T^{n_k}S_{n_k} \to \text{Id}_{Y_0}$ pointwise on $Y_0$.

then $T$ has a a dense subset of $M$-hypercyclic vectors in $M$.

**Theorem 2.6.** Let $T \in B(X)$, $M$ be a non-zero subspace of $X$ and $\{n_k\}_{k=1}^{\infty}$ be a sequence such that $T^{n_k}(M) \subseteq M$ for all $k$. If for each $U, V \subseteq M$ relatively open subsets of $M$ and for each $W$ neighborhood of zero in $M$, there exists a positive integer $N$, such that $T^{n_k}(U) \cap W \neq \emptyset$ and $T^{n_k}(W) \cap V \neq \emptyset$ for $k > N$ then $T$ is hereditarily M-transitive with respect to a subsequence of $\{n_k\}$.

**Theorem 2.7.** Let $T \in B(X)$ and $M$ be a non-zero subspace of $X$ and $\{n_k\}_{k=1}^{\infty}$ be a sequence such that $T^{n_k}(M) \subseteq M$. Then if $T$ satisfies $M$-hypercyclicity criterion with respect to a subsequence of $\{n_k\}$, then $T$ is hereditarily M-transitive with respect to a subsequence of $\{n_k\}$.

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Fixed point theorem in partially ordered metric spaces
and its application

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Abstract
In this paper, we present two fixed point theorems for weakly and generalized
weakly contractive mappings in the context of partially ordered complete metric
spaces. As an application, we discuss the existence of solution for some polynomial.
Our results extend some results in the literature.

Keywords: Fixed point, ordered set, weak $\phi$-contraction

Mathematics Subject Classification: 47H10, 54H25, 46J10

1 Introduction
A self-mapping $f$ on a metric space $(X, d)$ is said to be weak-$\phi$-contractive [2] if there exists a map
$\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(f(x), f(y)) \leq d(x, y) - \varphi(d(x, y)), \quad (1)$$

for all $x, y \in X$.

Later, Zhang and Song [6] introduced the notion of generalized weak-$\varphi$-contraction which is a
natural extension of the weak-$\varphi$-contraction. A self-mapping $f$ on a metric space $(X, d)$ is said to
be generalized weak-$\varphi$-contractive if there exists a map $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$ and
$\varphi(t) > 0$ for all $t > 0$ such that

$$d(f(x), f(y)) \leq N(x, y) - \varphi(N(x, y)), \quad (2)$$

for all $x, y \in X$, where,

$$N(x, y) = \max\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2}\}. \quad (3)$$

For more details on weak $\varphi$-contraction, we refer to e.g. [4] and [5].
At first we introduced some classes of functions.
Let $\Psi$ denotes the all functions $\psi : [0, +\infty) \to [0, 1)$, such that

$$\psi(t_n) \to 1 \quad implies \quad t_n \to 0. \quad (4)$$

Also let $\Phi$ denotes the all functions $\varphi : [0, +\infty) \to [0, +\infty)$, such that

$$\psi(t_n) \to 0 \quad implies \quad t_n \to 0. \quad (5)$$

It is clear that if $\varphi \in \Phi$.
At last let $\mathcal{A}$ stand for the class of those functions $\phi : [0, +\infty) \to [0, +\infty)$ which satisfies the
following conditions:
(i) \( \phi \) is nondecreasing,

(ii) \( \phi(x) < x \), for each \( x > 0 \),

(iii) \( \beta(x) = \frac{\phi(x)}{n} \in \Psi \).

Recently, Amini-Harani and Emami [1] proved a fixed point theorem in ordered metric spaces which extended the Harjani and Sadarangani theorem [3]. In this paper we extend this results.

2 Main Result

We present our main results in this section.

**Theorem 2.1.** Let \((X, \triangleright, \leq)\) be a partially ordered set and suppose that there exists a metric \(d\) in \(X\) such that \((X, d)\) is a complete metric space. Let \(f : X \to X\) be a nondecreasing mapping such that there exists an element \(x_0 \in X\) with \(x_0 \triangleright f(x_0)\). Suppose that there exists \(\varphi \in \Phi\) such that

\[
d(f(x), f(y)) \leq d(x, y) - \varphi(d(x, y)),
\]

for each \(x, y \in X\) with \(x \leq y\)(i.e., weak-\(\varphi\)-contractive). Suppose also that either

(a) \(f\) is continuous, or,

(b) for every nondecreasing sequence \(\{x_n\}\) if \(x_n \to x\) then \(x_n \triangleright x\) for all \(n \in \mathbb{N}\).

Then \(f\) has a fixed point. Moreover, if for each \(x, y \in F(f)\) there exists \(z \in X\) which is comparable to \(x\) and \(y\), then the fixed point of \(f\) is unique.

**Proof.** Since \(x_0 \leq f(x_0)\) and \(f\) is nondecreasing, we obtain that

\[x_0 \leq f(x_0) \leq f^2(x_0) \leq \cdots \leq f^n(x_0) \leq \cdots.\]  

(7)

Put \(x_n = f^n(x_0), n = 1, 2, 3, \cdots\). Then, (7) turns into

\[x_n \leq x_{n+1}\]  

for all \(n \in \mathbb{N}\). Thus, (6) yields

\[d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) - \varphi(d(x_n, x_{n-1})).\]  

Using this inequality we can prove that \(\lim_{n \to \infty} \varphi(d(x_n, x_{n-1})) = 1\). Then we claim that the iterative sequence \(\{x_n\}\) is Cauchy. So there exists \(x \in X\) such that \(\lim_{n \to \infty} x_n = x\).

If (a) hold, that is, if \(f\) is continuous, then

\[x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} f(x_{n-1}) = f(x).\]  

(10)

Suppose that (b) holds. Then, we get

\[d(x_{n+1}, f(x)) = d(f(x_n), f(x)) \leq d(x_n, x) - \varphi(d(x_n, x)) \leq d(x_n, x)\]  

by using (6). Letting \(n \to \infty\) in above inequality, we conclude that \(d(x, f(x)) = 0\). Hence \(x = f(x)\).

To finalize the proof, we show that this fixed point \(x\) is unique. Let \(y\) be an arbitrary fixed point of \(f\). Hence there exists \(z \in X\) such that \(z \leq x\) and \(y\). So \(f^n(z)\) is comparable to \(f^n(x) = x\) and \(f^n(y) = y\). Therefore

\[d(x, f^n(z)) = d(f^n(x), f^n(z)) \leq d(x, f^{n-1}(z)) - \varphi(d(x, f^{n-1}(z))).\]  

So \(\{d(x, f^n(z))\}\) is decreasing. Hence from (12), we conclude that \(\lim_{n \to \infty} d(x, f^n(z)) = 0\). Analogously, \(\lim_{n \to \infty} d(y, f^n(z)) = 0\). Therefore \(x = y\) and this completes the proof. \(\square\)
The proof of the following theorem is similar to the proof of above theorem.

**Theorem 2.2.** Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric $d$ in $X$ such that $(X, d)$ is a complete metric space. Let $f : X \rightarrow X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \leq f(x_0)$. Suppose that there exists $\varphi \in \Phi$ such that

$$d(f(x), f(y)) \leq N(x, y) - \varphi(N(x, y)), \quad (13)$$

for each $x, y \in X$ with $x \leq y$ (i.e., generalized weak-$\varphi$-contractive).

Suppose also that either

(a) $f$ is continuous, or,

(b) for every nondecreasing sequence $\{x_n\}$ if $x_n \rightarrow x$ then $x_n \leq x$ for all $n \in \mathbb{N}$.

Then $f$ has a fixed point. Moreover, if every $x, y \in F(f)$ are comparable, then the fixed point of $f$ is unique.

3 Application

In this section, we prove the existence of solution of some polynomials.

**Theorem 3.1.** Let $a_0, a_1, \ldots, a_{k-1} \in [0, +\infty)$, such that $a_1 + a_2 + \ldots + a_{k-1} < 1$ and $a_0 \geq 1$. Then the equation

$$y^k = a_{k-1}y^{k-1} + a_{k-2}y^{k-2} + \ldots + a_1y + a_0 \quad (14)$$

has a unique solution on $[\sqrt[k]{a_0}, +\infty)$.

**Proof.** Suppose that $f : [a_0, +\infty) \rightarrow [a_0, +\infty)$ is defined by $f(x) = a_{k-1}\sqrt[k]{x^{k-1}} + a_{k-2}\sqrt[k]{x^{k-2}} + \ldots + a_1\sqrt[k]{x} + a_0$. If $x \leq y$ then $f(x) \leq f(y)$. So $f$ is nondecreasing. Also for $x, y \in [a_0, +\infty)$ with $x \leq y$, we derive that

$$0 \leq f(y) - f(x) = a_{k-1}(\sqrt[k]{y^{k-1}} - \sqrt[k]{x^{k-1}}) + a_{k-2}(\sqrt[k]{y^{k-2}} - \sqrt[k]{x^{k-2}}) + \ldots + a_1(\sqrt[k]{y} - \sqrt[k]{x}). \quad (15)$$

Suppose that $g_i : [1, +\infty) \rightarrow \mathbb{R}$ is defined by $g_i(t) = t - t^{1-\frac{i}{k}}$, for $i = 1, 2, \ldots, k - 1$. Since $g_i(t) = 1 - (1 - \frac{1}{k})t^{\frac{i}{k}} \geq 0$, then $g_i$ is monotone nondecreasing. Hence, if $1 \leq x < y$, then $g_i(x) < g_i(y)$. So, $y^{1-\frac{i}{k}} - x^{1-\frac{i}{k}} < y - x$. Therefore from (15), we get

$$0 \leq f(y) - f(x) \leq (y - x) - \varphi(y - x), \quad (16)$$

where $\varphi(t) = [1 - (a_{k-1} + a_{k-2} + \ldots + a_1)t]$. Also $a_0 \leq f(a_0)$. Thus, using Theorem 2.1, the mapping $f$ has a unique fixed point $x \in [a_0, +\infty)$. Moreover, the sequence $\{f^n(a_0)\}$ converges to this fixed point.

Otherwise, there exists a unique $y \in [\sqrt[k]{a_0}, +\infty)$ such that $y^k = x$. So, from $x = f(x)$, we have $y^k = f(y^k)$ and therefore we find

$$y^k = a_{k-1}y^{k-1} + a_{k-2}y^{k-2} + \ldots + a_1y + a_0. \quad (17)$$

Also the sequence $\{\sqrt[k]{f^n(a_0)}\}$ converges to $y$ and this completes the proof. □
References


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Some fixed point results in cone Banach type spaces

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Abstract
In this paper, we give some generalized theorems on points of coincidence and common fixed point for two weakly compatible mappings on a cone Banach type space.

Keywords: cone normed type spaces, common fixed point, weakly compatible mappings

Mathematics Subject Classification: 47H10, 54H25, 55M20.

1 Introduction
In 1980, Rzepecki provided a generalization of metric spaces. He defined metric $d_E$ on a set $X$ by $d_E : X \times X \rightarrow S$, where $E$ is Banach space and $S$ is a normal cone in $E$ with partial order $\preceq$, and he generalized the fixed point theorems of Maia type. In 1987, Lin [8] introduced the notion of K-metric spaces and consider some results of Khan and Imdad [6] in K-metric spaces. In 2007, Huang and Zhang [7] introduced cone metric spaces and defined some properties of convergence of sequences and completeness in cone metric spaces, also they proved a fixed point theorem of cone metric spaces. Beginning around the year 2007, the fixed point theorems in cone metric spaces have been extensively proved by a number of authors and there are many interesting results concerning these theorems (see [1], [2], [4], and [10]).

In this paper, we introduced the notion of cone Banach type spaces and we proved the generalization of some known results on points of coincidence and the generalization of some common fixed point theorems for two compatible mappings in cone Banach type spaces.

2 preliminaries
Let $E$ be a real Banach space with norm $\|\cdot\|$ and $P$ be a subset of $E$. $P$ is called a cone if and only if the following conditions are satisfied: (P1) $P$ is closed, nonempty and (P1) $a,b \geq 0$ and $x,y \in P \Rightarrow ax+by \in P$ (P3) $x \in P$ and $-x \in P \Rightarrow x = 0$. Let $P \subset E$ be a cone, we define a partial ordering $\preceq$ on $E$ with respect to $P$ by $x \preceq y$ if and only if $y-x \in P$. We write $x < y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y-x \in \text{int}P(\text{interior of } P)$. The cone $P \subset E$ is called normal if there is a positive real number $K$ such that for all $x,y \in E$, $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$.

The least positive number satisfying the above inequality is called the normal constant of $P$. It is clear that $k \geq 1$. Rezapour and Hambarani proved that existence of an ordered Banach space $E$ with cone $P$ which is not normal but with $\text{int}P \neq \emptyset$.

Throughout this paper, we assume that $E$ is a real Banach space and $P$ is a cone such that $\text{int}P \neq \emptyset$.

Let $X$ be a nonempty set. A function $d : X \times X \rightarrow E$ is said to be a cone b-metric function on $X$ with the constant $K \geq 1$ if the following conditions are satisfied: (1) $0 \leq d(x,y)$, for
all \(x, y \in X\), and \(d(x, y) = 0\) if and only if \(x = y\), (2) \(d(x, y) = d(y, x)\) for all \(x, y \in X\), (3) \(d(x, z) \leq K \left( d(x, y) + d(y, z) \right)\) for all \(x, y, z \in X\). The pair \((X, d)\) is called the cone b-metric space (or cone metric type space (in brief CMTS)) [see [10]].

Let \(X\) be a vector space over \(\mathbb{R}\). Suppose the mapping \(||\cdot||_p : X \to E\) satisfies: (i) \(||x||_p > 0\) for all \(x \in X\), (ii) \(||x||_p = 0\) if and only if \(x = 0\), (iii) \(||x + y||_p \leq ||x||_p + ||y||_p\) for all \(x, y \in X\), (iv) \(||kx||_p = |k||x||_p\) for all \(k \in \mathbb{R}\), then \(||\cdot||_p\) is called cone norm type on \(X\), and the pair \((X, ||\cdot||_p)\) is called a cone normed space (in brief CNS). Note that each CNS is cone metric space (in brief CMS). Indeed, \(d(x, y) = ||x - y||_p\).

Similar to the definition of CMTS, we give the following definition:

Let \(X\) be a vector space over \(\mathbb{R}\). Suppose the mapping \(||\cdot||_p : X \to E\) satisfies: (i) \(||x||_p > 0\) for all \(x \in X\), (ii) \(||x||_p = 0\) if and only if \(x = 0\), (iii) \(||x + y||_p \leq K \left( ||x||_p + ||y||_p \right)\) for all \(x, y \in X\) and for constant \(K \geq 1\), (iv) \(||rx||_p = |r||x||_p\) for all \(r \in \mathbb{R}\), then \(||\cdot||_p\) is called cone normed type space (in brief CNTS). Note that each CNTS is CMTS. Indeed, \(d(x, y) = ||x - y||_p\). For example suppose that \(C_\theta(X) = \{ f : X \to \mathbb{C} : \sup_{x \in X}|f(x)| < \infty \}\). Define \(||\cdot||_p : C_\theta(X) \to \mathbb{R}\) by \(||f||_p = \sqrt[p]{\sup_{x \in X}|f(x)|^p}\), then \(||\cdot||_p\) satisfies the following properties: (i) \(||f||_p > 0\) for all \(x \in X\), (ii) \(||f||_p = 0\) if and only if \(f = 0\), (iii) \(||f + g||_p \leq \sqrt[p]{3} \left( ||f||_p + ||g||_p \right)\) for all \(x, y \in X\), (iv) \(||rf||_p = |r||f||_p\) for all \(r \in \mathbb{R}\).

Let \((X, ||\cdot||_p, K)\) be a CNTS, let \(\{x_n\}\) be a sequence in \(X\) and \(x \in X\). Then (i) \(\{x_n\}\) converges to \(x\) whenever for every \(c \in E\) with \(0 < c\) there is a natural number \(N\) such that \(||x_n - x||_p < c\) for all \(n > N\). It is denoted by \(\lim_{n \to \infty} x_n = x\) or \(x_n \to x\); (ii) \(x_n\) is a Cauchy sequence whenever for every \(c \in E\) with \(0 < c\) there is a natural number \(N\) such that \(||x_n - x_m||_p < c\) for all \(n, m > N\); (iii) \((X, ||\cdot||_p, K)\) is a complete cone normed type space if every Cauchy sequence is convergent. Complete cone normed type spaces will be called cone Banach type spaces.

**Lemma 2.1.** Let \((X, ||\cdot||_p, K)\) be a CNTS, \(P\) a normal cone with normal constant \(M\), and \(\{x_n\}\) a sequence in \(X\). Then,

(i) the sequence \(\{x_n\}\) converges to \(x\) if and only if \(||x_n - x||_p \to 0\), as \(n \to \infty\);

(ii) the sequence \(\{x_n\}\) is Cauchy if and only if \(||x_n - x_m||_p \to 0\) as \(n, m \to \infty\);

(iii) the sequence \(\{x_n\}\) converges to \(x\) and the sequence \(\{y_n\}\) converges to \(y\) then \(||x_n - y_n||_p \to ||x - y||_p\).

**Lemma 2.2.** Let \(\{y_n\}\) be a sequence in a cone Banach type space \((X, ||\cdot||_p, K)\) such that

\[d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n)\]

for some \(0 < \lambda < 1/K\), and all \(n \in \mathbb{N}\), where \(d(x, y) = ||x - y||_p\). Then \(\{y_n\}\) is a Cauchy sequence in \((X, ||\cdot||_p, K)\).

Let \(S, T\) be self-mappings on a cone b-metric space \((X, d)\). A point \(z \in X\) is called a coincidence point of \(S, T\) if \(Sz = Tz\), and it is called a common fixed point of \(S, T\) if \(Sz = z = Tz\). Moreover, a pair of self-mappings \((S, T)\) is called weakly compatible on \(X\) if they commute at their coincidence points, in other words

\[z \in X, \quad Sz = Tz \implies STz = TSz.\]

**Theorem 2.3.** Let \(C\) be a subset of a cone Banach type space \((X, ||\cdot||_p, K)\) and \(d : X \times X \to E\) be such that \(d(x, y) = ||x - y||_p\). Suppose that \(F, T : C \to C\) are two mappings such that, \(TC \subset FC\) and \(FC\) is closed and convex. If there exists some constant \(1 - \frac{1}{K} < 1\) such that

\[d(Fy, Ty) + rd(Fx, Fy) \leq d(Fx, Tx) \tag{1}\]

for all \(x, y \in C\). Then \(F\) and \(T\) have at least one point of coincidence. Moreover if, \(F\) and \(T\) are weakly compatible then \(F\) and \(T\) have a common unique fixed point.
Theorem 2.4. Let C be a subset of a cone Banach type space \((X, \| \cdot \|_p, K)\) such that \(1 < K \leq 2\). Let \(d : X \times X \rightarrow E\) be a mapping such that \(d(x, y) = \|x - y\|_p\). Suppose that \(F, T : C \rightarrow C\) are two mappings such that \(TC \subset FC\) and \(FC\) is closed and convex. If there exists some constant \(1 - \frac{1}{K} < \epsilon < 1\) such that
\[
d(Tx, Ty) + (1 - \frac{1}{K})d(Fy, Ty) + rd(Fx, Fy) \leq \frac{1}{2}d(Fx, Tx)
\]
for all \(x, y \in C\). Then \(F\) and \(T\) have at least one point of coincidence. Moreover if, \(F\) and \(T\) are weakly compatible then \(F\) and \(T\) have a unique common fixed point.

Theorem 2.5. Let \(C\) be a subset of a cone Banach space \((X, \| \cdot \|_p, K)\) such that \(d(x, y) = \|x - y\|_p\). If there exist \(a, b, s\) and \(F, T : C \rightarrow C\) be two mappings such that, \(TC \subset FC\) and \(FC\) is closed and convex which satisfies the condition
\[
0 < s + |a|K^{\frac{1}{2} + sgn(a)} - 2b < 2(aK^{-sgn(a)} + b),
\]
\[
ad(Tx, Ty) + b \left(d(Fx, Tx) + d(Fy, Ty)\right) \leq sd(Fx, Fy)
\]
for all \(x, y \in C\). Then \(F\) and \(T\) have at least one point of coincidence. Moreover if, \(a > s\) and \(F\) and \(T\) are weakly compatible then \(F\) and \(T\) have a unique common fixed point.

Theorem 2.6. Let \(C\) be a subset of a cone Banach type space \((X, \| \cdot \|_p, K)\) where \(d(x, y) = \|x - y\|_p\), and \(F, T : C \rightarrow C\) be two mappings such that \(TC \subset FC\) and \(FC\) is closed and convex which satisfies the condition
\[
ad(Fy, Ty) + d(Fx, Tx) \leq bd(Fx, Tx) + \frac{1}{K}d(Fx, Fy)
\]
for all \(x, y \in C\), where \(1 < b < 1 + \frac{(2a-1)K-1}{2K^2}, a > K+1\). Then \(F\) and \(T\) have a unique point of coincidence. Moreover if, \(K > 1\) and \(F\) and \(T\) are weakly compatible then \(F\) and \(T\) have a unique common fixed point.

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Alzer inequality for two operators in Hilbert spaces

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Abstract

In this paper, we give the Alzer inequality for Hilbert space operators as follows: Let $A, B$ be two selfadjoint operators on an Hilbert space $H$ such that $0 < A, B \leq \frac{1}{2}I$, where $I$ is identity operator on $H$ [3, 4]. Also, let $A := A\nabla B$ and $G := A\sharp B$ be arithmetic and geometric means of $A, B$, and $A' := A'\nabla B'$ and $G' := A'\sharp B'$ be arithmetic and geometric means of $A', B'$ where $A' := I - A$ and $B' := I - B$. Then we show that

$$A' - G' \leq A - G.$$  

Keywords: operator concavity, selfadjoint operator, arithmetic mean, geometric mean.

Mathematics Subject Classification: Primary: 47A63; Secondary: 15A42, 46L05, 47A30.

1 Introduction and preliminaries

Let $x_1, \cdots, x_n \in (0, \frac{1}{2}]$ and $\lambda_1, \cdots, \lambda_n > 0$ with $\sum_{j=1}^{n} \lambda_j = 1$. We denote by $A_n$ and $G_n$, the arithmetic and geometric means of $x_1, \cdots, x_n$ respectively, i.e

$$A_n = \sum_{j=1}^{n} \lambda_j x_j, \quad G_n = \prod_{j=1}^{n} x_j^{\lambda_j},$$

and also by $A'_n$ and $G'_n$, the arithmetic and geometric means of $1 - x_1, \cdots, 1 - x_n$ respectively, i.e.

$$A'_n = \sum_{j=1}^{n} \lambda_j (1 - x_j), \quad G'_n = \prod_{j=1}^{n} (1 - x_j)^{\lambda_j}.$$  

Alzer proved the following inequality and its refinement [1, 2]

$$A'_n - G'_n \leq A_n - G_n.$$  

(1)

Throughout the paper, let $B(H)$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and $I$ is the identity operator. A selfadjoint operator $A \in B(H)$ is called positive if $\langle Ax, x \rangle \geq 0$ holds for every $x \in H$ and then we write $A \geq 0$. Also, we said to be $A > 0$ if $\langle Ax, x \rangle > 0$ holds for every $x \in H$. For every selfadjoint operators $A, B \in B(H)$, we say $A \leq B$ if $B - A \geq 0$. Let $f$ be a continuous real valued function defined on an interval $[\alpha, \beta]$. The function $f$ is called operator decreasing if $A \leq B$ implies $f(A) \leq f(B)$ for all $A, B$ with spectra in $[\alpha, \beta]$. A function $f$ is said to be operator concave on $[\alpha, \beta]$ if

$$\lambda f(A) + (1 - \lambda)f(B) \leq f(\lambda A + (1 - \lambda)B)$$

for all $\lambda \in [0, 1]$. Then, for every operator concave function $f$ on $[\alpha, \beta]$, we have

$$f(A)_n - f(G')_n \leq f(A_n) - f(G)_n.$$  

(2)
for any selfadjoint operators $A, B \in \mathbb{B}(\mathcal{H})$ with spectra in $[\alpha, \beta]$ and all $\lambda \in [0, 1]$.

We recall that, the weighted arithmetic mean $\nabla_p$, the weighted harmonic mean $!_p$ and the weighted geometric mean (the $p$-power mean) $\ast_p$ defined for $0 < p < 1$:

$$A \nabla_p B := (1-p)A + pB,$$
$$A !_p B := \left((1-p)A^{-1} + pB^{-1}\right)^{-1},$$
$$A \ast_p B := A^{\frac{1}{2}} \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{p}{2}} A^{\frac{1}{2}}.$$

Also, we know that $A \ast_p B = B \ast_{1-p} A$.

Notice that if $p = \frac{1}{2}$ in above definitions, we have the classic arithmetic, harmonic and geometric means and denote its as follows:

$$A := A \nabla \frac{1}{2} B = A \nabla \frac{1}{2} B = \frac{1}{2} A + \frac{1}{2} B,$$
$$H := A !_\frac{1}{2} B = A !_\frac{1}{2} B = \left(\frac{1}{2} A^{-1} + \frac{1}{2} B^{-1}\right)^{-1},$$
$$G := A ^\frac{1}{2} B = A ^\frac{1}{2} B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}.$$

Also $A' = A' \nabla B'$, $H' = A' !_\frac{1}{2} B'$ and $G' = A' ^\frac{1}{2} B'$.

In this paper, we state Alzer inequality (1) for two operators in Hilbert spaces.

2 Main Results

In this section, we state a new property of arithmetic and geometric mean for positive operators.

**Theorem 2.1.** Assume that $A$ and $B$ are two positive operators in $\mathbb{B}(\mathcal{H})$ such that $\|B^{-\frac{1}{2}} AB^{-\frac{1}{2}}\| < 1$ and $\lambda \in (0, 1)$. Then we have

$$A \nabla_\lambda B - A \ast_\lambda B = \sum_{k=2}^{\infty} (-1)^{k-1} \left(\frac{1 - \lambda}{k}\right) B^{\frac{k-1}{2}} (B - A)^{k} B^{\frac{k-1}{2}}. \quad (2)$$

In the following theorem we state the Alzer inequality for two positive operator in $\mathbb{B}(\mathcal{H})$.

**Theorem 2.2** (Alzer Inequality). Suppose that $A, B \in \mathbb{B}(\mathcal{H})$ are operators such that $0 < A \leq B \leq \frac{1}{2} I$, and let $A' := I - A$ and $B' = I - B$. If $0 < \lambda < \frac{1}{2}$, then

$$A' \nabla_\lambda B' - A' \ast_\lambda B' \leq A \nabla_\lambda B - A \ast_\lambda B. \quad (3)$$

**Corollary 2.3.** With the above notations, we have

$$A' - G' \leq A - G.$$

**References**


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Some results on fundamental locally multiplicative
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Abstract
In this paper after studying nonvoidness of spectrum on fundamental locally multi-
plicative topological algebras, we obtain some new results concerning applications
of Gelfand spectrum and continuity of homomorphisms on these algebras. The
commutativity of such algebras is also studied.

Keywords: Fundamental topological algebra, FLM algebra, spectrum, homomor-
phism.

Mathematics Subject Classification: 46H05

1 Introduction and Definitions
The fundamental topological spaces (also algebras) has been introduced in [1] in 1990 extending
the meaning of locally bounded and locally convex algebras.
A topological linear space A is said to be fundamental one if there exists \( b > 1 \) such that for every sequence \( \{x_n\} \) of \( A \), the convergence of \( b^n(x_n - x_{n-1}) \) to zero in \( A \) implies that \( \{x_n\} \) is Cauchy.
A fundamental topological algebra is an algebra whose underlying topological linear space is fun-
damental.
A fundamental topological algebra is called to be locally multiplicative, if there exists a neighbour-
hood \( U_0 \) of zero such that for every neighbourhood \( V \) of zero, the sufficiently large powers of \( U_0 \)
lie in \( V \). We call such an algebra, an FLM algebra.
In this note, we have a discussion about nonvoidness of spectrum on FLM algebras. Also we obtain
some new results concerning applications of Gelfand spectrum and continuity of homomorphisms
on these algebras. Finally, the commutativity of FLM algebras is studied.

2 Main results
In this section we state and proof the following main results.

**Theorem 2.1.** let \( A \) be a complete metrizable FLM algebra with the unit element \( e \) and \( A' \) be a
topological dual space of \( A \).If \( A' \) separates the point of \( A \) then \( sp(x) \neq \emptyset \), \( \forall x \in A \).

**Proof.** Since \( Inv(A) \) is an open set in FLM algebra[2,4,3 ], then \( Inv(A) \) is a \( G_\delta \)-set in \( A \). By
[8,2,2,38] inversion is continuous for \( A \). Now suppose that \( sp(x) = \emptyset \) for some \( x \in A \). Then for
each \( \lambda \in C, (\lambda e - x) \) is invertible. We define \( R_x : C \to A \) with \( R_x(\lambda) = (\lambda e - x)^{-1}, \forall \lambda \in C \)
\( R_x \) is entire function on \( C \). Also for \( x \in A' \),
\( \varphi oR_x : C \to C \) with \( \varphi oR_x(\lambda) = \varphi(R_x(\lambda)) = \varphi(\lambda e - x)^{-1} \) is entire function. Set
\[ f(\lambda) = \varphi(R_x(\lambda)) = \varphi(\lambda e - x)^{-1} \]
then we have \( f(\lambda) \to 0 \) as \( \lambda \to \infty \), and so \( f = 0 \) by liouville’s theorem. In particular, \( f(0) = \varphi(-x^{-1}) = 0 \). Since \( A' \) separates the point of \( A \), thus \( x^{-1} = 0 \), which is a contradiction.

**Theorem 2.2.** Let \( A \) be a complete metrizable FLM algebra with unit element \( e \). If \( A' \) separates the point \( A \) and \( \text{Inv}(A) = A \setminus \{0\} \) then \( A = Ce \).

**Theorem 2.3.** (The Gelfand homomorphism). Let \( A \) be a commutative complete metrizable FLM algebra with unit element \( e \) and \( \phi_A \) be the Gelfand spectrum of \( A \). Then the following statements hold:

i) Every element \( a \in A \) satisfies: \( \text{sp}(a) = \{ \varphi(a) : \varphi \in \phi_A \} = \hat{a}(\phi_A) \)

ii) The Gelfand spectrum \( \phi_A \) is a compact Hausdorff space.

**Proof.** In [4,5.5] and [4,5.6] the proof of (i) and (ii) has been showed for FLM algebras.

**Theorem 2.4.** Let \( A \) be a commutative complete metrizable FLM algebra with unit element \( e \) and \( \phi_A \) be the Gelfand spectrum of \( A \). Then the following holds:

i) \( \text{sp}(x + y) \subseteq \text{sp}(x) + \text{sp}(y) \), ii) \( \text{sp}(xy) \subseteq \text{sp}(x)\text{sp}(y) \)

**Proof.** By preceding theorem we have,

\[ \text{sp}(x + y) = \text{Im}(x + y) = \text{Im}(\hat{x} + \hat{y}) \subseteq \text{Im}(\hat{x}) + \text{Im}(\hat{y}) = \text{sp}(x) + \text{sp}(y) \]

The analogous calculation holds for the product \( xy \).

**Theorem 2.5.** Let \( \varphi : A \to B \) be an algebra homomorphism between complete metrizable FLM algebras with unit elements \( e \) and \( e' \) respectively. If \( B \) is commutative and semi-simple and moreover \( \phi_A \) and \( \phi_B \) are Gelfand spectrum of \( A \) and \( B \) respectively, then \( \varphi \) is continuous.

**Proof.** Straight forward.

**Theorem 2.6.** Let \( A \) be a complete metrizable FLM algebra and \( B \) be a complete metrizable fundamental topological algebra. If \( B \) satisfies the following property:

for each sequence \( \{y_n\} \subseteq B \), \( y_n \neq 0 \) as \( n \to \infty \), there is a sequence \( \{\varphi_n\} \) of multiplicative linear functionals on \( B \) such that \( \inf_n |\varphi_n(y_n)| = \varepsilon > 0 \). Then each homomorphism \( f : A \to B \) is continuous.

**Proof.** Suppose \( f \) is not continuous. Let \( \{x_n\} \subseteq A \) be a sequence such that \( x_n \to 0 \) but \( f(x_n) \neq 0 \). Put \( y_n = f(x_n) \). We may assume that \( y_n \neq 0 \) for all \( n \geq 1 \) (otherwise choose subsequence). By hypothesis \( \inf_n |\varphi_n(y_n)| = \varepsilon > 0 \). So we have \( |\varphi_n(f(\varepsilon^{-1}x_n))| = |\varphi_n e^{-1} f(x_n)| \geq 1 \) for all \( n \geq 1 \).

Put \( z_n = \varepsilon^{-1}x_n \) then \( z_n \to 0 \) as \( n \to \infty \). Since \( \varphi_n f \) is multiplicative linear functional on \( A \), therefore by [2,4.5], it is continuous and so \( \varphi_n (f(\varepsilon^{-1}z_n)) \to 0 \). On the other hand, \( |\varphi_n f(z_n)| \geq 1 \). This give a contradiction and consequently \( f \) is continuous.

**Definition 2.7.** Let \( A \) be a complete metrizable FLM algebra with unit element \( e \) and \( A' \) separates the point of \( A \). We say that a linear continuous functional \( f \) satisfies property (c) on \( A \) if the following relation holds,

\[ f(e) = 1 \quad \text{and} \quad |f(x)| \leq \rho(x) \quad \text{for all} \quad x \in A. \]

**Theorem 2.8.** Let \( A \) be a complete metrizable FLM algebra with unit element \( e \) and \( A' \) separates the point of \( A \). If \( \varphi \in A' \) satisfies property (c) on \( A \), then \( A \) is a commutative.

**Proof.** Let \( x, y \in A \) and \( f : C \to A \) with \( f(\lambda) = \exp(\lambda x) y \exp(-\lambda x) \), \( \forall \lambda \in C \). So \( \varphi f(\lambda) = \varphi(f(\lambda)) = \varphi(\exp(\lambda x) y \exp(-\lambda x)) \). Since \( \varphi \) satisfies property (c), then we have

\[ |\varphi f(\lambda)| = |\varphi(\exp(\lambda x) y \exp(-\lambda x))| \leq \rho(\exp(\lambda x) y \exp(-\lambda x)) \]

Our assumption implies that the property \( \rho(xy) = \rho(yx) \forall x, y \in A \) is valid for FLM algebras, thus \( |\varphi(f(\lambda))| \leq \rho(y) \), i.e. \( \varphi f \) is a bounded. On the other hand \( \varphi f \) is differentiable on \( C \). By
using lioville’s theorem, we obtain that $\varphi of$ is a constant function. This implies that $(\varphi of)'(\lambda) = \varphi(f'(\lambda)) = 0$, since $A'$ separates the point of $A$. Then we have $f(\lambda)' = 0, \forall \lambda \in C$ and consequently, $x \exp(\lambda x) y \exp(-\lambda x) = \exp(\lambda x) y x \exp(-\lambda x) = 0 \forall \lambda \in C$. For $\lambda = 0$ we obtain, $xy - yx = 0$ or $xy = yx$, i.e. $A$ is commutative.

**Theorem 2.9.** Let $A$ and $B$ be complete metrizable FLM algebras with unit elements $e$ and $e'$ respectively. Also let $A'$ and $B'$ separate the point of $A$ and $B$ respectively. Suppose that $T$ is a continuous linear mapping from $A$ into $B$ such that $Te = e'$ and $\rho(Tx) \leq \rho(x)$ for all $x \in A$. If $f \in B'$ satisfies property (c) on $B$, then $foT \in A'$ satisfies property (c) on $A$ and moreover $A$ is commutative.

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Multiwavelets and multivariate wavelets

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Abstract

Wavelets have been invented since the late 1980s, and have found many applications. In parallel with the classical 1-dimensional wavelets, there are generalizations in different context and forms. Discussion of higher dimensional wavelets is one the approaches which is interesting for researchers. Here we try to briefly introduce and discuss this subject.

Keywords: Wavelets, refinement equation, multiwavelets, multivariate wavelets

Mathematics Subject Classification: 42C40

1 Introduction

Wavelets have been invented since the late 1980s, and have found many applications in operator theory, numerical analysis, signal and image processing and other fields. Classical wavelet theory is based on a refinement equation of the form

\[ \phi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \phi(2x - k), \]

which defines the scaling function \( \phi \). The scaling function leads to multiresolution analysis (MRAs), wavelets, fast decomposition and reconstruction algorithms. Though all wavelets are not arisen from an MRA, but it is still main tool in constructing wavelets and dealing with general cases, we also follow this approach. Generalizations include wavelet packets, multiwavelets, multivariate wavelets and other constructions. In some cases the generalization is straightforward, in other cases results are more complex, in unexpected ways. We briefly discuss some generalization topics.

2 Function space

One such generalizations which are mentioned above, are multiwavelets, which have been around since the early 1990s. We replace the scaling function \( \phi \) by a function vector

\[ \Phi(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{pmatrix}, \]

called a multiscaling function, and the refinement equation by

\[ \Phi(x) = \sqrt{m} \sum_{k \in \mathbb{Z}} H_k \Phi(mx - k), \]

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the reconstruction coefficients $H_k$ are now $n \times n$-matrices and we define $\text{Supp} \Phi = \bigcup_k \text{Supp} \phi_k$. The Multiscaling and multiwavelet functions are column vectors and coefficients are row vectors. We choose the dilation factor $m \geq 2$ in the multiwavelet case. Multiwavelets lead to MRAs and fast algorithms just like scalar wavelets but they have some advantages: they can have short support coupled with high smoothness and high approximation order, and they can be both symmetric and orthogonal.

**Proposition 2.1.** If $\Phi$ is a refinement scaling function with compact support then $\text{Supp} \Phi \subset \left[ \frac{k_0}{m-1}, \frac{k_1}{m-1} \right]$. 

**Definition 2.2.** An MRA of $L^2(\mathbb{R})$ is a doubly infinite nested sequence of closed subspaces of $L^2$, $\ldots \subset V_{-1} \subset V_0 \subset V_2 \subset \ldots$ which have these properties:

1. $\bigcup_j V_j = L^2(\mathbb{R})$,
2. $\bigcap_j V_j = \{0\}$,
3. $f(\cdot) \in V_j \Leftrightarrow f(m \cdot) \in V_{j+1}, \forall j \in \mathbb{Z}$,
4. $f(\cdot) \in V_j \Leftrightarrow f(\cdot - m^{-j}k) \in V_j, \forall j, k \in \mathbb{Z}$,
5. There is a function vector $\Phi \in L^2$ so that $\{\phi_l(x-k) : l = 1, 2, \ldots, n\}$ forms a Riesz basis of $V_0$.

**Proposition 2.3.** Assume that $\Phi_1, \Phi_2$ are orthogonal multiscaling functions with compact support. Then they span the same space $V_0$ if and only if $\Phi_2(x) = \sum_k A_k \Phi_1(x-k)$, where $A(\xi) = \sum_k A_k e^{-ik\xi}$ is a paraunitary matrix polynomial.

Considering orthogonal projections of spaces $V_j$ the multiwavelet functions $\Psi^{(s)}$ can be represented as

$$\Psi^{(s)}(x) = \sqrt{m} \sum_k G_k^{(s)} \Phi(mx-k),$$

for some coefficients $G_k^{(s)}$. Note that there is no simple formula like the scalar case and multiwavelet functions are not unique. Another generalization for wavelets is multivariate wavelets which have been around since 1995s. This class of wavelets provides a multidimensional basis instead of 1-dimensional basis. As we have mentioned above multivariate wavelets appear in many areas of mathematics and physics, and their construction is based on two main approaches that will be stated follow.

**Separable MRAs**

**Case 1:**
First let us consider the tensor product of one dimensional wavelets.

**Definition 2.4.** If $\{\psi_{jk}\}_{j,k \in \mathbb{Z}}$ is a wavelet basis for $L^2(\mathbb{R})$, then by switching $j,k$ to $J = (j_1,j_2,\ldots,j_n)$ and $K = (k_1,k_2,\ldots,k_n)$, define the $n$-dimensional wavelet basis for $L^2(\mathbb{R}^n)$ by

$$\psi_{jk} := \bigotimes_{m=1}^n \psi_{j_mk_m}. $$

Though this approach works good enough in some applications and examples, such as Haar system, as a multivariate wavelet basis, but because of some technical problems specially in defining their support, we need another approach for them.

**Case 2:**
Another way for constructing multidimensional wavelets is based on the tensor product of one-dimensional MRAs.

**Definition 2.5.** Let $\{V_j^{(m)}\}_{j \in \mathbb{Z}}$ and $m = \{1, 2, \ldots, n\}$ be MRAs in $L^2(\mathbb{R})$. Define $V_j \subset L^2(\mathbb{R}^n)$ as follow:

$$V_j = \text{span}\{f = \bigotimes_{m=1}^n f_m \ s.t. f_m \in V_j^{(m)}\}_{j \in \mathbb{Z}}$$
Note that \( f : \mathbb{R}^d \rightarrow \mathbb{C} \) and \( f(x) = f_1(x_1) \cdots f_n(x_n) \).

**Proposition 2.6.** The above definition provides an MRA for \( L^2(\mathbb{R}^n) \).

Following these definition and proposition, for mother wavelet \( \Psi \) we have

\[
\Psi_E = (\otimes_{v \in E} \psi(v))(\otimes_{u \in E} \phi(u)),
\]

on which \( E \subset \oplus \{1, 2, \ldots, n\} \).

**Nonseparable MRAs**

Till now we’ve discussed the separable MRAs with a special scaling function. But we are interested to construct multivariate wavelets independent of tensor product that guide us to nonseparable MRAs, based on matrix dilation.

**Definition 2.7.** A collection of closed and nested subspaces \( \ldots \subset V_{-1} \subset V_0 \subset V_2 \subset \ldots \) of \( L^2(\mathbb{R}^n) \) is called an MRA of \( L^2(\mathbb{R}^n) \) with a matrix dilation \( M \), if the following conditions hold:

1. \( \bigcup_j V_j = L^2(\mathbb{R}^n) \),
2. \( \bigcap_j V_j = \{0\} \),
3. \( f(.) \in V_j \iff f(M^{-j} .) \in V_0, \forall j \in \mathbb{Z} \),
4. \( f(.) \in V_j \iff f(\cdot - M^{-j} k) \in V_j, \forall j, k \in \mathbb{Z} \),
5. There is a function \( \phi \in V_0 \) so that \( \{\phi(\cdot + k) : k \in \mathbb{Z}^n\} \) forms a Riesz basis of \( V_0 \). In this case the refinement equation is represented by \( \phi(x) = \sum_k h_k \phi(Mx + k), x \in \mathbb{R}^n \).

**Proposition 2.8.** If a refinement equation has a solution \( \phi \) which is in \( L(\mathbb{R}^n) \) or \( L^2(\mathbb{R}^n) \), then such \( \phi \) is unique up to a factor \( c \).

**Example 2.9.** Consider dilation matrix \( M = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & 2 \end{pmatrix}_{n \times n} \) with \( N > n \). We will consider so holds B-splines as

\[
\hat{B}_N(\xi) = 2^n \prod_{k=1}^N \frac{1 - e^{2\pi i (a_k \xi)}}{2\pi i (a_k \xi)},
\]

on which \( a_k \in \mathbb{R}^n, k = 1, 2, \ldots, N \) and 0 < \( |a_k| < 2 \). Then \( \hat{B}_N(\xi) \) as an scaling function provides an MRA.

**References**


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1 Introduction

Frame for Hilbert space were formally defined by Duffin and Schaeffer [2] in 1952 for studying some problem in non harmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossman and Meyer [1]. In [7] Wenchang Sun introduced a generalization of frames and showed that this includes more other cases of generalizations of frame concept and proved that many basic properties can be derived within this more general context.

The concept of frames especially the g-frame was introduced in Hilbert $C^*$-module, and some of their properties were investigated in [3] and [5]. A g-Riesz basis always has a canonical dual which is necessary a g-Riesz basis. In this paper we obtain a necessary and sufficient condition for a dual of a g-Riesz basis to be again a g-Riesz basis.

2 Preliminaries

In the following we briefly recall some definitions and basic properties of Hilbert $C^*$-module and g-frames in Hilbert $C^*$-module. We first give some notations which we need later. Throughout this paper $J$ is finite or countably index set. $A$ is unital $C^*$-algebra with identity $1_A$, $U$ and $V$ are finitely or countably generated Hilbert $C^*$-module and $\{V_j\}_{j \in J}$ be a sequence of closed Hilbert submodules of $V$. For each $j \in J$, $End_A^* (U, V_j)$ is the collection of all adjointable $A$-linear maps from $U$ to $V_j$.

We also denote

$$\bigoplus_{j \in J} V_j = \{ g = \{ g_j \} : g_j \in V_j and \sum_{j \in J} \langle g_j, g_j \rangle \text{ is norm convergent in } A \}$$

For any $f = \{ f_j \}$ and $g = \{ g_j \}$, if the $A$-valued inner product is define by $\langle f, g \rangle = \sum_{j \in J} \langle f_j, g_j \rangle$ and the norm is defined by $\| f \| = \| \langle f, f \rangle \|^\frac{1}{2}$, then $\bigoplus_{j \in J} V_j$ is a Hilbert $A$-module (see [6]).

**Definition 2.1.** Let $H$ be a left $A$-module such that the linear structure of $A$ and $H$ are compatible, $H$ is called a pre-Hilbert $A$-module if $H$ is equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle : H \times H \rightarrow A$ such that
1. \( \langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle \) \( f, g, h \in \mathcal{H}, \alpha, \beta \in \mathbb{C} \);
2. \( \langle af, g \rangle = a \langle f, g \rangle \) \( f, g \in \mathcal{H}, a \in \mathcal{A} \);
3. \( \langle f, g \rangle = \langle g, f \rangle^* \) \( f, g \in \mathcal{H} \);
4. \( \langle f, f \rangle \geq 0; \) if \( \langle f, f \rangle = 0 \) then \( f = 0 \).

For every \( f \in \mathcal{H} \), we define \( \| f \| = \| \langle f, f \rangle \|^{\frac{1}{2}} \) and \( \| f \| = \langle f, f \rangle^{\frac{1}{2}} \). If \( \mathcal{H} \) is complete with respect to the norm, it is called a Hilbert \( \mathcal{A} \)-module or a Hilbert \( \mathcal{C}^\ast \)-module over \( \mathcal{A} \).

**Definition 2.2.** We call a sequence \( \{ \Lambda_j \in \text{End}_\mathcal{A}(\mathcal{U}, \mathcal{V}_j) : j \in J \} \) a \( g \)-frame in Hilbert \( \mathcal{A} \)-module \( \mathcal{U} \) with respect to \( \{ \mathcal{V}_j \}_{j \in J} \), if there exist two positive constant \( C, D \) such that:

\[
C \langle f, f \rangle \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq D \langle f, f \rangle \quad \forall f \in \mathcal{U} \quad (2)
\]

**Remark 2.3.** For any \( j \in J \), if we let \( \mathcal{V}_j = \mathcal{V} = \mathcal{A} \) and \( \Lambda_j f = \langle f, f_j \rangle \) for any \( f \in \mathcal{U} \), in this case the \( g \)-frame is just a frame for \( \mathcal{U} \).

**Definition 2.4.** A \( g \)-frame \( \{ \Lambda_j \in \text{End}_\mathcal{A}(\mathcal{U}, \mathcal{V}_j) : j \in J \} \) in Hilbert \( \mathcal{A} \)-module \( \mathcal{U} \) with respect to \( \{ \mathcal{V}_j \} \) is said to be a \( g \)-Riesz basis if and only if

(i) \( \Lambda_j \neq 0 \);
(ii) If an \( \mathcal{A} \)-linear combination \( \sum_{j \in K} \Lambda_j^* g_j \) is equal to zero, then every summand \( \Lambda_j^* g_j \) equal to zero, where \( \{ g_j \}_{j \in K} \in \bigoplus_{j \in K} \mathcal{V}_j \) and \( K \subset J \).

**Definition 2.5.** Suppose that \( \{ \Lambda_j \}_{j \in J} \) be a (standard) \( g \)-frame and \( \{ \Gamma_j \}_{j \in J} \) a sequence for \( \mathcal{U} \) with respect to \( \{ \mathcal{V}_j \} \). Then \( \{ \Gamma_j \}_{j \in J} \) is said to be a (standard) dual \( g \)-sequence of \( \{ \Lambda_j \}_{j \in J} \) if

\[
f = \sum_{j \in J} \Gamma_j^* \Lambda_j \quad (3)
\]
holds for all \( f \in \mathcal{U} \), where the sum in 3 converges in norm.

The pair \( \{ \Lambda_j \}_{j \in J} \) and \( \{ \Gamma_j \}_{j \in J} \) are called a dual \( g \)-frame pair when \( \{ \Gamma_j \}_{j \in J} \) is also a \( g \)-frame.

**3 Main Result**

In this section we study the relations between \( g \)-Riesz basis and their dual \( g \)-frame in Hilbert \( \mathcal{C}^\ast \)-module. We begin this section with a characterization of \( g \)-Riesz basis.

**Theorem 3.1.** Suppose that \( \{ \Lambda_j \}_{j \in J} \) be a \( g \)-frame of a finitely or countably generated Hilbert \( \mathcal{A} \)-module \( \mathcal{U} \) with respect to \( \{ \mathcal{V}_j \} \). Let \( T_\Lambda : \mathcal{U} \rightarrow \bigoplus_{j \in J} \mathcal{V}_j \) be the analysis operator of \( \{ \Lambda_j \}_{j \in J} \) with close rang. Then \( \{ \Lambda_j \}_{j \in J} \) is \( g \)-Riesz basis if and only if \( \Lambda_n \neq 0 \) and \( P_n(\text{Rang} T_\Lambda) \subseteq \text{Rang} T_\Lambda \), where \( P_n \) is the projection on \( \bigoplus_{j \in J} \mathcal{V}_j \).

**Proof.** Suppose first \( \{ \Lambda_j \}_{j \in J} \) is a \( g \)-Riesz basis. Let \( g = \{ g_j \}_{j \in J} \in (\text{Rang} T_\Lambda)^\perp \) then for all \( f \in \mathcal{U} \), we have

\[
0 = \langle \{ g_j \}_{j \in J}, T_\Lambda f \rangle = \langle T_\Lambda^* \{ g_j \}_{j \in J}, f \rangle = \langle \sum_{j \in J} \Lambda_j^* g_j, f \rangle \quad (4)
\]
So \( \sum_{j \in J} \Lambda_j^* g_j = 0 \) then \( \Lambda_j^* g_j = 0 \) hold for all \( j \in J \).

Now let \( f \in \mathcal{U} \) we have

\[
0 = \langle \{ g_j \}_{j \in J}, P_n(T_\Lambda) \rangle = \langle \{ g_j \}_{j \in J}, P_n(\{ \Lambda_j f \}_{j \in J}) \rangle = \langle g_n, \Lambda_n f \rangle = \langle \Lambda_n^* g_n, f \rangle = 0. \quad (5)
\]
So \((\text{Rang}T_{\Lambda})^\perp \subseteq P_n(\text{Rang}T_{\Lambda}) \subseteq \text{Rang}T_{\Lambda}\).

Suppose now that \(P_n(\text{Rang}T_{\Lambda}) \subseteq \text{Rang}T_{\Lambda}\) for each \(n\). Suppose that \(\sum_{j \in J} \Lambda_j^* g_j = 0\) and fix \(n \in J\), then \(P_n T_{\Lambda} f \in \text{Rang}T_{\Lambda}\), so there exist \(f \in \mathcal{U}\) such that \(P_n T_{\Lambda} f = T_{\Lambda} f_n\).

Now for any \(f \in \mathcal{U}\) we have
\[
(f, \Lambda_n^* g_n) = (\Lambda_n f, g_n) = \sum_{j \in J} \langle \Lambda_j f_n, g_j \rangle = \langle f_n, \sum_{j \in J} \Lambda_j^* g_j \rangle = 0,
\]
then \(\Lambda_j^* g_j = 0\) hold for all \(n \in J\).

\textbf{Corollary 3.2.} Let \(\{\Lambda_j\}_{j \in J}\) is a \(g\)-frame of \(\mathcal{U}\) with respect to \(\{V_j\}_{j \in J}\) and let \(\text{Rang}T_{\Lambda}\) is closed, then \(\{\Lambda_j\}_{j \in J}\) is a \(g\)-Riesz basis if and only if
(i) \(\Lambda_j \neq 0\);
(ii) if \(\sum_{j \in J} \Lambda_j^* g_j = 0\) for some sequence \(\{g_j\} \in \bigoplus_{j \in J} V_j\), then \(\Lambda_j^* g_j = 0\) for each \(j \in J\).

\textbf{Proof.} See the proof of theorem 3.1.

We have the following equivalent definition for \(g\)-frame in Hilbert \(C^*\)-module.

\textbf{Proposition 3.3.} let \(\mathcal{U}\) be a finitely or countably generated Hilbert \(A\)-module over a unital \(C^*\)-algebra \(A\) and \(\{\Lambda_j\}_{j \in J}\) be a \(g\)-sequence. Then \(\{\Lambda_j\}_{j \in J}\) is a \(g\)-frame of \(\mathcal{U}\) with respect to \(\{V_j\}_{j \in J}\) with bounds \(C\) and \(D\) if and only if
\[
C\|f\|^2 \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq D\|f\|^2 \quad \forall f \in \mathcal{U}.
\]

We now give a necessary and sufficient condition about the uniqueness of dual \(g\)-frame in Hilbert \(C^*\)-module.

\textbf{Theorem 3.4.} Suppose that \(\{\Lambda_j\}_{j \in J}\) is a \(g\)-frame for \(\mathcal{U}\) with respect to \(\{V_j\}_{j \in J}\). Then the following statements are equivalent
(i) \(\{\Lambda_j\}_{j \in J}\) has a unique dual \(g\)-frame;
(ii) \(\text{Rang}T_{\Lambda} = \bigoplus_{j \in J} V_j\);
(iii) if \(\sum_{j \in J} \Lambda_j^* g_j = 0\) for some sequence \(\{g_j\}_{j \in J} \in \bigoplus_{j \in J} V_j\), then \(\Lambda_j^* g_j = 0\) for each \(j\).

\textbf{Proof.} The proof is the same as proof theorem 3.10 in [4].

\textbf{Corollary 3.5.} In case the equivalent conditions in theorem 3.4 are satisfies, \(\{\Lambda_j\}_{j \in J}\) is a \(g\)-Riesz basis.

\textbf{References}


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The Hankel operator on $l^2$ space

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Abstract

In this paper, we apply the Hankel operator and this action on Hardy spaces and by using the equality $H_g = P_+ M_g C$ which is known to hold in $l^2$ context [2]. Thus obtaining a Nehari-type Theorem for Hankel operators on Hardy spaces, we obtain an upper bound for Hankel operator on $l^2$ space.

Keywords: Hankel operator, Hardy spaces, Riesz(Szego) projection, Multiplication operator, Fourier-coefficients.

Mathematics Subject Classification: 47B35, 30H10, 47B06, 42A16.

1 Introduction

A Hankel operator on the space $l^2$ of all square-summable complex sequences is an operator defined by a matrix whose entries $a_{n,k}$ depend only on the sum of the coordinates: $a_{n,k} = c_n + k$ for some sequence $(c_n)_{n=0}^{\infty}$.

Hankel operators on different spaces are related to many areas such as the theory of moment sequences, orthogonal polynomials, Toeplitz operators or analytic Besov spaces. In the theory that will say we will use the identity: $H_g = P_+ M_g C$ that in this identity $H_g$ is a Hankel operator associated with a function $g \in L^\infty(T)$ and $T$ is an unit circle and $P_+$ is the Riesz(Szego) projection from $L^p(T)$ onto $H^p$.

In the proof of this theory we will use the flip operator that is a isometric conjugation operator for the functions on obvious that $C$ is an isometry from $H^p$ into $L^p(T)$. Throughout the text $D = \{ z \in \mathbb{C} : |z| < 1 \}$ unit disk in the complex plan $\mathbb{C}$ and $H(D)$ will signify the algebra of holomorphic functions in $D$.

2 The Hankel Operator on $l^2$ Space

Definition 2.1. The Hardy space $H^p (0 < p < \infty)$ is the space of all $f$ in $H(D)$ for which $\|f\|_{H^p} = \lim_{r \to 1^-} M_p(r,f) < \infty$ where the integral means $M_p(r,f)$ for $f$ in $H(D)$ and $0 < r < 1$ are defined by

$$M_p(r,f) = \left( \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}.$$

Definition 2.2. The Riesz(Szego) projection $P_+$ from $L^p(T)$ onto $H^p$ is defined by

$$P_+ u(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{u(t)}{1 - ze^{-it}} dt$$

For all $z \in D$. 

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Definition 2.3. The multiplication operator is an operator by the essentially bounded function $g : M_{g}u = gu$.  

Theorem 2.4. Let $g \in L^\infty(0, 2\pi)$. We define the associated Hankel operator $H_{g}$ by $H_{g}f(z) = \sum_{n=0}^{\infty}(\sum_{k=0}^{\infty}\hat{g}(n+k)\hat{f}(k))z^n$ in $D$ where $\hat{g}(n) = \frac{1}{2\pi}\int_{0}^{2\pi}e^{-int}g(t)dt$, $n \geq 0$ is the Fourier coefficients with non-negative indices. The operator $H_{g}$ which is defined as above is a bounded on $l^2$ and consequently

$$\|H_{g}\|_{\text{HP} \to \text{HP}} \leq \frac{\|g\|_{\infty}}{\sin^{\frac{1}{2}}p}.$$  

Proof. For the proof of this Theorem, at first we prove above theorem for any space $\text{HP}$, $1 < p < \infty$ then by applying an isomorphism of Hilbert spaces which is hold between $\text{H}^{2}$ and $l^2$ such as [6] we will obtain the statement.  

Given $f \in \text{HP}$, denote by $f_{m}$ it is mth Taylor polynomial: $f_{m}(z) = \sum_{k=0}^{m}\hat{f}(k)z^{k}$. The following result [7], will be useful: if $1 < p < \infty$ and $f \in \text{HP}$ then $\|f_{m} - f\|_{\text{HP}} \to 0$ as $m \to \infty$.  

Given $f \in \text{HP}$, at first we verify that the power series for $H_{g}f$ convergence in $D$, so we prove that definition of $H_{g}$ with action on $f$ is well-defined.  

To this end it is suffices to show that $|\sum_{k=0}^{\infty}\hat{g}(n+k)\hat{f}(k)| \leq \|g\|_{\infty}\|f\|_{\text{HP}}$.  

For $f_{m}$ instead of $f$, this follows immediately by recalling that $C$ is an isometry of $\text{HP}$ into $L^{p}(\text{T})$ and applying Holder’s inequality:

$$|\sum_{k=0}^{m}\hat{g}(n+k)\hat{f}(k)| = |\int_{0}^{2\pi}g(t)e^{-int}\sum_{k=0}^{m}\hat{f}(k)e^{-ikt}\frac{dt}{2\pi}|$$

$$= |\int_{0}^{2\pi}g(t)e^{-int}(cf_{m})(e^{it})\frac{dt}{2\pi}|$$

$$\leq \|g\|_{\infty}\|f_{m}\|_{\text{HP}}.$$  

A similar argument applied to the differences $f_{m} - f_{n}$ shows that $(\sum_{k=0}^{m}\hat{g}(n+k)\hat{f}(k))_{m=0}^{\infty}$ is a cauchy sequence uniformly in $n$, so it is legitimate to let $m \to \infty$ to obtain an above result.  

We let the identity $H_{g}f = P_{+}M_{g}f$ for all $f \in \text{HP}$ $1 < p < \infty$ holds, then by applying above identity and by theorem of Hollenbeck and verbitsky [3] which they denoted that $\|P_{+}\|_{L^{p}(\text{T}) \to \text{HP}} \geq \frac{1}{\sin^{\frac{1}{2}}p}$:

$$\|H_{g}\|_{\text{HP} \to \text{HP}} = \|P_{+}M_{g}C\|_{\text{HP} \to \text{HP}} \leq \|P_{+}\|_{L^{p}(\text{T}) \to \text{HP}}\|M_{g}\|_{L^{p}(\text{T}) \to \text{HP}}$$

$$\leq \|P_{+}\|_{L^{p}(\text{T}) \to \text{HP}}\|g\|_{\infty}$$

$$\leq \frac{\|g\|_{\infty}}{\sin^{\frac{1}{2}}p}.$$  

Finally, by applying the standard $\text{HP}$ pointwise estimate $|f(z)| \leq (1 - |z|^{2})^{\frac{1}{2}}\|f\|_{\text{HP}}$ [2] and Fatou’s lemma the statement follows. \hfill \qed

References


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Geometry and Topology
Some topologies on a spacetime such as a poset

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Abstract

In this seminar, we want to study the order relation on a spacetime and topologies related with order relation on this spacetime. Then, we try to compare the topologies on spacetime such as a poset or a manifold.

Keywords: Manifold, Spacetime, order relation, topology, poset

Mathematics Subject Classification: 51P05

1 Introduction

In [1] and [3] introduced the notion of posets and topologies on posets and in [4] there are the definitions and results about the Lorentzian manifolds and spastimes. There are some conditions for spacetime such that are useful for comparison between the topologies on a spacetime such as a poset and a manifold. For example if $M$ is globally hyperbolic, then $(M, \leq)$ is a continuous poset. We want to study topologies related on a spacetime.

There are some interesting topology on a poset that related to its order relation. The following definitions and result is necessary for study these topologies on a poset.

Definition 1.1. A poset a partially ordered set, i.e., a set together with a reflexive, antisymmetric and transitive relation.

Definition 1.2. Let $(P, \leq)$ be a partially ordered set. A nonempty subset $S \subseteq P$ is directed if $(\forall x, y \in S)(\exists z \in S) x, y \leq z$.

Definition 1.3. For a subset $X$ of a poset $(P, \leq)$, set

$\uparrow X:=\{y \in P: (\exists x \in X)x \leq y\}$,

$\downarrow X:=\{y \in P: (\exists x \notin X)y \leq x\}$

Definition 1.4. For elements $x, y$ of a poset $(P, \leq)$, write $x \ll y$ iff for all directed sets $S$ with a supremum

$y \leq \sqcup S \Rightarrow (\exists s \in S)x \leq s$

Definition 1.5. Let $(P, \leq)$ be a poset. Then

$\downarrow x = \{a \in P; a \ll x\}$ and $\uparrow x = \{a \in P; x \ll a\}$

Definition 1.6. A basis for a poset $(P, \leq)$ is a subset $B$ such that $B \cap \downarrow x$ contains a directed set with supremum $x$ for all $x \in P$.

Definition 1.7. A poset is continuous if it has a basis.
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**Definition 1.8.** subset $U$ of a poset $P$ is Scott open if
(i) $U$ is an upper set: $x \in U$, $x \leq y \Rightarrow y \in U$
(ii) $U$ is inaccessible by directed suprema: For every directed $S \subseteq P$ with a supremum $\sqcup S \in U \Rightarrow S \cap U \neq \emptyset$

The collection of all Scott open sets on $P$ is called the Scott topology.

**Theorem 1.9.** The collection $\{\uparrow x : x \in P\}$ is a basis for the Scott topology on a continuous poset.

**Definition 1.10.** The Lawson topology on a continuous poset $P$ has as a basis all sets of the form $\uparrow x \uparrow F$, for $F \in P$ are finite.

## 2 Main Result

Now we consider to a spacetime such that a poset and for this we must introduce an order relation for it

**Definition 2.1.** A manifold $M$ is a locally Euclidean Hausdorff space and has a countable basis. A Lorentz metric on a manifold is a symmetric, nondegenerate tensor field of type $(0, 2)$ whose signature is $(- + + +)$.

**Definition 2.2.** A spacetime is a real four-dimensional smooth manifold $M$ with a Lorentz metric.

**Definition 2.3.** For $p,q \in M$,
1) $p \ll q$ means there exists a future directed timelike curve in $M$ from $p$ to $q$.
2) $p < q$ means there exists a future causal curve in $M$ from $p$ to $q$.
We shall use the notation $p \leq q$ to mean $p=q$ or $p < q$.

Given any point in spacetime $M$, the timelike future and causal future of $p$, denoted $I^+(p)$ and $J^+(p)$, respectively are defined as,

$I^+(p) = \{q \in M : p \ll q\}$ and $J^+(p) = \{q \in M : p \leq q\}$.

In a spacetime, the relation $\leq$ set is a partial order which is

**Reflexive:** For all $p \in M$, we have $p \leq p$
**Antisymmetric:** For all $p, q \in M$ we have $p \leq q \leq p \Rightarrow p = q$
**Transitive:** For all $p, q, z \in M$, we have $p \leq q \leq z \Rightarrow p \leq z$

**Remark 2.4.** If $M$ be a spacetime, then $(M, \leq)$ would be a poset. Then one can study topologies related to this poset.

**Definition 2.5.** A spacetime $M$ is globally hyperbolic if it is strongly causal and if $J^+(p) \cap J^-(q)$ is compact in the manifold topology, for all $p, q \in M$.

**Proposition 2.6.** Let $M$ be a globally hyperbolic spacetime. Then

$p \ll q \Leftrightarrow q \in I^+(p)$

for all $p, q \in M$

**Theorem 2.7.** If $M$ is globally hyperbolic, then $(M, \leq)$ is a continuous poset.

**Corollary 2.8.** $I^-(p)$ is a directed set on a spacetime poset $M$. $I^+(p)$ is a basis for Scott topology on a spacetime poset $M$. 

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Lemma 2.9. If \((x_n)\) is a sequence in \(\mathcal{M}\) with \((x_n) \leq x\) for all \(n\), then
\[
\lim_{n \to \infty} x_n = x \implies \bigcup_{n \geq 1} x_n = x
\]

Proof. See [1].

Lemma 2.10. For any \(x \in \mathcal{M}\), \(I^-(x)\) contains an increasing sequence with supremum \(x\).

Proof. See [1].

Theorem 2.11. If \(\mathcal{M}\) is globally hyperbolic, then \((\mathcal{M}, \leq)\) is a bicontinuous poset with \(\ll \subseteq I^+\) whose interval topology is the manifold topology.

Proof. See [1].

Definition 2.12. A spacetime \(\mathcal{M}\) is a causally simple if \(J^+(x)\) and \(J^-(x)\) are closed for all \(x \in \mathcal{M}\).

Theorem 2.13. Let \(\mathcal{M}\) be a spacetime and \((\mathcal{M}, \leq)\) a continuous poset with \(\ll \subseteq I^+\). The following are equivalent:
(i) \(\mathcal{M}\) is causally simple.
(ii) The Lawson topology on \(\mathcal{M}\) is a subset of the interval topology on \(\mathcal{M}\).

Proof. See [1].

Remark 2.14. In this seminar we try to compare the topologies on spacetime such as a poset or a manifold.

References


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On the causal character of the orbits in cohomogeneity one Minkowski space $\mathbb{R}^3_1$

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Abstract
In this paper we study the causal character of principal orbits in cohomogeneity one Minkowski space $\mathbb{R}^3_1$. Among other results, we show that, if the action is proper, each principal orbit may be space-like, degenerate or Lorentzian, and the union of the the orbits with the same causal character is open and dense in $\mathbb{R}^3_1$. Furthermore, if the action is proper, we prove that there is no light-like orbit.

Keywords: Minkowski space, cohomogeneity one

Mathematics Subject Classification: 53C50, 57S25

1 Introduction

A cohomogeneity one $G$-manifold is a manifold $M$ with a smooth action of a compact connected Lie group $G$ which has a codimension one orbit. The study of such manifolds has been started by T. Nagano ([10]) and P.S. Mostert ([9]), who studied the topology of the manifold $M$ and of its orbits and its orbit space $M/G$. A new impulse to the research was given by L. B. Bergery in 1982 ([7]), who supposed that $M$ is a Riemannian manifold and $G$ is a closed and connected subgroup of the isometry group of $M$ which acts by cohomogeneity one on $M$, so called cohomogeneity one Riemannian manifold. In this case, he showed that the isotropy subgroups are compact and the orbit space is Hausdorff, and the study of cohomogeneity one Riemannian manifolds followed by many mathematicians, see for instance [5, 6, 7, 11, 12, 13]. The problem is still an active one. For the indefinite metric one can find very few results (see [1, 2, 3, 8]).

An action of a Lie group $G$ on a smooth manifold $M$ is said to be proper if the mapping $\phi: G \times M \to M \times M, (g, x) \mapsto (g.x, x)$ is proper. As a well known fact, there is a proper action of a Lie group $G$ on a manifold $M$, if and only if there is a complete $G$-invariant Riemannian metric on $M$ (see [4]). If the action of $G$ on $M$ is proper, then the isotropy subgroups are compact and, we say, the orbits $G(x)$ and $G(y)$ have the same orbit type if $G_x$ and $G_y$ are conjugate in $G$. This defines an equivalence relation among the orbits of $G$ on $M$. We denote by $[G(x)]$ the corresponding equivalence class, which is called the orbit type of $G(x)$. By $D$ we denote the set of orbit types of the action of $G$ on $M$. A fundamental feature of proper actions is the existence of slice, which enables one to define a partial ordering on the set of orbit types. The partial ordering on $D$ is defined by, $[G(y)] \leq [G(x)]$ if and only if $G_x$ is conjugate in $G$ to some subgroup of $G_y$. If $\Sigma$ is a slice at $y$, it implies that $[G(y)] \leq [G(x)]$ for all $x \in \Sigma$. Since $M/G$ is connected, there is a largest orbit type in $D$. Each representative of this largest orbit type is called a principal orbit. In other words, an orbit $G(x)$ is principal if and only if for each point $y \in M$ the isotropy subgroup $G_x$ is conjugate to some subgroup of $G_y$ within $G$. Other orbits are called singular. We say that $x \in M$ is a principal point if $G(x)$ is a principal orbit.

Let $M$ be a semi-Riemannian manifold and $G$ be a connected Lie subgroup of $\text{Iso}(M)$ which acts on $M$ by cohomogeneity one. For $x \in M$, the orbit $G(x)$ is space-like (resp. time-like) if
each tangent vector $v$ to $G(x)$ is space-like (resp. time-like), i.e. $\langle v, v \rangle > 0$ (resp. $\langle v, v \rangle < 0$), and it is said to be light-like if each tangent vector $v$ to $G(x)$ is null, i.e. $\langle v, v \rangle = 0$, and it is called Lorentzian (resp. degenerate) if the restriction of the metric on $G(x)$ is Lorentzian (resp. degenerate). The category into which a given orbit falls is called its causal character.

In this paper we assume that $G$ is a closed and connected subgroup of $\text{Iso}(\mathbb{R}^3_1)$ which acts properly and by cohomogeneity one on $\mathbb{R}^3_1$ and we study the causal character of the orbits. We show that there is no light-like orbit and the union of the principal orbits with the same causal character is open and dense in $\mathbb{R}^3_1$.

**Notation:** $\mathbb{R}^3_1$ denotes the 3-dimensional real vector space $\mathbb{R}^3$ with a scaler product given by

$$\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

## 2 Main Result

As indicated in the title of the paper, the aim of this paper is to determine the causal character of the orbits in cohomogeneity one Minkowski space $\mathbb{R}^3_1$. We state our result as follows.

**Theorem 2.1.** Let $\mathbb{R}^3_1$ be of cohomogeneity one under the proper action of a connected Lie subgroup $G \subseteq \text{Iso}(\mathbb{R}^3_1)$. Then there is no light-like orbit.

**Theorem 2.2.** Let $\mathbb{R}^3_1$ be of cohomogeneity one under the proper action of a connected Lie subgroup $G \subseteq \text{Iso}(\mathbb{R}^3_1)$. If there is a space-like (resp. Lorentzian) principal orbit, then the union of space-like (resp. Lorentzian) principal orbits are open and dense in $\mathbb{R}^3_1$. Indeed, the union of principal orbits with the same causal character (even degenerate orbits) is open and dense in $\mathbb{R}^3_1$.

**Theorem 2.3.** Let $\mathbb{R}^3_1$ be of cohomogeneity one under the proper action of a connected Lie subgroup $G \subseteq \text{Iso}(\mathbb{R}^3_1)$. If there is a singular orbit, then it is a totally geodesic time-like submanifold and each other orbit is a Lorentzian cylinder.

**References**


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**Poster**

D–Recurrent Hopf hypersurfaces of Sasakian space form

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**Abstract**

In this paper, we are studying recurrent Hopf hypersurfaces in the Sasakian space form and prove that such hypersurface is the product of the Sasakian space form and the geodesic curve.

**Keywords:** Recurrent hypersurfaces, Sasakian manifold.

**Mathematics Subject Classification:** 53C25, 53C40.

1 Introduction

A differentiable manifold $\widetilde{M}^{2m+1}$ is said to have an almost contact structure if it admits a (non-vanishing) vector field $\xi$, a one-form $\eta$ and a $(1, 1)$–tensor field $\phi$ satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,$$

where $I$ denotes the field of identity transformations of the tangent spaces at all points. These conditions imply that $\phi\xi = 0$ and $\eta \circ \phi = 0$, and that the endomorphism $\phi$ has rank $2m$ at every point in $\widetilde{M}^{2m+1}$. A manifold $\widetilde{M}^{2m+1}$, equipped with an almost contact structure $(\phi, \xi, \eta)$ is called an almost contact manifold and will be denoted by $(\widetilde{M}^{2m+1}, (\phi, \xi, \eta))$.

Suppose that $\widetilde{M}^{2m+1}$ is a manifold carrying an almost contact structure. A Riemannian metric $g$ on $\widetilde{M}^{2m+1}$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields $X$ and $Y$ is called compatible with the almost contact structure, and $(\widetilde{M}^{2m+1}, (\phi, \xi, \eta, g))$ is said to be an almost contact metric structure on $\widetilde{M}^{2m+1}$. It is known that an almost contact manifold always admits at least one compatible metric. Note that putting $Y = \xi$ yields

$$\eta(X) = g(X, \xi)$$

for all vector fields $X$ tangent to $\widetilde{M}^{2m+1}$, which means that $\eta$ is the metric dual of the characteristic vector field $\xi$.

A manifold $\widetilde{M}^{2m+1}$ is said to be a contact manifold if it carries a global one-form $\eta$ such that

$$\eta \wedge (d\eta)^m \neq 0$$

everywhere on $M$. The one-form $\eta$ is called the contact form.

A submanifold $M$ of a contact manifold $\widetilde{M}^{2m+1}$ tangent to $\xi$ is called an invariant (resp. anti-invariant) submanifold if $\phi(T_p M) \subset T_p M, \forall p \in M$ (resp. $\phi(T_p M) \subset T_p^\perp M, \forall p \in M$).
A submanifold $M$ tangent to $\xi$ of a Riemannian contact manifold $\tilde{M}^{2n+1}$ is called a contact CR-submanifold if there exists a pair of orthogonal differentiable distributions $D$ and $D^\perp$ on $M$, such that:

1. $TM = D \oplus D^\perp \oplus \mathbb{R}\xi$, where $\mathbb{R}\xi$ is the 1-dimensional distribution spanned by $\xi$;
2. $D$ is invariant by $\phi$, i.e., $\phi(D_p) \subset D_p, \forall p \in M$;
3. $D^\perp$ is anti-invariant by $\phi$, i.e., $\phi(D^\perp_p) \subset T^\perp_p M, \forall p \in M$.

Let $(\tilde{M}, \phi, \xi, \eta, \tilde{g})$ be a $(2n+1)$-dimensional contact manifold such that:

\[
\nabla_X\xi = -\phi X, \quad (\nabla_X\phi)Y = \tilde{g}(X, Y)\xi - \eta(Y)X
\]

where $\nabla$ is Levi-Chivita connection of $\tilde{M}$, then $\tilde{M}$ is called a Sasakian manifold.

A plane section $\pi$ in $T_pM$ is called a $\phi$--section if it is spanned by $X$ and $\phi X$, where $X$ is a unit tangent vector orthogonal to $\xi$. The sectional curvature of a $\phi$--section is called a $\phi$--sectional curvature. A Sasakian manifold with constant $\phi$--sectional curvature $c$ is said to be a Sasakian space form and is denoted by $\tilde{M}(c)$.

The curvature tensor of $\tilde{M}(c)$ of a Sasakian space form $\tilde{M}(c)$ is given by [2]

\[
\tilde{R}(X,Y)Z = \frac{c+3}{4} \{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y\}
\]

\[
-\frac{c-1}{4}\{\eta(Z)[\eta(Y)X - \eta(X)Y] + [\tilde{g}(Y,Z)\eta(X) - \tilde{g}(X,Z)\eta(Y)]\xi
\]

\[
-\tilde{g}(\phi Y, Z)\phi X + \tilde{g}(\phi X, Z)\phi Y + 2\tilde{g}(\phi X, Y)\phi Z
\]

for any tangent vector fields $X, Y, Z$ on $\tilde{M}(c)$.

**Definition 1.1.** Let $A$ be the shape operator of hypersurface $M$ in $\tilde{M}$ and the plane spanned by $\{\xi, U\}$ being invariant subspace of $A$ then the hypersurface $M$ is called a Hopf hypersurface of $\tilde{M}$.

**Definition 1.2.** Let $T$ to be a $(1,1)$ tensor field on the Riemannian manifold $M$. Then $T$ is called recurrent tensor field if $(\nabla_X T)Y = \omega(X)TY$ where $\omega$ is a one-form and $X, Y$ are vector fields on $M$.

**Definition 1.3.** Let $(M, g)$ to be a Riemannian manifold. Let $S_pM$ be the set of unit vectors in $T_pM$. That is

\[
S_pM = \{z \in T_pM \mid g(z, z) = 1\}.
\]

Then

\[
SM = \bigcup_{p \in M} S_pM = \{z \in TM \mid g(z, z) = 1\}
\]

is called the unit sphere bundle of $(M, g)$.

**Definition 1.4.** Let $(M, g)$ to be a Riemannian manifold and $z \in SM$. Then the restriction $R_z : z^\perp \to z^\perp$ of the linear map $R(\cdot, z)z$ to $z^\perp$ is called the Jacobi operator with respect to $z$, that is $R_z x = R(x, z)z$, where $x \in z^\perp$.

**Definition 1.5.** Let $M$ to be hypersurface in $\tilde{M}$ and for all $X$ in $D$ the Jacob operator with respect to $\xi$ be recurrent then the hypersurface $M$ is called $D$--Recurrent hypersurface of $\tilde{M}$.
2 Main Result

Let \((M, g)\) to be a real hypersurface tangent to \(\xi\) of Sasakian space form \(\overline{M}(c)\) and \(N\) be a unit normal vector field on \(M\). Then we have

\[
TM = D \oplus D^\perp \oplus \mathcal{R}_\xi
\]

where \(D\) is \(\phi\)-invariant subspace and \(D^\perp\) is one dimensional subspace where spanned by \(U = \phi(N)\) such that, it is orthogonal component of \(D\). If \(M\) be a \(D\)-Recurrent Hopf hypersurface of \(\overline{M}\), In the other words:

\[
(\nabla_X R_{\xi})Y = \omega(X) R_{\xi}(Y)
\]

for all \(X\) in \(D\) and \(Y\) in span \(\{\xi, \phi X\}\)^⊥ where \(\omega\) is a one-form on \(\overline{M}\).

**Theorem 2.1.** Let \(\overline{M}\) to be Sasakian space form and \(M\) be a a hopf hypersurfaces of \(\overline{M}\) where for all vector fields \(X\) and \(Y\) tangent to \(M\) applies to in the condition (1), then \(M\) is locally product of \(M' \times \gamma\), which \(M'\) is totally geodesic Sasakian space form and \(\gamma\) is a geodesic curve of \(M\).

References


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Lie superalgebras with integrable left invariant para-hypercomplex structures

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Abstract
In this paper, the author will manage to classify real Lie superalgebras $g = g_0 \oplus g_1$ admitting an integrable, left invariant, para-hypercomplex structure. An special metric of split signature $(2,2)$ will be given, which is very convenient in understanding the structure of $g = g_0 \oplus g_1$.

Keywords: Para hypercomplex structure, hypercomplex structure, complex product structure

Mathematics Subject Classification: Primary: 17B60; Secondary: 17B70, 17B30.

1 Introduction

Invariant hypercomplex structures on Lie algebras and Lie superalgebras are important from a geometric view point as well as from algebraic view point. For example, Snow [9] and Ovando [7] classified the invariant complex structures on 4-dimensional, solvable, simply-connected real Lie algebras. Recently, Andrada and Salamon [1] have shown that any para-hypercomplex structure on a real Lie algebra $g$ rise to a hypercomplex structure on its complexification $g^C$ (considered as a real Lie algebra).

Definition 1.1. (1) Let $V$ be a real supervector space. A complex structure on $V$ is an endomorphism $J_1$ of $V$ satisfying the condition $J_1^2 = -I$.

(2) A product structure on $V$ is an endomorphism $J_2$ of $V$ satisfying the conditions $J_2^2 = 1, J_2 \neq \pm 1$.

(3) A para-hypercomplex structure on $V$ is a pair $(J_1, J_2)$ of anti-commuting complex structure $J_1$ and product structure $J_2$, i.e. satisfying the relations $J_1^2 = -1, J_2^2 = 1, J_1 J_2 = -J_2 J_1$.

(4) If both structures $J_1$ and $J_2$ are complex then the pair $(J_1, J_2)$ is called a hypercomplex structure on $V$.

Existence of a complex structure implies that $V$ has to be of an even dimension. In the sequel, we concentrate on the case of para-hypercomplex structure.

It is customary to denote $J_3 = J_1 J_2$. Note that the structure $J_3$ is a product structure. The Lie subsuperalgebra of $\text{End}(V)$ spanned by $J_1$, $J_2$ and $J_3$ is isomorphic to $sl_2(R)$.

For any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, one can define a structure by the formula $J_x := x_1 J_1 + x_2 J_2 + x_3 J_3$. 

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Denote by \( <x, y> = x_1y_1 - x_2y_2 - x_3y_3 \) for \( x = (x_1, x_2, x_3), \ y = (y_1, y_2, y_3) \), an inner product in \( R^3 = R^{(1, 2)} \) and by
\[ x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) \]
the usual cross product.

The structure \( J_x \) is a complex structure provided that \( <x, x> = x_1^2 - x_2^2 - x_3^2 = 1 \), and it is a product structure if \( <x, x> = x_1^2 - x_2^2 - x_3^2 = -1 \). Hence, a para-hypercomplex structure \((J_1, J_2)\) defines a 2-sheeted hyperboloid \( S^- \) of complex structures and a 1-sheeted hyperboloid \( S^+ \) of product structure.

**Proposition 1.2.** If \((J_1, J_2)\) is a para-hypercomplex structure on a super vector space \( V \), then:

(i) \( J_xJ_y = -<xy > I + (-1)^{|x|}J_x \times y \).

(ii) The pair \((J_x, J_y)\) in \( S^- \times S^+ \) is a para-hypercomplex structure if and only if \( x \perp y \).

**Proof.** From the relations \( J_1J_2 = J_1 - J_2J_1, \ J_1J_3 = -J_2J_1, \ J_2J_3 = -J_1J_2\) the statement i) follows by a direct calculation. Since \( J_x \) is a complex structure and \( J_y \) is a product structure, the pair \((J_x, J_y)\) is a para-hypercomplex structure if and only if \( J_x \) and \( J_y \) anti-commute. Using the relation i) and the anti-commutativity of the cross product we have
\[ 0 = J_xJ_y + J_yJ_x = -2\langle x, y \rangle \cdot 1 \]
Hence, the statement ii) is proved.

\[ \square \]

## 2 Main results

There are so many results that we give some of them for instance.

**Proposition 2.1.** Let \((J_1, J_2)\) be an integrable para-hypercomplex structure on a Lie superalgebra \( g = g_0 \oplus g_1 \), then

(i) The product structure \( J_3 = J_1J_2 \) is integrable;

(ii) Any compatible para-hypercomplex structure \((J_x, J_y)\) is integrable.

In the case that \( g = g_0 \oplus g_1 \) has a non-trivial center, we have the following results.

**Theorem 2.2.** The restriction of an para-hypercomplex structure on the Lie superalgebra \( g = g_0 \oplus g_1 \) on \( g_0 \) is an para-hypercomplex structure on which as Lie algebra.

**Corollary 2.3.** The only non-solvable, real 4-dimensional Lie superalgebra admitting a para-hypercomplex structure is Lie superalgebras with even part \( R \oplus sl_2(\mathbb{R}) \).

In the case of solvable Lie superalgebra \( g = g_0 \oplus g_1 \) with \( \dim g_0 \leq 2 \) we have:

**Proposition 2.4.** Let \( g = g_0 \oplus g_1 \) be a Lie superalgebra, \( g_0' \) is the derivation of \( g_0 \). If \( \dim g_0' = 1 \) then \( g_0 = R \oplus h_3 \) or \( R \oplus aff(\mathbb{R}) \). If \( \dim g_0' = 2 \) then \( g_0 \) is the direct sum of \( h_3 \) and \( R \oplus aff(\mathbb{R}) \).

In the case of solvable Lie superalgebra \( g = g_0 \oplus g_1 \) with \( \dim g_0' = 3 \) we have:

**Theorem 2.5.** Let \( g = g_0 \oplus g_1 \) be a Lie superalgebra and let \( g_0 \) be a 4-dimensional solvable Lie algebra with non-trivial center and \( \dim g_0' = 3 \). If \( g \) admit a para-hypercomplex structure the \( g_0 \) it is Lie algebra \( \mathfrak{a}_4 \).

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Some Lie superalgebras with complex product structures

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Abstract
The aim of this paper is to study the real Lie supergroups which admit compatible hypercomplex and product structures, by means of associated Lie superalgebra. If $g = g_0 \oplus g_1$ is a Lie superalgebra with such a product structure then its complexification does admit a hypercomplex structure.

Keywords: Product structure, hypercomplex structure, complex product structure

Mathematics Subject Classification: Primary: 17B60; Secondary: 17B70, 17B30.

1 Introduction
A complex product structure on a Lie superalgebra is an analogue of a hypercomplex structure, i.e. a pair of anticommuting complex structures. Indeed, a complex product structure is a latent hypercomplex structure in the sense that a complex product structure on a Lie superalgebra $g = g_0 \oplus g_1$ determines a hypercomplex structure on the real Lie superalgebra $(g^C)_R = (g_0^C)_R \oplus (g_1^C)_R$ underlying $g^C$.

Definition 1.1. An almost complex structure on a Lie algebra $g$ is a linear endomorphism $J : g \to g$ satisfying $J^2 = -1$. If $J$ satisfies the condition $J[X,Y] = [JX,Y] + [X,JY] + J[JX,JY]$ for all $X,Y \in g$, we will say that $J$ is integrable and we will call it a complex structure on $g$. Note that the dimension of a Lie algebra carrying an almost complex structure must be even.

we define another kind of structure on a Lie algebra which is analogous to a complex structure

Definition 1.2. An almost product structure on $g$ is a linear endomorphism $E : g \to g$ satisfying $E^2 = 1$ (and not equal to $\pm 1$). It is said to be integrable if $E[X,Y] = [EX,Y] + [X, EY] + E[JX, JY]$ for all $X,Y \in g$. An integrable almost product structure will be called a product structure.

In this article we are interested in another structure on $g = g_0 \oplus g_1$ which arises from the combination of the two above. Namely, a complex product structure on a real Lie superalgebra $g$

Definition 1.3. A complex product structure on the Lie algebra $g$ is a pair $(J, E)$ of a complex structure $J$ and a product structure $E$ satisfying $JE = -EJ$.

we now define complex product structures on Lie superalgebras

Definition 1.4. Given $g = g_0 \oplus g_1$ a real Lie superalgebra, a complex structure on $g$ is endomorphism $J : g \to g$ satisfying the conditions:
(i) $J^2 = -(-1)^k$, $(k = 0, 1)$ and
(ii) $J[X,Y] = [JX,Y] + [X,JY] + J[JX,JY]$
Definition 1.5. A complex product structure on the Lie superalgebra $g$ is a pair $\{J, E\}$ of a complex structure $J$ and a product structure $E$ satisfying $JE = (-1)^{\alpha}EJ$ ($\alpha = 0, 1$)

One can see clearly that:

Theorem 1.6. The restriction of an complex product structures on the Lie superalgebra $g = g_0 \oplus g_1$ on $g_0$ is an complex product structures on which as Lie algebra.

1.1 Hypercomplex structures from complex product structures

In this paper we explain how a standard complex product structure on $gl(2n, \mathbb{R})$ gives rise to a hypercomplex structure on $gl(2n, \mathbb{C})$. In general, $(g^n)\mathbb{R} = (g_0^n)\mathbb{R} \oplus (g_1^n)\mathbb{R}$ also admits a complex product structure, so the process can be continued so as to obtain an infinite family of hypercomplex structures

Definition 1.7. A hypercomplex structure on the Lie algebra $g$ is a pair $J_1, J_2$ of complex structures on $g$ satisfying $J_1J_2 = -J_2J_1$. Note that $J_3 := J_1J_2$ is another complex structure on $g$ and $J_1, J_2, J_3$ satisfy

$J_1J_2 = J_3, \quad J_2J_3 = J_1, \quad J_3J_1 = J_2$

Definition 1.8. Given $g = g_0 \oplus g_1$ a real Lie superalgebra, a hypercomplex structure (abbreviated HCS) on $g$ is a family $\{J_k\}_{k=1,2}$ of endomorphisms of $g$ satisfying the conditions:

(i) $J_k^2X = -(-1)^kX, \quad X \in g_{\alpha}, (\alpha = 0, 1), (k = 1, 2)$ and

$J_1J_2X = -(-1)^{\alpha}J_2J_1(X), (\alpha = 0, 1)$

(ii) $N_k = 0, \quad (k = 1, 2)$,

where $I$ is the identity and $N_k$ is the Nijenhuis tensor corresponding to $J_k$ as:

$N_k(X, Y) = -(-1)^{\alpha\beta}([J_kX, J_kY] - J_k([X, J_kY] + [J_kX, Y] - [X, Y])$ ,

$X \in g_{\alpha}, Y \in g_{\beta}$, $\alpha, \beta = 0, 1$.

Throughout this paper, we shall use the notation

$\hat{g} = (g^C)\mathbb{R} = (g_0^n)\mathbb{R} \oplus (g_1^n)\mathbb{R}$.

We note that multiplication by $i$ in $g^C = g \otimes \mathbb{C}$ defines a complex structure $I$ on $g$ satisfying $[IX, Y] = I[X, Y]$ for all $X, Y \in g$, (1.1)

Now suppose that $g = g_0 \oplus g_1$ has dimension $2n$ and a complex structure $J$ set $J|g = J$ and extend the definition of $J$ to $I_g$ by $J(IgX) = I(JgX)$. It is easy to verify the integrability of $J$, using (1.1).

Theorem 1.9. If $g = g_0 \oplus g_1$ has a complex product structure then $\hat{g}$ has a hypercomplex structure $\{I, J\}$ with $I$ and $J$ as above.

Proof. We have a decomposition $g = g_+ \oplus g_-$, with $g_+$ and $g_- = Jg_+$ subalgebras of $g$. By complexifying, we obtain

$\hat{g} = (g_+ \oplus Ig_+) \oplus (g_- \oplus Ig_-)$, where $g_+ \oplus Ig_+$ and $g_- \oplus Ig_-$ are complex subalgebras of $\hat{g}$. By the lemma(1.1), the endomorphism $I$ defined by $I|g_+ \oplus Ig_+ = I, I|g_- \oplus Ig_- = -I$

is a complex structure on $\hat{g}$ Also, it is clear that $I$ and $J$ anticommute. \qed

2 Main results

Proposition 2.1. $gl(2n, \mathbb{R})$ has a complex product structure for all $n \geq 1$.

Proposition 2.2. There is no complex product structure $\{J, E\}$ on $sp(n, \mathbb{R})$ such that $Ja_+ = a_-$.

Complex product structures on 4-dimensional Lie superalgebras

In this case we shall consider some concrete examples of 4-dimensional Lie superalgebras carrying complex product structures.

Solvable case
Example 2.3. Let $g = g_0 \oplus g_1$ be the Lie superalgebra defined by $g_0 = \text{span}\{A, B, C, D\}$ with non-zero Lie bracket relations given by


$g$ lies in the class $A_2$ of the classification made in [26], with $\lambda = -1$. It follows that $g$ admits only one complex structure $J$, up to isomorphism, given by

$$JA = B, JC = D.$$ 

Consider the following subalgebras of $g$:

$$g_+ = \text{span}A - D, C, \quad g_- = \text{span}B + C, D.$$ 

Set $E|_{g_+} = 1, E|_{g_-} = -1$; the endomorphism $E$ of $g$ is clearly a product structure on $g$ which anticommutes with the complex structure $J$, giving rise then to a complex product structure on $g = g_0 \oplus g_1$. Thus, there exists an LSA structure on $g_+$ and $g_-$, which can be extended to a bilinear product on $g = g_0 \oplus g_1$. We already know that this product satisfies condition (10) but we will show that it does not satisfy (9). Denote by $L_v$ the endomorphism of $g$ given by left-multiplication with $v \in g$. Using equation (20), we obtain

$$L_{A-D}(A-D) = A-D, \quad L_{B+C}(A-D) = 2C,$$

$$L_{A-D}(C) = -C, \quad L_{B+C}(C) = 0,$$

$$L_{A-D}(B+C) = B+C, \quad L_{B+C}(B+C) = 2D,$$

$$L_{A-D}(D) = -D, \quad L_{B+C}(D) = 0.$$ 

and $L_C \equiv 0, L_D \equiv 0$. Now, setting $x = A-D, y = B+C, z = A-D$ in (9), we get

$$x.(y,z) - (x.y).z = -4C,$$

while

$$y.(x,z) - (y.x).z = 2C,$$

and hence (9) does not hold.

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\section*{Abstract}

In this paper we will consider the deformation theory of compact \(G\)-manifolds, where \(G = G_2\). We will prove that the moduli space of torsionfree \(G\)-structures is a smooth manifold.also proved smoothness of the moduli space on compact \(G\)-manifolds for any of the Ricci-flat holonomy groups \(G_2\) in a fairly uniform way. The arguments used here are geared to make it easier to generalise to the asymptotically cylindrical case.

**Keywords:** \(EAC\) manifolds, \(G_2\)-manifolds, cylindrical

\section{Introduction}

A way to obtain irreducible compact \(G_2\)-manifolds is by gluing a pair of noncompact \(G_2\)-manifolds which are asymptotically cylindrical. A manifold is said to have cylindrical ends if it is homeomorphic to a cylinder outside a compact piece. An asymptotically cylindrical manifold is a Riemannian manifold with cylindrical ends for which the metric is asymptotic to a product metric on the cylindrical ends. Asymptotically cylindrical manifolds are easier to work with than arbitrary non-compact manifolds. many analysis results for elliptic operators on compact manifolds can be generalised to statements about asymptotically translation-invariant elliptic operators acting on suitable spaces of sections on an asymptotically cylindrical manifold. In some arguments it is helpful to impose a stronger condition, requiring the manifold to be exponentially asymptotically cylindrical (EAC).Given a pair of \(EAC\) \(G_2\)-manifolds whose cylinders match one can form a generalised connected sum by truncating the cylinders after some large but finite length and gluing them together. If the neck length is sufficiently large then the \(EAC\) \(G_2\)-structures can be glued to form a torsion-free \(G_2\)-structure on the connected sum. This is a gluing construction for compact \(G_2\)-manifolds. Kovalev proves an \(EAC\) version of the Calabi conjecture to produce \(EAC\) Calabi-Yau 3-folds. By multiplying with circles reducible \(EAC\) \(G_2\)-manifolds are obtained, which can be glued to form irreducible compact \(G_2\)-manifolds of different topological types from those constructed by Joyce.

\textbf{Definition 1.1.} Let \(X^6\) be a compact manifold, and denote by \(i\) the R-coordinate on the cylinder \(X \times \mathbb{R}\). Let \(M\) be a Riemannian manifold with \(\text{HOL}(M) \subseteq H\) and \(\rho\) a representation of \(H\). The Lichnerowicz Laplacian on \(E_p\) is the formally self-adjoint operator

\[ \Delta_p = \nabla \ast \nabla - 2(D_\rho)^2(R) : \Gamma(E_p) \rightarrow \Gamma(E_p) \]

where \(\nabla\) is the connection on \(E_p\) induced by the Levi-Civita connection on \(M\).

\textbf{Definition 1.2.} A \(G_2\)-structure \(\varphi\) on \(X \times \mathbb{R}\) is cylindrical if it is translation-invariant and the associated metric is a product metric \(g_\varphi = g_x + dt^2\)
**Definition 1.3.** A manifold $M$ is said to have cylindrical ends if it is a union of two pieces $M_0$ and $M_\infty$ with common boundary $X$, where $M_0$ is compact, and $M_\infty$ is identified with $X \times \mathbb{R}^+$ by a diffeomorphism (identifying $\partial M_\infty$ with $X \times \{0\}$) $X$ is called the cross-section of $M$.

**Definition 1.4.** A tensor field or differential operator on $X \times \mathbb{R}$ is called translation invariant if it is invariant under the obvious $\mathbb{R}$-action on $X \times \mathbb{R}$.

**Definition 1.5.** An asymptotically cylindrical manifold is a Riemannian manifold with cylindrical ends for which the metric is asymptotic to a product metric on the cylindrical ends.

**Definition 1.6.** A metric $g$ on a manifold $M$ with cylindrical ends is said to be EAC if it is exponentially asymptotic to a product metric on $X \times \mathbb{R}^+$. An EAC manifold is a manifold with cylindrical ends equipped with an EAC metric.

**Proposition 1.7.** Let $M^7$ an EAC $G_2$-manifold with cross-section $X$. Then $H^2_0(X) = A^6_6 \oplus E^2_6$, $H^4_0(X) = A^4_6 \oplus E^4_6$ and the same are orthogonal. Furthermore

(i) $H^2_1(X) \to H^2_0(X)$, $[\alpha] \to * [\alpha]$ maps $A^6_6$ to $E^2_6$ and $E^2_6$ to $A^6_6$

(ii) $H^4_1(X) \to H^4_0(X)$, $[\alpha] \to [\alpha] \cup [\Omega]$ maps $A^4_6$ to $A^4_6$ and $E^4_6$ to $E^4_6$

(iii) $H^4_2(X) \to H^4_0(X)$, $[\alpha] \to [\alpha] \cup [t \omega^2]$ maps $A^4_6$ to $A^4_6$ and $E^4_6$ to $E^4_6$

**Proof:** (i) is obvious, since $*$ maps $A^m_6 \leftrightarrow E^{6-m}_6$.

$[\alpha] \to [\alpha] \cup [\Omega]$ is a bijection $H^1_1(X) \to H^0_0(X)$. It maps $A^1_6$ into $A^1_6$ and $E^1_6$ into $E^1_6$. It follows that $A^1_6 \to A^1_6$ and $E^1_6 \to E^1_6$ are both surjective and that $H^0_0(X)$ splits as $A^4_6 \oplus E^4_6$. $H^2_0(X)$ splits too by (i).

(iii) easily follows from (i) and (ii) in the same way.

**Lemma 1.8.** Let $M$ be a Ricci-flat EAC manifold:

(i) If $M$ has a finite normal cover homeomorphic to a cylinder then $M$ or a double cover of $M$ is homeomorphic to a cylinder

(ii) If $\pi_1(M)$ is infinite then $M$ has a finite cover $\tilde{M}$ with $b_1(\tilde{M}) > 0$

**Proof:** (i) If $M$ is a finite normal cover of $M$ homeomorphic to a cylinder then it is isometric to a product cylinder $Y \times \mathbb{R}$. $M$ is a quotient of $Y \times \mathbb{R}$ by a finite group $A$ of isometries. The isometries are products of isometries of $Y$ and of $\mathbb{R}$ (since they preserve the set of globally distance minimising geodesics $\{(y) \times \mathbb{R} : y \in Y\}$). The elements of $A$ have finite order, so they must act on the $\mathbb{R}$ factor as either the identity or as reflections. Therefore the subgroup $B \subseteq A$ which acts as the identity on $\mathbb{R}$ is either all of $A$, in which case $M$ is the cylinder $(\mathbb{R}^2 \times \mathbb{R}) \times \mathbb{R}$, or a normal subgroup of index 2, in which case $(\mathbb{R}^2 \times \mathbb{R}) \times \mathbb{R}$ is a cylindrical double cover of $M$.

(ii) Let $G_0 \subseteq \pi_1(M)$ be a nilpotent subgroup of finite index. $G_0$ is solvable, so the derived series $G_{i+1} = [G_i, G_i]$ reaches 1. Therefore there is a largest $i$ such that $G_i \subseteq \pi_1(M)$ has finite index. Let $M_\ell$ be the cover of $M$ corresponding to $G_i \subseteq \pi_1(M)$. $\frac{G_i}{G_{i+1}}$ is an infinite Abelian group, so has non-zero rank.

**Theorem 1.9.** Let $M_-$ be $M_+$ with its orientation reversed and $(\varphi_+, \varphi_-)$ a matching pair of $G_2$-structures. If $\varphi_+$ and $\varphi_-$ define the same metric then $M_+$ has a double cover isometric to a cylinder.

**Proof.** $\varphi_-$ is a torsion-free $G_2$-structure on $M_+$ which defines the same metric as $\varphi_+$. The matching condition for $\varphi_+$ and $\varphi_-$ implies that the parallel section is asymptotic to $[\frac{\partial}{\partial t}]$. In other words either $M_+$ or a double cover of $M_+$ has a parallel vector field asymptotic to $\pm [\frac{\partial}{\partial t}]$. Now this is impossible for a manifold with a single end, so $M_+$ has a double cover which is isometric to a cylinder. 

1.10. Let $M_{+}$ denote the moduli space of torsion-free EAC $G_2$-structures on $M_{+}$ and $N$ the moduli space of Calabi-Yau structures on their common cross-section $X$. We can define a subset $M_{+} \subseteq M_{+} \times M_{-}$ consisting of pairs which have matching images in $N$. While we can apply our
2 Main Result

if $C = C_5 + C_{14}$ is a skew-symmetric tensor, then the evolution of the skew-symmetric tensor $P(C)$ under the flow equation:

$$\frac{\partial}{\partial t} C_{ij} = h_{ij} \partial \varphi_{ijkl} + h_{ik} \partial \varphi_{ilj} + X^l \varphi_{ijkl} + X^i \varphi_{jkl} + X^j \varphi_{ilk} + X^k \varphi_{ijl}$$

is given by:

$$\frac{\partial}{\partial t} (P(C))_{ij} = (P(\frac{\partial}{\partial t} C))_{ij} + 6 \pi \{ (h, C_{14}) \}_{ij} - 6 \pi \{ (h, C_7) \}_{ij}$$

where $\pi$ and $\pi_4$ denote the projections onto $\Omega^2_2$ and $\Omega^4_2$ respectively.

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Some result about $G_2$-manifolds and its application in Soliton equation

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Abstract

we are going to show that on a compact 7-dimensional manifold which admits a $G_2$-structure soliton solutions to the Laplacian flow of R. Bryant can only be shrinking or steady. We also show that the space of symmetries (vector fields that annihilate via the Lie derivative) of a torsion-free $G_2$-structure on a compact 7-manifold is canonically isomorphic to $H^1(M, R)$. Some comparisons with Ricci solitons are also discussed, along with some future directions of exploration.

Keywords: $G_2$-manifolds, Soliton, symmetry

1 Introduction

Let M be a 7-dimensional manifold that admits a $G_2$-structure given by a non-degenerate 3-form $\varphi$. A natural geometric flow when M is compact is the Laplacian flow first suggested by R. Bryant in:

$$\frac{\partial \varphi}{\partial t} = -\Delta \varphi$$

for a family $\varphi = \varphi(t)$ of $G_2$ -structure, where $\Delta \varphi$ denotes the Hodge Laplacian with respect to the metric induced by $\varphi(t)$ The original intention of the equation (1) is to flow $\varphi$ to a torsion-free $G_2$ -structure, since $\varphi$ being torsion-free is equivalent to being harmonic with respect to the metric $g_\varphi$ nR it induces.

Definition 1.1. let $\varphi$ be a $G_2$-structure and X a vector field on M. We say that $(\varphi, x)$ is a Laplacian soliton if equation $\rho \varphi + L_x \varphi = -\Delta \varphi$ is satisfied for some constant $\rho \in \mathbb{R}$

Definition 1.2. Consider the differential 3-form $\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}$ in $\mathbb{R}^7$ where $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$. The group $G_2$ can be defined as the subgroup in $GL(7, \mathbb{R})$ that preserves $\varphi_0$. From a principal bundle point of view, a $G_2$-structure on a 7-dimensional manifold $M$ is just a sub-frame bundle over $G_2$, of the $GL(7, \mathbb{R})$- frame bundle over $M$.

Definition 1.3. We say that a $G_2$-structure is torsion-free if $\varphi$ solves the nonlinear system of partial differential equations $\nabla \varphi = 0$ where $\nabla$ is the covariant derivative induced by $g_\varphi$. It was shown that a $G_2$-structure is torsion free if and only if it is closed and co-closed. (with respect to the hodge star induced by $g_\varphi$)

Definition 1.4. A 7-manifold $M$ that admits a torsion-free $G_2$-structure has its Riemannian holonomy (with respect to $g_\varphi$) a subgroup of $G_2$, and such manifolds are simply known as $G_2$ -manifolds.
Proposition 1.5. Let $M$ be a compact $7$-manifold. For any $G_2$-structure $\varphi$ on $M$, vector field $X$, and $f \in C^\infty(M)$, we have: $\int_M L_X \varphi \wedge *f \varphi = -3 \int_M df \wedge i_X \varphi$

proof: We have $L_X \varphi = i_X d\varphi + d i_X \varphi$. From the decomposition of $d \varphi$ we see that
\[
i_X df \wedge *f \varphi = \tau_0 f(i_X * \varphi) \wedge * \varphi + 3 f(i_X (\tau_1 \wedge \varphi)) \wedge * \varphi + f(i_X * \tau_3) \wedge * \varphi
= 3 f(i_X (\tau_1 \wedge \varphi)) \wedge * \varphi + f(i_X * \tau_3) \wedge * \varphi
= -3 f(\tau_1 \wedge \varphi) \wedge (i_X * \varphi) + f * \tau_3 \wedge i_X * \varphi
= -3 f \tau_1 \wedge (i_X * \varphi)
= -3 f \tau_1 \wedge (-4 * i_X \varphi)
= -12 f \tau_1 \wedge i_X \varphi
\]
where we have used the identity $\varphi \wedge (i_X * \varphi) = -4 * i_X \varphi$ in the fifth equality and also the point-wise orthogonality of the $G_2$-decomposition of differential forms in the second and fourth equalities above. On the other hand, from the decomposition of $d * \varphi$ we have
\[
\int_M d(i_X \varphi) \wedge *f \varphi = \int_M i_X \varphi \wedge *d f \varphi
= -\int_M i_X \varphi \wedge d * f \varphi
= -\int_M i_X \varphi \wedge (df \wedge * \varphi + fd * \varphi)
= -\int_M i_X \varphi \wedge df \wedge * \varphi - \int_M f(i_X \varphi) \wedge (4 \tau_1 \wedge * \varphi + * \tau_2)
= -\int_M df \wedge * \varphi \wedge (i_X \varphi) - 4 \int_M f(i_X \varphi) \wedge \tau_1 \wedge * \varphi
= -\int_M df \wedge * \varphi \wedge (i_X \varphi) - 4 \int_M f \tau_1 \wedge * \varphi \wedge (i_X \varphi)
= -4 \int_M df \wedge 3 * i_X \varphi - 4 \int_M f \tau_1 \wedge 3 * i_X \varphi
= -3 \int_M df \wedge i_X \varphi - 12 \int_M f \tau_1 \wedge * i_X \varphi
\]
where we have also used the identity $* \varphi \wedge (i_X * \varphi) = 3 * i_X \varphi$. Integrating (1) and adding to (2), the lemma now follows.

Lemma 1.6. let $\varphi$ be a closed $G_2$-structure. Then $\varphi$ is an eigen-form if and only if $\Delta_\varphi \in \Omega^4$.

proof: The only if part is trivial. For the other direction, note that if $\Delta_\varphi = f \varphi$ then since $d \varphi = 0$ we must have $d(f \varphi) = 0$ as well. In other words, $df \wedge \varphi = 0$. Recall that the equation above, along with the special form that $\varphi$ takes with respect to a local orthonormal frame, shows in a straight-forward way that $f$ must be constant. Thus $\varphi$ must be an eigenform.

Theorem 1.7. If $M^7$ is compact, then there are no expanding or steady soliton solutions of $-\Delta d \psi + L_X \psi = d(i_X \psi)$ other than the trivial case of a torsion-free $G_2$-structure in the steady case.

Proof. We take the wedge product of both sides of $\Delta d \psi + L_X \psi = d(i_X \psi)$ with $\psi = d(i_X \psi)$ $\varphi = * \psi$ and integrate over $M$ to obtain $\int_M (\Delta d \psi, \psi) vol + \lambda \int_M ||i_X \psi||^2 vol + \int_M (d(i_X \psi), \psi) vol = 0$ Since $M$ is compact, we have $\int_M (d(i_X \psi), \psi) vol = \int_M (i_X \psi, df \psi) vol$ But the $G_2$-structure is coclosed, so $\tau_1 = 0$ and hence $\psi = i_X \psi = * d \psi = * d \varphi = * (\tau_0 \psi + * \tau_3) = \tau_0 + \tau_3$. Therefore $d \psi$ lies in the space $\Lambda^3 \oplus \Lambda^7$, so $i_X \psi$ lies in $\Lambda^7$. Since this decomposition of $\Lambda^3$ is pointwise orthogonal with respect to the metric $g$, we see that the last term in:
\[
\int_M (\Delta d \psi, \psi) vol + \lambda \int_M ||i_X \psi||^2 vol + \int_M (d(i_X \psi), \psi) vol = 0
\]
vanishes. Since $\langle (\Delta d \psi, \psi) \rangle + 7 \lambda \int_M vol = ||d \psi||^2 + 7 vol(M) = 0$ again using the fact that $d \psi = 0$. Thus we cannot have $\lambda > 0$ and if $\lambda = 0$ then the $G_2$-structure must be torsion-free. In the latter case $X$ must be a vector field generating a $G_2$-symmetry: $L_X \psi = 0$. Since $M$ is compact, there will be no such nonzero $X$ unless $M$ has reducible holonomy.
2 Main Result

the space of symmetries of \( \varphi \) is isomorphic to \( H^1(M, R) \). Proof. To prove the if part, we will employ the general relation below for the Levi-Civita connection:

\[
L_X \varphi(Y_1, Y_2, Y_3) = \nabla_X \varphi(Y_1, Y_2, Y_3) + \varphi(\nabla_{Y_1} X, Y_2, Y_3) + \varphi(Y_1, \nabla_{Y_2} X, Y_3) + \varphi(Y_1, Y_2, \nabla_{Y_3} X)
\]

for any vector fields \( X, Y_1, Y_2, Y_3 \). The torsion-free condition is defined by \( \nabla_X = 0 \). Furthermore, since any \( G_2 \)-manifold must have zero Ricci curvature everywhere, by the Bochner's Theorem we know that any harmonic 1-form must be parallel. Then we must have \( \nabla_X = 0 \). Combining these facts into above equation we get the desired result.

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Defect and area in Beltrami-Klein model of hyperbolic geometry

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Abstract
Ungar introduced into hyperbolic geometry the concept of defect based on relativity addition of A. Einstein and he defines the area of a triangle in Beltrami-Klein model which has not the additive property for a triangle (cf., [7]). Another approach is from Karzel for the relation between the K-loop and the defect of an absolute plane (cf., [2]) in the sense [1]. Our main concern is to introduce a systematical exact definition for defect and area in the Beltrami-Klein model of hyperbolic geometry by combining the ideas and methods of Karzel and Ungar which have the additive property.

Keywords: Hyperbolic Geometry, Beltrami-Klein Model, Defect, Area, Gyrogroup, K-loop

Mathematics Subject Classification: 51M09, 51M25

1 Introduction
In this paper let $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ denote the real, the complex numbers, the quaternions and the octonions respectively and let $K$ be one of $\mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. Then $(K, \mathbb{R})$ is a vector space and let $k := \dim(K, \mathbb{R})$ ($k$ is one of the values 2, 4 or 8). If $z \in K$ let $\bar{z}$ denote the conjugate element of $z$. Recall that $z \mapsto \bar{z}$ is an involutory antiautomorphism (for $\mathbb{C}$ even an automorphism) fixing exactly the elements of $\mathbb{R}$ and that the map $z \mapsto z\bar{z}$ is a positive definite form. We set:

$|z| := \sqrt{z\bar{z}}$, $S := \{z \in K \mid |z| = 1\}$, $D := \{z \in K \mid |z| < 1\}$ and if $z \in D$ let $||z|| := \sqrt{1 - z\bar{z}}$.

By the fundamental theorems of absolute geometry we know (cf. e.g. [1]):

1. Let $(P, G, \equiv, \zeta)$ be a classical absolute plane, then $(P, G, \equiv, \zeta)$ is either a Euclidean or hyperbolic plane.

2. All classical Euclidean planes are isomorphic to the following analytic model (where $K = \mathbb{C}$):

$P = K$, $G := \{a + Rb \mid a, b \in K, b \neq 0\}$, $(a, b) \equiv (c, d) \iff |b - a| = |d - c|$ and $\zeta(c \mid a, b) = -1$ (i.e. the point $c$ is between the points $a$ and $b$)

$\equiv \iff \exists \lambda \in [0, 1[\mid c = \lambda a + (1 - \lambda) b$. If $K = \mathbb{H}$ or $K = \mathbb{O}$ then we obtain by this definition an Euclidean space of dimension 4 or 8.

3. All classical hyperbolic planes are isomorphic to the Beltrami-Klein model $(P_h, G_h, \equiv_h, \zeta_h)$ which is defined inside the Euclidean plane: $P_h := D$ the set of points, $G_h := \{G \cap D \mid G \in G : G \cap D \neq \emptyset\}$, the set of lines, and the betweenness structure $\zeta_h$ is just the trace structure of the Euclidean plane.

By Theorems 2.1 and 4.1 of [6] we have:

**Theorem 1.1.** Let $a, b \in D$, $\|a\| := \sqrt{1 - aa}$, $<a, b> := \frac{1}{2}(ab + ba)$ and let $U$ be a line of the hyperbolic geometry with $o \in U$ then:
1. The reflection in the point \( a \) is given by:

\[
\tilde{a} : \mathbb{D} \rightarrow \mathbb{D} ; \; x \mapsto \frac{(a\bar{x} - 1)x + (2 - (a\bar{x} + \bar{x}a))a}{1 + a\bar{a} - (a\bar{x} + \bar{x}a)}.
\]

2. \( a' = \frac{1}{1 + \sqrt{1-a^2}} a = \frac{a}{1 + \|a\|^2} \) is the midpoint of \( a \) and \( a \).

3. \( \forall a, b \in U \quad a \oplus b = \frac{1}{1 + ab}(a + b) \).

4. If \( a, b \in \mathbb{D} \) then the midpoint \( m \) of \( a \) and \( b \) is given by:

\[
m = \frac{\|b\|_h}{\|a\|_h + \|b\|_h} a + \frac{\|a\|_h}{\|a\|_h + \|b\|_h} b.
\]

5. \( a \oplus b = \frac{a + b}{1 + \langle a, b \rangle} + \frac{1}{1 + \|a\|_h} \left( \frac{\langle a, b \rangle - a - |a|^2 b}{1 + \langle a, b \rangle} \right) \).

Which is well known as the Einstein’s velocity addition of \( a \) and \( b \).

6. \( (1 + \langle a, b \rangle) \|a \oplus b\|_h = \|a\|_h \|b\|_h \).

7. If \( a \perp b \) then \( a \oplus b = a + \|a\|_h b \) and \( \|a \oplus b\|_h = \|a\|_h \|b\|_h \).

**Theorem 1.2.** Let \( \mathbb{D} \) be the Beltrami-Klein model of hyperbolic geometry with Einstein’s velocity addition \( \oplus \). Then

1. \( (\mathbb{D}, \oplus) \) is a K-loop( or gyrocommutative gyrogroup)(e.g. cf. [6]), hence a non-associative loop which is defines the so called precession function \( \delta_{a,b} = ((a \oplus b)^+)^{-1} o a^+ o b^+ \), where \( a^+ \in Sym(\mathbb{D}) \) and \( a^+(x) = a \oplus x \) and satisfying the following properties:

(i) \( \delta_{a,b} \in Aut(\mathbb{D}, \oplus) \)

(ii) \( \delta_{a,b} = \delta_{b,a} \circ a \)

(iii) \( \circ (a \oplus b) = (\circ a) \oplus (\circ b) \)

2. For any \( a, b \in \mathbb{D}, \delta_{a,b}(b \oplus a) = a \oplus b \).

3. \( \delta_{a,b} \) coincides with the defect of the triangle \( \triangle(a, a, -b) \) ( cf. (4.4.1), [2]).

4. Let \( \triangle(a, b, c) \) be a triangle in \( \mathbb{D} \) with \( \gamma \) be the angle \( \angle(a, c, b) \). Then by DefinitionV of [7] we have:

\[
\cos(\gamma) = \frac{\langle \circ c \oplus a, \circ c \oplus b \rangle}{\|\circ c \oplus a\| \circ c \oplus b||}.
\]

**2 Main Result**

In this section we obtain a rigorous formula for defect and area of a triangle and area of a circle in the Beltrami-Klein model of hyperbolic geometry. Since we can move any triangle by a proper motion to origin, so it is enough to obtain the defect of a triangle with a vertex to be \( o \).

**Theorem 2.1.** Let \( \triangle(a, b, o) \) be a triangle in the Beltrami-Klein model of hyperbolic geometry with defect \( \delta \). Then

\[
\tan(\frac{\delta}{2}) = \frac{\langle a, b^\perp \rangle}{(1 + \|a\|_h)(1 + \|b\|_h) - \langle a, b \rangle}
\]

where \( b^\perp = ib, i = \sqrt{-1} \)(cf. [5]).
The following theorem is equivalent with the additive property of defect for triangles.

**Theorem 2.2.** Let $a, b, c \in \mathbb{D}$ be three collinear points, then $\delta_{a, -b} = \delta_{a, -c} \circ \delta_{c, -b}$ (cf. [5]).

Ungar defines in [7] the area for a triangle $\triangle(a, b, c)$ in the Beltrami-Klein model of hyperbolic geometry equal to $2 \tan(\frac{\delta}{2})$, where $\delta$ is the defect of $\triangle(a, b, c)$. But by this definition fails the additive property of area for triangles. We prefer to define the area for a triangle like most authors as follows:

**Definition 2.3.** The area of a triangle $\triangle(a, b, c)$ in the Beltrami-Klein model of hyperbolic geometry is the defect of $\triangle(a, b, c)$.

By this definition and theorem 2.2 we can obtain the following theorem:

**Theorem 2.4.** Let $C_r$ be a circle with radius $r$ in Beltrami-Klein model of hyperbolic geometry with circumference $P$ and area $S$. Then $P = \frac{2\pi r}{\sqrt{1-r^2}}$ and $S = \pi r^2 \frac{2}{\sqrt{1-r^2}}$ (cf., [5]).

It is now clear when $r \to 0$ then $\frac{1}{\sqrt{1-r^2}} \to 1$ and the formulas for area and circumference of a circle in hyperbolic plane tends to the classic formulas for area and circumference of a circle in Euclidean plane.

**References**


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On soft connected topological spaces

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Abstract

In this paper, we introduce and study the notions of soft connected topological spaces after a review of preliminary definitions.

Keywords: Soft connected, soft open set, soft closed set, soft topological space

Mathematics Subject Classification: 53C60, 53C25.

1 Preliminaries

In this section, we recall some definitions and concepts discussed in \cite{1, 2, 3, 4, 5, 6}. Let $U$ be an initial universe and $E$ be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of $U$ and $A$ be a nonempty subset of $E$. A pair $(F, A)$ is called a soft set over $U$, where $F$ is a mapping given by $F : A \rightarrow \mathcal{P}(U)$. For two soft sets $(F, A)$ and $(G, B)$ over common universe $U$, we say that $(F, A)$ is a soft subset $(G, B)$ if $A \subseteq B$ and $F(e) \subseteq G(e)$, for all $e \in A$. In this case, we write $(F, A) \subseteq (G, B)$ and $(G, B)$ is said to be a soft super set of $(F, A)$. Two soft sets $(F, A)$ and $(G, B)$ over a common universe $U$ are said to be soft equal if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$. A soft set $(F, A)$ over $U$ is called a null soft set, denoted by $\Phi_A$, if for each $e \in A$, $F(e) = \emptyset$. Similarly, it is called absolute soft set, denoted by $\bar{U}$, if for each $e \in A$, $F(e) = U$. The union of two soft sets $(F, A)$ and $(G, B)$ over the common universe $U$ is the soft set $(H, C)$, where $C = A \cup B$ and for each $e \in C$,

$$H(e) = \begin{cases} F(e) & e \in A \\ G(e) & e \in B \\ F(e) \cup G(e) & e \in A \cap B \end{cases}$$

We write $(F, A) \cup (G, B) = (H, C)$. Moreover, the intersection $(H, C)$ of two soft sets $(F, A)$ and $(G, B)$ over a common universe $U$, denoted by $(F, A) \cap (G, B)$, is defined as $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for each $e \in C$.

Let $Y$ be a nonempty subset of $X$. Then $\tilde{Y}$ denotes the soft set $(Y, E)$ over $X$ for each $Y(e) = Y$, where $e \in E$. In particular, $(X, E)$ will be denoted by $\tilde{X}$.

Let $(F, E)$ be a soft set over $X$ and $x \in X$. We say that $x \in (F, E)$, whenever $x \in F(e)$, for each $e \in E$.

The relative complement of a soft set $(F, A)$ is denoted by $(F, A)'$ and is defined by $(F, A)' = (F', A)$ where $F : A \rightarrow \mathcal{P}(U)$ is a mapping given by $F'(e) = U - F(e)$, for each $e \in A$.

Let $\tau$ be the collection of soft sets over $X$. Then $\tau$ is called a soft topology on $X$ if $\tau$ satisfies the following axioms:

(i) $\Phi_X, \tilde{X}$ belong to $\tau$.

(ii) The union of any number of soft sets in $\tau$ belong to $\tau$.

(iii) The intersection of any two soft sets in $\tau$ belong to $\tau$.

The triple $(X, \tau, E)$ is called a soft topological space over $X$. The member of $\tau$ are said to be
soft open in $X$, and the soft set $(F, E)$ is called soft closed in $X$ if its relative component $(F, E)'$ belongs to $\tau$.

Let $SS(X)_E$ be the collection of all soft sets with set of parameter $E$, over $X$. The cartesian product of soft sets $(F, A) \in SS(X)_A$ and $(G, B) \in SS(Y)_B$ is a soft set $(F \times G, A \times B)$ in $SS(X \times Y)_{A\times B}$ where $F \times G : A \times B \to \mathcal{P}(X) \times \mathcal{P}(Y)$ is a mapping given by $(F \times G)(a, b) = F(a) \times G(b)$ for each $(a, b) \in A \times B$.

\section{Main Results}

In this section, we are going to define some new concepts for soft topological spaces and study some properties related to these spaces.

Let $(X, \tau, E)$ be a soft topological space over $X$. A soft separation of $\bar{X}$ is a pair $(F, E), (G, E)$ of no-null soft open sets over $X$ such that

$$\bar{X} = (F, E) \cup (G, E), \quad (F, E) \cap (G, E) = \Phi_E.$$

A soft topological space $(X, \tau, E)$ is said to be soft connected if there does not exist a soft separation of $\bar{X}$.

**Theorem 2.1.** A soft topological space $(X, \tau, E)$ is soft connected if and only if the only soft sets in $SS(X)_E$ that are both soft open and soft closed over $X$ are $\Phi_E$ and $\bar{X}$.

Let $(U, \tau', A)$ and $(V, \tau', B)$ be soft topological spaces. Let $f_{pu} : SS(U)_A \to SS(V)_B$ be a function. Then $f_{pu}$ is said to be soft $pu$-continuous if for each $(F, B) \in \tau'$ we have $f_{pu}^{-1}(F, B) \in \tau$.

**Theorem 2.2.** Let $f_{pu}$ be a soft $pu$-continuous function carrying the soft connected space $(U, \tau, A)$ onto the soft space $(V, \tau', B)$. Then $(V, \tau', B)$ is soft connected.

Let $(F, E)$ be a soft set over $X$ and $Y$ be a nonempty subset of $X$. Then the sub soft set of $(F, E)$ over $Y$ denoted by $(Y^F, E)$, where $Y^F(e) = Y \cap F(e)$, for each $e \in E$. In other word $(Y^F, E) = \bar{Y} \cap (F, E)$.

Now, suppose that $(X, \tau, E)$ be a soft topological space over $X$ and $Y$ be a nonempty subset of $X$. Then $\tau_Y = \{ (Y^F, E) | (F, E) \in \tau \}$, is said to be the soft relative topology on $Y$ and $(Y, \tau_Y, E)$ is called a soft subspace of $(X, \tau, E)$.

**Theorem 2.3.** Let $(Y, \tau_Y, E)$ be a soft subspace of the soft space $(X, \tau, E)$. Then a soft set $(F, E)$ is soft closed over $Y$ if and only if it equals the intersection of a soft closed set over $X$ with $(Y, E)$.

**Corollary 2.4.** Let $(Y, \tau_Y, E)$ be a soft subspace of the soft space $(X, \tau, E)$. If $(A, E)$ is soft closed over $Y$ and $(Y, E)$ is soft closed over $X$, then $(A, E)$ is soft closed over $X$.

**Proposition 2.5.** If the soft sets $(F, E)$ and $(G, E)$ form a soft separation of $\bar{X}$, and $(Y, \tau_Y, E)$ is a soft connected subspace of $(X, \tau, E)$, then $\bar{Y}$ lies entirely within either $(F, E)$ or $(G, E)$.

**Theorem 2.6.** The union of a collection of soft connected subspace of $(X, \tau, E)$ that have non-null intersection is soft connected.

**Definition 2.7.** Let $(X, \tau, E)$ be a soft topological space over $X$ and $(F, E)$ be a soft set over $X$. Then the soft closure of $(F, E)$ denoted by $(\bar{F}, E)$ is the intersection of all soft closed super sets of $(F, E)$.

**Proposition 2.8.** Let $(X, \tau, E)$ be a soft topological space over $X$ and $(F, E)$ be a soft set over $X$. If $x \in (F, E)$, then every soft open set $(U, E)$ containing $x$ intersects $(F, E)$.

The following example shows that the converse of Proposition 2.8 is not true.
Example 2.9. Let

$$X = \{h_1, h_2, h_3\}, \ E = \{e_1, e_2\}, \ \tau = \{\Phi_E, \tilde{X}, (F_1, E), (F_2, E), \ldots, (F_{30}, E)\},$$

where $F_1, F_2, \ldots, F_{30}$ are given in Example 9 of [5]. Then $(X, \tau)$ is a soft topological space over $X$. We consider the soft set $(F_{25}, E)$, where

$$F_{25}(e_1) = \{h_2\}, \ F_{25}(e_2) = X.$$

It is easy to see that the following hold

$$(F_{25}, E) = (F_{25}, E), \ h_1 \notin (F_{25}, E).$$

But for every soft open set $(F, E)$ containing $h_1$ we have $(F, E) \cap (F_{25}, E) \neq \Phi_E$.

Theorem 2.10. Let $(Y, \tau_Y, E)$ be a soft connected subspace of $(X, \tau, E)$. If $(Y, E) \subseteq (Z, E) \subseteq (Y, E)$, then $(Z, \tau_Z, E)$ is also soft connected.

Remark: There are some differences between topological space and soft topological spaces. The following examples exhibit some of them.

Example 2.11. Let $X$ be a nonempty set, $E = \{e_1, e_2\}$ and $\tau = \{\Phi_E, \tilde{X}, (F_1, E), (F_2, E)\}$ where

$$F_1(e_1) = \emptyset, \ F_1(e_2) = X, \ F_2(e_1) = X, \ F_2(e_2) = \emptyset.$$

Then $(X, \tau, E)$ is a soft topological space and it is easy to see that $(F_1, E), (F_2, E)$ is a soft separation of $(X, E)$. Therefore a soft space $(X, \tau, E)$ with $|X| = 1$ can be soft disconnected.

Example 2.12. Let $X$ be a nonempty set, $Y = \{h\}, \ E = \{e_1, e_2\}, \ \tau_1 = \{(F_1, E), (F_2, E)\}$ and $\tau_2 = \{(G_1, E), (G_2, E), (G_3, E), (G_4, E)\}$ where, $F_1(e_1) = F_1(e_2) = X, \ F_2(e_1) = F_2(e_2) = \emptyset, G_1(e_1) = G_2(e_2) = G_3(e_1) = G_3(e_2) = Y, G_1(e_2) = G_2(e_1) = G_4(e_1) = G_4(e_2) = \emptyset$. Then $(X, \tau_1, E)$ and $(Y, \tau_2, E)$ are soft spaces. It is easy to see that the soft space $(X \times Y, E \times E)$ is not soft connected.

Theorem 2.13. Let $(X, \tau_1, E)$ and $(Y, \tau_2, E)$ be two soft connected topological spaces. Let each soft subset $(\{x\}, E)$ be soft connected as a soft subspace of $(X, \tau_1, E)$. Then the soft cartesian product of these two soft spaces is soft connected.

Question. At the end we pose a natural question here: Is the soft cartesian product of two soft connected spaces soft connected?

References


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Numerical Analysis and Differential Equations
Local well-posedness for the initial-value problem for the nonlinear Schrödinger equation

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Abstract
Let us consider the initial-value problem for the nonlinear Schrödinger equation
\[
\begin{aligned}
    iu_t &= -\Delta u + V(x)u + f(|u|^2)u, \\
    u(0) &= u_0,
\end{aligned}
\]  
(1)
where \(\Delta = \partial_{x_1}^2 + \ldots + \partial_{x_d}^2\) is the Laplacian in \(d\) dimensions, \(V(x) : \mathbb{R}^d \to \mathbb{R}\) is a bounded potential, and \(f(|u|^2) : \mathbb{R} \to \mathbb{R}\) is a real analytic function. The main question is: if \(u_0 \in H^s(\mathbb{R}^d)\) for some \(s \geq 0\), does there exist a unique local solution \(u(t)\) to the initial-value problem (1), which remains in \(H^s(\mathbb{R}^d)\) for all \(t \in [0, T]\) and, if it is so, does it extend globally for all times \(t \in \mathbb{R}_+\)? The classical theory of partial differential equations considers smooth solutions for large values of \(s \geq 0\). It was only recently this theory was extended to solutions of low regularity (for smaller values of \(s\)). The following theorem gives local well-posedness of initial-value problem (1) in \(H^s(\mathbb{R}^d)\) for \(s > \frac{d}{2}\).

Keywords: Nonlinear schrödinger equation, Local well-posedness.

Mathematics Subject Classification: 37K10

1 Introduction
The initial-value problem for a differential equation is said to be \textit{locally well-posed} in a Banach space \(X\) if for any initial data from \(X\) there exists a unique function in \(X\) over time interval \([0, T]\) for some \(T > 0\), which solves the differential equation in some sense and depends continuously on the initial data. If the maximal time is infinite, we say that the initial-value problem is \textit{globally well-posed}. If no solution exists for \(T > 0\), or the solution is not unique, or the solution is not continuous with respect to the initial-value problem is said to be \textit{ill-posed} [4]. Physicists have typically a very good intuition on whether the initial-value problem for a given equation is well-posed or ill-posed. Without making long arguments with physicists on the issue, we would like to explain a simple and rather universal method of how to prove local of the Gross-Pitaevskii equation. As first step of the Picard-Kato method [3], the differential equation is written in the integral form, e.g. by decomposing the vector field into linear and nonlinear parts, introducing the fundamental solution operator for the linear part, and using Duhamels principle for the nonlinear term. The integral equation is then shown to be Lipschitz continuous map from a local neighborhood of an initial point in a suitably chosen Banach space \(X\) to itself, which is also a contraction if a non-empty time interval \([0, T]\) is sufficiently small. Existence of a unique fixed point of the integral equation in a complete metric space \(C([0, T], X)\) follows by the Banach Fixed-Point Theorem [1]. This gives also continuous dependence on initial data in \(X\) and the existence of continuously differentiable solution in time on \([0, T]\) in a larger Banach space \(Y\) that enclosed \(X \subset Y\). To extend the maximal existence
time to $\infty$, if possible, and to prove global well-posedness, one needs to use other properties of the nonlinear equation such as the conserved quantities.

2 Analytical technique

**Step 1: Linear estimates.** We shall interpret the linear potential term $V(x)u$ and the nonlinear term $f(|u|^2)u$ as a perturbation to the linear evolution problem

$$
\begin{cases}
  iS_t = -\Delta S, \\
  S(0) = S_0
\end{cases}
$$

(2)

where $S_0 \in H^s(\mathbb{R}^d)$ for some $s \geq 0$. Since the linear Schrödinger equation has constant coefficients and is defined on an unbounded domain, it can be solved uniquely in $H^s(\mathbb{R}^d)dx$ by the Fourier transform. By the Fourier transform

$$
\hat{S}_0(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} S_0(x)e^{-ik \cdot x}dx, \quad k \in \mathbb{R}^d.
$$

The Fourier transform of the initial-value problem (2) gives

$$
\begin{cases}
  i\hat{S}_t = -|k|^2 \hat{S}, \\
  \hat{S}(0) = \hat{S}_0.
\end{cases}
$$

(3)

Let us denote the solution operator by $S(t) = e^{it\Delta}S_0$. By the Plancherel identity for the Fourier transform [2], the solution operator is unitary (norm-preserving) in the sense

$$
\|S(t)\|_{H^s} = \|e^{it\Delta}S_0\|_{H^s} = \|e^{-|k|^2} \hat{S}_0\|_{L^2} = \|\hat{S}_0\|_{L^2} = \|S_0\|_{H^s}, \quad s \geq 0.
$$

(4)

**Step 2: Duhamel’s principle.** Let us now write the initial-value problem (1) as the inhomogeneous equation

$$
\begin{cases}
  iu_t = -\Delta u + F(t), \\
  u(0) = u_0.
\end{cases}
$$

(5)

where $F(t) = V(t) + f(|u(t)|^2)u(t)$. We substitute $u(t) = e^{it\Delta}v(t)$ and note that $e^{it\Delta}$ is invertible and commutes with $\Delta$ for all $v \in H^s(\mathbb{R}^d)$ thanks to the norm preservation (4). As a result, we obtain

$$
v(t) = u_0 - i \int_0^t e^{-i\lambda \Delta}F(u(\lambda))d\lambda.
$$

Thus, the initial-value problem (1) can be written as the fixed-point problem for the integral equation

$$
u(t) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-\lambda)\Delta}(Vu(\lambda) + f(|u(\lambda)|^2)u(\lambda))d\lambda.
$$

(6)

**Step 3: Fixed-point iteration.** Let us recall the Banach Fixed-Point Theorem. We shall choose the Banach space $X = C([0,T], H^s(\mathbb{R}^d))$ for some $T > 0$ and $s \geq 0$ and closed non-empty set $M = B_\delta(X) \subset X$, the ball of radius $\delta > 0$ in $X$ centered at $0 \in X$ together with its boundary. Let $\mathcal{A}(u)$ denote the right-hand-side of the integral equation (6). We need to prove that $\mathcal{A}$ maps an element in $M$ to an element in $M$ and that is a contraction operator. In other words, we need to prove that

\begin{align*}
(i) \quad & \forall u(t) \in C([0,T], H^s(\mathbb{R}^d)) : \sup_{t \in [0,T]} \|u(t)\|_{H^s} \leq \delta \quad \Rightarrow \sup_{t \in [0,T]} \|\mathcal{A}(u(t))\|_{H^s} \leq \delta,
\end{align*}

(\text{for some } \delta > 0)
and there is $q \in (0, 1)$ such that

$$(ii) \ \forall u(t), v(t) \in C([0, T], H^p(\mathbb{R}^d)) : \quad \sup_{t \in [0, T]} \|u(t)\|_{H^p}, \sup_{t \in [0, T]} \|v(t)\|_{H^p} \leq \delta \Rightarrow \sup_{t \in [0, T]} \|A(u(t)) - A(v(t))\|_{H^p} \leq \delta.$$

Then the integral operator $A(u)$ in the integral equation (6) maps $B_\delta(C([0, T], H^p(\mathbb{R}^d)))$ to itself if $T > 0$ satisfies the inequality

$$T(C_v + C_u(\delta)) \leq \frac{1}{2},$$

(7)

Where $C_v, C_u(\delta)$ are obtained for $s > \frac{d}{2}$ because $H^p(\mathbb{R}^d)$ is a Banach algebra with respect to multiplication.

To achieve a contraction of the integral $A(u)$, if $T > 0$ satisfies another inequality

$$T(C_v + C_{uv}(\delta)) \leq 1,$$

(8)

where

$$C_{uv}(\delta) = \sum_{k=1}^{\infty} |f_k| C_s^{2k} \delta^{2k}.$$

By the Banach Fixed-Point Theorem, there exists a unique solution in the ball of radius $\delta > 0$ in $C([0, T], H^p(\mathbb{R}^d))$ with $s > \frac{d}{2}$, where $T > 0$ and $\delta > 0$ are chosen to satisfy the constraints (7) and (8) simultaneously. The continuous dependence from $u_0 \in H^p(\mathbb{R}^d)$ also follow the Banach Fixed-Point Theorem.

**Step 4: Differentiability in time.** If $u \in H^p$, then $\Delta u \in H^{p-2}$ for any $s \geq 0$. As result, if $u$ is a solution of the integral equation (6), then

$$-\Delta u + V u + f(|u|^2)u \in C([0, T], H^{p-2}(\mathbb{R}^d)).$$

Integrating the evolution equation (1) in time, we obtain $u(t) \in C([0, T], H^{p-2}(\mathbb{R}^d))$. The end points of $[0, T]$ can be included since $[0, T]$ constructed in step 3 is not maximal existence interval. Therefore, $u \in C([0, T], H^p(\mathbb{R}^d)) \cap C^4([0, T], H^{p-2}(\mathbb{R}^d))$.

### 3 Local well-posedness

By the four steps, we have

**Theorem 3.1.** Fix $s > \frac{d}{2}$ and let $u_0 \in H^p(\mathbb{R}^d)$. Assume that $V \in C^4(\mathbb{R}^d)$ and $f$ is analytic in variable $|u|^2$ with $f(0) = 0$. There exists a $T > 0$ and a unique solution $u(t)$ of the initial-value problem (1) such that

$$u(t) \in C([0, T], H^p(\mathbb{R}^d)) \cap C^4([0, T], H^{p-2}(\mathbb{R}^d)),$$

and $u(t)$ depends continuously on initial data $u_0$.

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Spectral collocation method for the numerical solution of the Fitzhugh-Nagumo equation

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Abstract
A numerical technique, the spectral collocation method based on Lagrange polynomials and Chebyshev polynomials, are applied to obtain solutions of the Fitzhugh-Nagumo equation. This method is one of the most effective method which applied for different kinds of nonlinear partial differential equations. The problems are reduced to a system of ordinary differential equation that are solved by RK45 method. The numerical results shows that, Chebyshev polynomials approximates solution with higher accuracy than Lagrange polynomials.

Keywords: Numerical solution, Spectral collocation method, Lagrange polynomials, Chebyshev polynomials.

1 Introduction
The nonlinear equation proposed by Hodgkin and Huxley is the most widely accepted mathematical description of the excitation and propagation of nerve impulses, that is

\[ u_t = u_{xx} + u(u - \alpha)(1 - u), \]  

with the initial condition

\[ u(x, 0) = f(x), \quad x \in D \]  

where \( D = \{ x : a < x < b \} \), \( \delta D \) is its boundary and \( \alpha \) is an arbitrary constant. This generally called FitzHugh- Nagumo (FN) equation.

In this paper, we aim to employ the Lagrange spectral collocation method and Chebyshev spectral collocation (ChSC) method for solving FitzHugh- Nagumo equation. The method is accomplished through, starting with Lagrange and Chebyshev approximation for the approximate solution and generating approximations for the higher-order derivatives through successive differentiation of the approximate solution. This method has introduced by Khater et al.[2] for solving Burgers’-type equations.

2 Spectral collocation method
Spectral collocation methods have become increasingly popular for solving differential equations and also they are very useful in providing highly accurate solutions to differential equations. In
the matrix version of the collocation method, the solution \( u \) is approximate as

\[
u(x, t) = \sum_{n=0}^{N} L_n(x)u_n(t), \tag{3}\]

where \( u_n(t) = u(x_n, t) \) and the \( L_n(x) \) are the Lagrange polynomials of degree \( N \) or Chebyshev polynomials. The derivatives of the approximate solution \( u \) are then estimated at the collocation points by differentiating (7) and evaluating the resulting expression, see [2].

The first three derivatives of approximate solution \( u \) at the Chebyshev-Gauss-Lobatto points in the interval \([a, b]\)

\[
x_n = \frac{1}{2} \left( (a + b) - (b - a) \cos \left( \frac{\pi n}{N} \right) \right), \quad n = 0, 1, \ldots, N,
\]

are given

\[
u_x(x_i, t) = d_1(t) + \sum_{n=1}^{N-1} [A_x]_{in}u_n(t),
\]

\[
u_{xx}(x_i, t) = \bar{d}_1(t) + \sum_{n=1}^{N-1} [B_x]_{in}u_n(t),
\]

\[
u_{xxx}(x_i, t) = \tilde{d}_1(t) + \sum_{n=1}^{N-1} [C_x]_{in}u_n(t), \tag{4}\]

where

\[
d_1(t) = D_0^1 u_0(t) + D_N^1 u_N(t),
\]

\[
\bar{d}_1(t) = D_0^1 u_0(t) + D_N^1 u_N(t) + [B_x]_{00} u_0(t) + [B_x]_{NN} u_N(t),
\]

\[
\tilde{d}_1(t) = D_0^2 u_0(t) + D_N^2 u_N(t) + [C_x]_{00} u_0(t) + [C_x]_{NN} u_N(t).
\]

With substituting (4), into Fitzhugh-Nagumo equation equation (13), we obtain

\[
\dot{u}(t) = F(t, u(t)), \tag{5}
\]

\[
u(0) = u_0.
\]

Eq.(11) forms a system of ordinary differential equations (ODEs) in time. Therefore to advance the solution in time, we use ODE solver such as the RK45 because it is an explicit method which gives a good accuracy and extends trivially to nonlinear.

3 Numerical results

In this section we apply the Lagrange spectral collocation method and Chebyshev spectral collocation method to FitzHugh- Nagumo equation. we provide a table of results including the absolute error between exact solution and result obtained by the present methods.

Example 3.1. Consider the FitzHugh- Nagumo equation

\[
u_t = \nu_{xx} + u(u - 2)(1 - u), \tag{6}\]
with the exact solutions

\[ u(x,t) = \frac{1}{2} \left( 1 - \cosh\left(\frac{-x}{2\sqrt{2}} + \frac{3t}{4} + c\right) \right). \quad (7) \]

Where \(c\) is an arbitrary constant.

In Table 4, we give the absolute error between the exact solution and the result obtained by the present methods for \(a = -5, b = 5, \alpha = 2, N = 10\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(t)</th>
<th>Chebyshev</th>
<th>Lagrange</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>5.70E-7</td>
<td>5.43E-3</td>
</tr>
<tr>
<td>1.05</td>
<td>1</td>
<td>1.24E-7</td>
<td>3.70E-3</td>
</tr>
<tr>
<td>4.06</td>
<td>1</td>
<td>3.48E-8</td>
<td>3.45E-3</td>
</tr>
<tr>
<td>5.02</td>
<td>1</td>
<td>2.65E-9</td>
<td>2.46E-3</td>
</tr>
</tbody>
</table>

4 Conclusions

In this paper the spectral collocation methods are extended to obtain numerical solutions for FitzHugh–Nagumo equation in a bounded domain. Using these methods the problem is reduced to a system of ODEs that are solved by RK45 method. The numerical results shows that, Chebyshev polynomials approximates solution with higher accuracy than Lagrange polynomials.

References


An improved formulation for the dual reciprocity boundary element method

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Abstract
We present an improved formulation for the dual reciprocity method (DRM) which is a generalized boundary element method (BEM) for a class of PDEs. The classical formulation for the BEM for non-homogeneous equations involves some matrix operations including evaluation of an inverse matrix. In the proposed formulation, the computational efficiency of the method is improved by avoiding the matrix inversion.

Keywords: Time dependent equations, Boundary element method, Dual reciprocity method.

Mathematics Subject Classification: 65N38

1 Introduction
The dual reciprocity method is a generalization of the BEM which has been widely used solving non-homogeneous PDEs [1]. In this method, the PDE is split into two parts: the homogeneous part which is treated by the classical BEM and the non-homogeneous part which is considered by a generalization of the particular solution method [2]. Despite the good performance of the DRM, the formulation of the method involves multiplication of some full matrices and evaluation an inverse matrix [4]. There has been some attempt to remove the matrix inversion to improve the computational efficiency of the method [3]. In the present work, we describe a new formulation for DRM without using the inverse matrix. While the method is computationally improved, the accuracy of the numerical results is preserved.

2 Dual Reciprocity Method
We describe the method for the PDE
\[ \nabla^2 T(X,t) = \frac{1}{k} \frac{\partial T(X,t)}{\partial t}, \quad X \in \Omega, \quad t > 0, \quad k = \frac{\lambda}{\rho c} \]
with an initial condition and a Neumann boundary condition \( \frac{\partial T}{\partial n} = \bar{q}(t) \), where \( \bar{q}(t) \) is a known function and \( k \) is the thermal diffusivity.

The classical BEM often deals with the above equation by applying the fundamental solution of the Laplace operator and the other part of equation is approximated by some radial basis functions (RBFs), \( \phi^j(X) \) as
\[ T(X,t) \approx \sum_{j=1}^{L+N} \phi^j(X)\alpha^j(t), \]

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where \( N \) and \( L \) are the number of boundary and internal points, respectively. Differentiating of equation (2) with respect to time gives

\[
\dot{T}(X, t) \approx \sum_{j=1}^{L+N} \phi^j(X) \dot{\alpha}^j(t),
\]

where the dots indicates the derivatives. Let the functions \( \psi^j(X) \) can be found such that the \( \nabla^2 \psi^j(X) = \phi^j(X) \), then equation (3) can be rewritten

\[
\dot{T}(X, t) \approx \sum_{j=1}^{L+N} \nabla^2 \psi^j(X) \dot{\alpha}^j(t) = \nabla^2 \left( \sum_{j=1}^{L+N} \psi^j(X) \dot{\alpha}^j(t) \right),
\]

which means that \( \sum_{j=1}^{L+N} \psi^j(X) \dot{\alpha}^j(t) \) is a particular solution of the main equation.

Substituting equation (3) into (1) leads to

\[
\nabla^2 T(X, t) = \sum_{j=1}^{L+N} \nabla^2 \psi^j(X) \dot{\alpha}^j(t)
\]

(5)

We know apply the BEM to the above equation, that is, multiplying both sides by the fundamental solution of the Laplace equation, \( T^* \). For the two-dimensional case \( T^* = \frac{1}{2\pi} \log(\frac{1}{r}) \) and for the three dimensional case \( T^* = \frac{1}{4\pi r} \).

Using the Green’s theorem, we obtain the following integral equation,

\[
c(X_i)T(X_i, t) + \int_{\Gamma} q^*(X_i, X)T(X, t)d\Gamma(X) - \int_{\Gamma} T^*(X_i, X)q(X, t)d\Gamma(X) = \\
= \sum_{j=1}^{L+N} \frac{1}{k} \left( c(X_i)\psi(X_j) - \int_{\Gamma} T^*(X_i, X)\eta^j(X)d\Gamma(X) \right) \dot{\alpha}^j(t),
\]

where \( X_i \) is the source point located on the boundary and \( \eta^j(X) \) is the normal derivative of \( \psi^j(X) \) and \( c(X_i) \) is a constant depending on the geometry. This equation is reduced to the linear system of equations

\[
HT - GQ = \frac{1}{k} (H\psi - G\eta) \dot{\alpha},
\]

(6)

where \( T = F\dot{\alpha} \) or \( \dot{\alpha} = F^{-1}T \) and \( F_{ij} = \phi^j(X_i) \). After imposing the boundary conditions and transferring the known values of \( T \) and \( q \) to the right side, a simple linear system of equations is obtain whose solution is nodal values of \( T \) and \( q \) on the boundary. As is observed, the right hand side of the above system includes an inverse matrix multiplied by some full matrices. This decreases the computational efficiency of the method. We know explain a method to avoid constructing the inverse matrix.

### 3 Formulation without matrix inversion

We explain the method for the case of a Neumann boundary condition. In this case the unknown values of the linear system (6) includes only the values of \( T \). Transferring the known value to the right side, we obtain

\[
HT = f(t) + C\dot{\alpha}
\]

(7)
where \( f(t) = Gq \) and \( C = \frac{1}{k} (H\psi - G\eta) \). Substituting \( T = F\dot{\alpha} \) into (7) leads to a set of differential equations with respect to the time dependent \( \alpha \), that is,

\[
D\alpha = f(t) + C\dot{\alpha},
\]

where \( D = HF \). We now use a time marching process as

\[
D(\theta \alpha_{m+1} + (1 - \theta)\alpha_m) = f(\theta t_{m+1} + (1 - \theta)t_m) + C(\alpha_{m+1} - \alpha_m) \frac{1}{\Delta t}
\]

(8)

where \( m \) and \( m+1 \) represent two subsequent steps and \( \Delta t \) is the time step length. This equation can be solved recursively. In each step, first, the values of \( \alpha_m \) is obtain and then the unknown function \( T \) is evaluated by \( T_{m+1} = F\dot{\alpha}_{m+1} \).

The formulation of the method for the case of Dirichlet boundary condition is different and will be presented later.

4 Numerical example

We consider the PDE

\[
\frac{\partial u(x,t)}{\partial t} = \nabla^2 u(x,t), \quad x \in \Omega
\]

(9)

where \( \Omega = \{(x,y) : 0 < (x,y) < 1\} \). A Neumann boundary condition is employed. The exact solution for this equation is given by

\[
u(x,y,t) = e^{-2t}\sin(x)\cos(y).
\]

We solve this equation by two methods, standard method and the proposed method. The numerical error is presented in Table 1. As is observed, the new method produces the same accuracy as that of the standard one, while in the new method the inverse matrix is not constructed and this considerably improves the efficiency of the method.

Table 1: The numerical errors of two methods are compared

<table>
<thead>
<tr>
<th>Standard method</th>
<th>Proposed method</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.8069 × 10^{-1}</td>
<td>3.8064 × 10^{-1}</td>
<td>0.05</td>
</tr>
<tr>
<td>2.8202 × 10^{-1}</td>
<td>2.8079 × 10^{-1}</td>
<td>0.2</td>
</tr>
<tr>
<td>2.0893 × 10^{-1}</td>
<td>2.0689 × 10^{-1}</td>
<td>0.35</td>
</tr>
<tr>
<td>1.7105 × 10^{-1}</td>
<td>1.6826 × 10^{-1}</td>
<td>0.45</td>
</tr>
</tbody>
</table>

References

A hybrid iterative method for large non-Hermitian linear systems

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Abstract
Many problems in scientific computing give rise to large scale linear systems of algebraic equations $Ax = b$ and plenty of iterative solution schemes are presented for solving large non-Hermitian systems. In this paper, we present an algorithm to improve efficiency and convergence of the solution of such systems. The algorithm is based on a hybrid scheme in which a few steps of GMRES are followed by a Richardson iteration employing an implicitly polynomial constructed by GMRES, with the factors ordered in a Leja sequence for stability.

Keywords: Hybrid method, Krylove subspace, CGNR, GMRES, CGS, Stability.

Mathematics Subject Classification: 65F10

1 Introduction
We consider a large non-Hermitian system of linear equations

$$Ax = b, \quad A \in \mathbb{C}^{N \times N}, \quad x, b \in \mathbb{C}^N.$$  \hspace{1cm} (1)

Let $x_0$ denote the approximation to the solution $x = A^{-1}b$ at the beginning of an iterative process. For sake of simplicity, we use the notation, $x_n$, $e_n = A^{-1}b - x_n$ and $r_n = Ae_n = b - Ax_n$ for the solution, error and residual at nth iteration, respectively.

The hybrid GMRES algorithm includes two phases. In the phase I, we use the GMRES method until $\|r_n\|$ drops by a suitable amount while in the phase II we apply this GMRES polynomial one or more times by means of a cyclic Richardson iteration

$$p_{kv}(z) = [p_n(z)]^k, \quad k = 1, 2, ...$$

The hybrid GMRES algorithm delivers phase II iterates $x_n$ satisfying

$$x_n = x_0 + q_{n-1}(A)r_0, \quad q_{n-1} \in P_{n-1}$$  \hspace{1cm} (2)

and

$$e_n = p_n(A)e_0, \quad r_n = p_n(A)r_0, \quad p_n \in P_n, \quad p_n(0) = 1,$$

here $p_n(z) = 1-zq_{n-1}(z)$ known as residual polynomials. Our goal is to make $\|p_n(A)r_0\|$ reasonably small.

The recurrence of the form

$$x_j = x_{j-1} + \tau_{j-1}r_{j-1} = x_{j-1} + \tau_{j-1}(b - Ax_{j-1}), \quad j = 1, 2, ..., n,$$  \hspace{1cm} (3)
is known as a Richardson iteration where \( \tau_0, \ldots, \tau_{n-1} \) are a cycle of iteration parameters and \( n \) is the period. If \( x \) is the solution of (1), an elementary algebraic manipulation along with (3), yields

\[
e_n = \prod_{j=1}^{n} (I - \tau_{j-1} A) c_0 = p_n(A)c_0,
\]

By induction based on the iteration (3), we have \( x_j - x_0 \in \langle r_0, \ldots, A^{j-1} r_0 \rangle \). Therefore \( x_j - x_0 = (a_0 + a_1 A + \ldots + a_{j-1} A^{j-1}) r_0 = q_{j-1}(A) r_0 \). Multiplying \( x_j - x_0 = x_j - x + x - x_0 = q_{j-1}(A) r_0 \) by \( A \) gives \( r_j = A q_{j-1}(A) r_0 \). Since \( r_j = A v_j = A p_j(A) c_0 = p_j(A) r_0 \) it follows that \( q_{j-1}(z) = (1 - p_j(z)) / z \) is a polynomial with \( p_j(0) = 1 \).

The Arnoldi/GMRES [1] process gives an orthonormal basis \( V_n = (v_1, v_2, \ldots, v_n) \) of the Krylov subspace \( K_n(A,r_0) = \langle r_0, A r_0, \ldots, A^{n-1} r_0 \rangle \). Then, we apply the iterative formula

\[
v_{n+1} = h_{n+1}^{-1}(A v_n - V_n h_n), \quad h_n = (h_{1n}, \ldots, h_{nn})^T,
\]

where the numbers \( h_{ij} \) are the element of a Hessenberg matrix of inner products. Since \( K_n(A,r_0) = \text{span}(v_1, v_2, \ldots, v_n) \), we get \( V_n = K_n C_n \), for some upper triangular matrix \( C_n \). We then generate the elements of \( C_n \) column by column in the GMRES iteration,

\[
\begin{pmatrix}
  c_{1,n+1} \\
  \vdots \\
  c_{n+1,n+1}
\end{pmatrix}
= h_{n+1}^{-1}
\begin{pmatrix}
  c_{1,n} \\
  \vdots \\
  c_{n,n}
\end{pmatrix}
- h_{n+1,n}^{-1} \begin{pmatrix}
  C_n h_n \\
  0
\end{pmatrix}.
\]

Having solved a Hessenberg least-squares problem at step \( n = v \), GMRES produces an iterate \( x_v \) of the form \( x_v = x_0 + V_v y \), for some vectors \( y \) of dimension \( v \). Due to \( V_v y = K_v C_v y \), it follows that the vector \( C_v y \) contains the coefficients of the polynomial \( q_{v-1}(z) = a_0 + a_1 z + \ldots + a_{v-1} z^{v-1} \), i.e., \( (a_0, \ldots, a_{v-1})^T = C_v y \). Since \( p_v(z) = 1 - z q_{v-1}(z) \), this gives us the coefficients of \( p_v(z) \) as well.

Phase I is completed and we computed \( q_{v-1}(z) \) and \( p_v(z) \) implicitly in the GMRES. We now face the question of how best to re-apply these polynomials for the further iterations of phase II. Many ideas have been advanced for this phase of a hybrid algorithm, one of the simplest one is the Horner iteration of Elman and Streit [4]. Alternatively, we prefer to factor \( p_v(z) \) numerically, i.e.,

\[
p_v(z) = \prod_{i=1}^{v} (1 - z / \xi_{i-1}), \quad (4)
\]

and then carry out a first-order Richardson iteration with \( \tau_{j-1} = 1 / \xi_{j-1} \)

\[
x_j = x_j - 1 / \xi_{j-1}, \quad j = 1, 2, \ldots, v.
\]

The factorization (4) offers a choice of the order in which to label the roots and this ordering is important for stability [3]. Leja point is defined by the condition

\[
|\xi_j| \prod_{i=0}^{j-1} |\xi_i - \xi_j| = \max_{j \leq i} \prod_{i=0}^{j-1} |\xi_i - \xi_j|,
\]

\( j = 0, 1, \ldots, v - 2 \).

The Richardson iteration with Leja ordering also has the appealing property since the polynomial tends to decrease steadily in norm.

The crucial element for successful application of the product hybrid GMRES algorithm revolves around the choice of \( v \). The same strategy for choosing \( v \) for the hybrid GMRES algorithm is suggested here:

\[
\text{Goal: equal amounts of work in phase I and phase II.} \quad (6)
\]
Where the work is measured by vector operations and a vector operation is defined to be the cost of an "axpy" operation $ax + y$ involving a scalar $a$ and $n$-vector $x$ and $y$. Suppose in phase I the residual has been reduced by the factor of $\|r_v\|/\|r_0\| = \sigma$ and our desired accuracy is $\|r_{final}\|/\|r_0\| = \varepsilon$, $(\varepsilon \leq \sigma)$. Assume that one matrix-vector multiplication costs $\delta$ vector operation for some $\delta > 0$. Therefore phase I work is $v(v + 3 + \delta)$ vector operations and phase II work is $v(1 + \delta)(\frac{\log \varepsilon}{\log \sigma} - 1)$ vector operations. The condition (6) can be realized by equating the phase II work and phase I work, i.e., $(v + 3 + \delta) = (1 + \delta)(\frac{\log \varepsilon}{\log \sigma} - 1)$.

2 Main Result

If $A$ is real, then it is reasonable to assume that the coefficients of $p_v(z)$ are real. If so, then the roots of $p_v(z)$ occur in complex conjugate pairs. It is shown that for a real problem (1) the iteration (5) can be carried out with real arithmetic alone, even when there are complex $\xi_j$, see [2],[7] for details. We provide experimental results of using the hybrid GMRES algorithm to solve (1) and compare its performance with some existent methods. Each of our experiments compares four algorithms, namely,

1. Hybrid GMRES (solid curves),
2. Restarted GMRES($v$) (solid curves),
3. CGN1 (dashed curves),
4. CGS2 (dots).

Example 2.1. Consider the tridiagonal Toeplitz matrix.

$$A = \begin{pmatrix} 5.1 & 3 & & & \\
2 & 5.1 & 3 & & \\
2 & 5.1 & 3 & & \\
& 2 & 5.1 & 3 & \\
& & 2 & 5.1 & \\
\end{pmatrix} (1000 \times 1000).$$

The convergence tolerance is $\varepsilon = 10^{-5}$, and the right-hand side $b$ and the initial guess $x_0$ are random real vectors with independent normally distributed elements.
The first plot shows $\log_{10}\|r_n\|$ as a function of the iteration number $n$ and the second plot shows $\log_{10}\|r_n\|$ as a function of work measured by vector operations. In this example the convergence of the Richardson iteration of Phase II is disappointing. Nevertheless, the second plot reveals that the hybrid iteration is the fastest.

References


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Operational matrix approach for solving nonlinear stochastic differential equations

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Abstract
An efficient numerical method is proposed for solving nonlinear stochastic differential equations, using operational matrix of block pulse functions. By using this approach, the stochastic differential equation reduces to a nonlinear system of algebraic equations which can be solved by Newton’s iterative method. Accuracy and efficiency of the method are shown with an example.

Keywords: Block pulse function, Brownian motion, Itô integral, Operational matrix, nonlinear stochastic differential equation.

Mathematics Subject Classification: Primary: 65C30, 60H35, 65C20; Secondary: 60H20, 68U20

1 Introduction
In many fields of science and engineering there are a large number of problems which are mathematically modeled by stochastic differential equations[1,2]. The topic of our study will be the following stochastic differential (SD) equation, in integral form as follows

\[ x(t) = x_0 + \int_0^t k_1(t, s)[x(s)]^\alpha ds + \int_0^t k_2(t, s)[x(s)]^\beta dB(s), \quad x_0 = x(0), \quad t \in [0,1), \]

in which, \( B(t), t \geq 0 \) is a Brownian motion and \( x_0 \) is a random variable independent of \( B(t) \), also, stochastic process \( x \) is a strong solution of Eq.(1), it is adapted to \( \{\mathcal{F}_t, t \geq 0\} \), furthermore, all Lebesgue’s and Itô’s integrals in the integral form of Eq.(1) are well defined[2]. \( \alpha, \beta \) are positive integers.

In the present paper, by using operational matrix and stochastic operational matrix the problem reduces to a nonlinear system of algebraic equations, which can be solved by Newton’s iterative method.

2 A brief review of block pulse functions
Block pulse functions(BPFs) have been applied for solving different problems[3]. In this paper, it is assumed that \( T=1, \) so BPFs are defined over \([0,1), \) and \( h = \frac{1}{m}. \) We define m-set of BPFs as

\[ \phi_i(t) = \begin{cases} 1 & (i-1)h \leq t < ih, \\ 0 & otherwise, \end{cases} \quad i = 1, \ldots, m, \]
From the definition of BPFs, it is clear that they are disjoint, orthogonal, and complete. Let

\[ \Phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_m(t))^T, \quad t \in [0, 1), \]

so,

\[ \Phi(t)\Phi^T(t)v = \tilde{v}\Phi(t), \]  \hspace{1cm} (2)

where, \( v \) is an \( m \)-vector, \( \tilde{v} \) is \( m \times m \) matrix and \( \tilde{v} = diag(v) \). It is easy to see that every \( m \times m \) matrix \( A \)

\[ \Phi^T(t)A\Phi(t) = \Phi^T(t)\tilde{A}, \]  \hspace{1cm} (3)

where, \( \tilde{A} \) is an \( m \)-vector with elements equal to the diagonal entries of matrix \( A \).

An arbitrary real bounded function \( f(t) \), which is square integrable in the interval \( t \in [0, 1) \), can be expanded as

\[ f(t) \simeq \tilde{f}_m(t) = \sum_{i=1}^{m} f_i \phi_i(t) = \Phi^T(t)F, \]  \hspace{1cm} (4)

where \( f_i = \frac{1}{T} \int_{0}^{1} f(t)\phi_i(t)dt \) are block pulse coefficients. Let \( k(t, s) \) be a function of two variables in \( L^2([0, 1] \times [0, 1]) \). It can be similarly expanded with respect to BPFs

\[ k(t, s) \simeq \Phi^T(t)K\Psi(s), \]  \hspace{1cm} (5)

where, \( \Psi(s) \) and \( \Phi(t) \) are \( m_1, m_2 \) dimensional BPFs vectors respectively, and \( K \) is the \( m_1 \times m_2 \) block-pulse coefficient matrix with

\[ k_{ij} = m_1 m_2 \int_{0}^{1} \int_{0}^{1} k(t, s)\phi_i(t)\psi_j(s)dtds. \quad i = 1, 2, \ldots, m_1, \quad j = 1, 2, \ldots, m_2. \]

For convenience, we put \( m_1 = m_2 = m \). Also we have

\[ \int_{0}^{t} \Phi(s)ds \simeq P\Phi(t), \]  \hspace{1cm} (6)

and

\[ \int_{0}^{t} \Phi(s)dB(s) \simeq P_s\Phi(t), \]  \hspace{1cm} (7)

where operational matrix \( P \) and \( P_s \) are given in [3].

### 3 Implementation in SD

We propose a method for solving Eq.(1) numerically. As usual, approximations of \( x(0), k_1(t, s), k_2(t, s), x(t), [x(t)]^\alpha, \ [x(t)]^\beta \) with respect to BPFs may be written as

\[ x_0 \simeq X_0^T\Phi(t) = \Phi^T(t)X_0, \]  \hspace{1cm} (8)

\[ x(t) \simeq X^T\Phi(t) = \Phi^T(t)X, \]  \hspace{1cm} (9)

\[ [x(t)]^\alpha \simeq X_\alpha^T\Phi(t) = \Phi^T(t)X_\alpha, \]  \hspace{1cm} (10)

\[ [x(t)]^\beta \simeq X_\beta^T\Phi(t) = \Phi^T(t)X_\beta, \]  \hspace{1cm} (11)

\[ k_i(t, s) \simeq \Phi^T(t)K_i\Phi(s), \quad i = 1, 2, \]  \hspace{1cm} (12)

where \( m \)-vectors \( X_0, X, X_\alpha, X_\beta, \) and \( m \times m \) matrix \( K_i \) are BPFs coefficients. Also \( X_\alpha \) and \( X_\beta \), are vectors whose elements are \( \alpha \)th and \( \beta \)th power of elements of The vector \( X \) respectively.

From Eqs.(10),(12),(2) and (6) the integral part in Eq.(1) can be approximated as

\[ \int_{0}^{t} k_1(t, s)[x(s)]^\alpha ds \simeq \Phi^T(t)K_1 \int_{0}^{t} \Phi(s)\Phi^T(s)X_\alpha ds \simeq \Phi^T(t)K_1 \tilde{X}_\alpha P\Phi(t), \]  \hspace{1cm} (13)
and also from Eqs. (11), (12), (2) and (7) for Itô integral term, we have
\[
\int_0^t k_2(t,s)[x(s)]^\beta dB(s) \approx \Phi^T(t)K_2 \int_0^t \Phi(s)\Phi^T(s)X_\beta dB(s) \approx \Phi^T(t)K_2 \tilde{X}_\beta P_2 \Phi(t),
\]
where $\tilde{X}_\alpha = diag(X_\alpha)$ and $\tilde{X}_\beta = diag(X_\beta)$. From Eq. (3) we can write
\[
\Phi^T(t)K_1 \tilde{X}_\alpha P \Phi(t) = \Phi^T(t)A_1,
\]
and
\[
\Phi^T(t)K_2 \tilde{X}_\beta P_2 \Phi(t) = \Phi^T(t)A_2,
\]
where, $A_1$ and $A_2$ are m-vectors with elements equal to the diagonal entries of matrices $K_1 \tilde{X}_\alpha P$ and $K_2 \tilde{X}_\beta P_2$, respectively.

By substituting relations (8), (9), (10) and (16) in Eq. (1), we get
\[
\Phi^T(t)X = \Phi^T(t)X_0 + \Phi^T(t)A_1 + \Phi^T(t)A_2,
\]
or,
\[
X - A_1 - A_2 = X_0.
\]
Eq. (18) is a nonlinear system of algebraic equations. Components of unknown vector $X$ can be obtained by solving this system using Newton’s iterative method.

### 4 Numerical example

To illustrate efficiency and accuracy of presented method we solve below example. Let $X_i$ denote the block pulse coefficient of exact solution and $Y_i$ be the block pulse coefficient of computed solutions by presented method. The error is defined as

\[
\|E\|_\infty = max_{1 \leq i \leq m}|X_i - Y_i|.
\]

**Example 4.1.** Consider the nonlinear stochastic Volterra integral equation as follows

\[
dx(t) = \frac{1}{2} a^2 n [x(t)]^{2n-1} dt + a[x(t)]^n dB(t), \quad t \in [0,1),
\]

with the exact solution $x(t) = \left(x_0^{1-n} - a(n-1)B(t)\right)^{\frac{1}{n-1}}$. See[1].

The numerical results are shown in Table 1 for $n = a = 2$. $\bar{E}$ is the errors mean and $s_E$ is the standard deviation of errors in 1000 iteration.

| $t_i$ | $\bar{E}$  | $s_E$  | \begin{tabular}{c|c} 0.95 Confidence Interval \\ Lowerbound & Upperbound \end{tabular} |
|-------|-------------|--------|-------------------------------|
| 0.2   | 0.002305    | 0.038921 | 0.020830 0.027739 |
| 0.4   | 0.002899    | 0.059176 | 0.005864 0.006262 |
| 0.6   | 0.077175    | 0.121112 | 0.065538 0.098611 |
| 0.8   | 0.141556    | 0.140713 | 0.125337 0.156775 |

### 5 Conclusion

The aim of present work is applying a method for solving nonlinear stochastic differential equations. The properties of the BPFs are used to reduce the problem to a system of nonlinear algebraic equations. The benefit of this method is low cost of setting up the equations without applying any projection method. For showing efficiency, the method is applied for an example.
References


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On computing root of matrices

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Abstract
In general we may define many functions for a square matrix. In this work, we introduce a square root function for a complex-valued matrix. The proposed square root is analytic, simple and can be compared with the well known methods. Then we present a few examples for square roots.

Keywords: Matrix square roots, Matrix functions, Factorization.

Mathematics Subject Classification: 15A16, 15A24, 65F60

1 Introduction
In this section we recall a few definitions from [1]. Let $A_{n\times n}$ be a complex-valued matrix. We first introduce Jordan canonical form of $A$.

Definition 1: A matrix, called Jordan Block, if all it’s elements are zero except along the diagonal and super diagonal, with each element of the diagonal consisting of a single number lambda, and each element of the super diagonal consisting of a 1.

Example 1. The following matrices are Jordan Block:

\[
\begin{bmatrix}
\lambda & 1 \\
0 & \lambda \\
\end{bmatrix}, \begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda \\
\end{bmatrix}, \begin{bmatrix}
\lambda & 1 & 0 & 0 \\
0 & \lambda & 1 & 0 \\
0 & 0 & \lambda & 1 \\
0 & 0 & 0 & \lambda \\
\end{bmatrix}.
\]

Definition 2: The Jordan canonical form, called a special type of block matrix that each block consists of Jordan blocks where diagonals elements are different.

Example 2. The following matrices are Jordan canonical form that have one, two and three Jordan blocks respectively:

\[
\begin{bmatrix}
\lambda_1 & 1 \\
0 & \lambda_1 \\
\end{bmatrix}, \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_2 \\
\end{bmatrix}, \begin{bmatrix}
\lambda_1 & 1 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & \lambda_3 \\
\end{bmatrix}.
\]

Definition 3: The Jordan matrix decomposition is the decomposition of a square matrix $A_{n\times n}$ into the form $A = ZJZ^{-1}$ where $J$ is a matrix of Jordan canonical form, and $Z$ is nonsingular. We can write $Z^{-1}AZ = J = \text{diag}(J_1, J_2, \ldots, J_p)$ where $J_k = J_k(\lambda_k)$.
and $m_1 + m_2 + ... + m_p = n$ that $m_k$ is number of repeat $\lambda_k$ and $k = 1, 2, ..., n$.

**Remark 1.** To compute $Z$, in Definition 3 we must compute vector $\lambda$ that contain eigenvalues of $A$. If $\lambda_k$’s be different then columns of $Z$ are eigenvectors of $A$. Otherwise suppose $m_k$ be number of repeat $\lambda_k$ and $q = n - \text{rank}(A - I)$ be number of independent vector for $\lambda_k$. Also $e = m - q$ be number of extended vector. Finally by $(\lambda_k I - A)x_k = x_{k-1}$ $x_k = (\lambda_k I - A)^{-1}x_{k-1}$.

Now we propose examples for this remark.

**Example 3.** Let $A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$ then $\lambda = (1, 2, 4, 4)^t$ and $x_1 = (-1, 1, 0, 0)^t$, $x_2 = (1, -1, 0, 1)^t$, $x_3 = (1, 0, -1, 1)^t$, where $x_k$ is eigenvectors of $\lambda_k$, $m = 2$, $q = 4 - 3 = 1$, $e = 2 - 1 = 1$ thus $x_4 = (\lambda_2 I - A)^{-1}x_3$ so $x_4 = (1, 0, 0, 0)^t$ then

$$Z = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{thus} \quad J = Z^{-1}AZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

**Example 4.** Let $A = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$ then $\lambda = (0.3028, -3.3028)^t$ and $Z = \begin{bmatrix} 0.9172 & -0.7933 \\ 0.3983 & 0.6089 \end{bmatrix}$ thus

$$J = Z^{-1}AZ = \begin{bmatrix} 0.3028 & 0 \\ 0 & -3.3028 \end{bmatrix}.$$

## 2 Main Result

Let $f$ be a complex-valued function. Now we can define $f(A)$ by Jordan canonical form of $A_{n \times n}$. In general $f(A)$ is not $f(a_{ij})$ when $a_{ij}$ is element of matrix $A$.

**Definition 4:** Let $f$ be a complex-valued function and $A = ZJZ^{-1}$ then $f(A) = Zf(J)Z^{-1} = Z\text{diag}(f(\lambda_k))Z^{-1}$, where

$$f(J_k) = \begin{bmatrix} f(\lambda_k) & f'(\lambda_k) & \cdots & f^{(m_k-1)}(\lambda_k) \\ f(\lambda_k) & f'(\lambda_k) & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ f(\lambda_k) & \cdots & f'(\lambda_k) & f(\lambda_k) \end{bmatrix},$$

and $m_k$ is number of repeat $\lambda_k$.

**Example 5.** Let $J = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $f(x) = x^3$ then $f(J) = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$.

**Properties 1.**

1. $f(A^t) = (f(A))^t$
2. $f(XAX^{-1}) = Xf(A)X^{-1}$
3. If $h(t) = f(t) + g(t)$ then $h(A) = f(A) + g(A)$
4. If $h(t) = f(t)g(t)$ then $h(A) = f(A)g(A)$
5. $f(A) = f(A)$

Now let $f(x) = \sqrt{x}$; there are many ways to compute $\sqrt{A}$. Björek and Hammarling presented Schur method, [2]. We propose a new formula to compute $\sqrt{A}$.

Case $n = 2$:
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} m & p \\ r & x \end{bmatrix}$, assume that $B = \sqrt{A}$. It can be shown that:

$$B = \begin{bmatrix} \sqrt{a - \frac{bc}{T + 2\sqrt{T^2 + 4T\Delta}}} & \frac{b}{\sqrt{T + 2\sqrt{T^2 + 4T\Delta}}} \\ \frac{c}{\sqrt{T + 2\sqrt{T^2 + 4T\Delta}}} & \sqrt{d - \frac{bc}{T + 2\sqrt{T^2 + 4T\Delta}}} \end{bmatrix}$$

where $T = a + d$, $\Delta = \det(A) = ad - bc$.

Example 6. Let $A = \begin{bmatrix} 1 & 4 \\ -7 & 5 \end{bmatrix}$ then $B = \begin{bmatrix} 1.6128 & 0.9565 \\ -1.6738 & 2.5692 \end{bmatrix}$.

Case $n = 3$:
Let $B = \sqrt{A}$ where $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} m & n & p \\ r & s & w \\ x & y & z \end{bmatrix}$, after not long algebraic calculations we have:

if $\det(A - \Omega I) \neq 0$ then

$$B = (tA + \sqrt{\Delta} I)(A - \Omega I)^{-1},$$

where $\Delta = \det(A)$ and $I$ is identity matrix. Also $t$ is trace of $B$ and $\Omega$ is function of elements of $B$.

Example 7. Let $A = \begin{bmatrix} 1 & 4 & -5 \\ -7 & 4 & 5 \\ 1 & 0 & 2 \end{bmatrix}$ then $B = \begin{bmatrix} 1.6007 & -1.0072 & 1.0670 \\ -1.8586 & 2.4521 & 1.8350 \\ 0.2902 & -0.0768 & 1.3533 \end{bmatrix}$.

Conclusion: If we want to compute matrix $B = \sqrt{A}$ by Definition 4 we must compute eigenvalues, eigenvectors and Jordan canonical form $(J)$ and in general it is not easy. It is relatively hard and boring. There are analytical and numerical methods to compute $B$. However, the proposed method does not need to complex mathematical calculations, it has analytic form and it is very simple to use.

References


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Some results of positive solutions for fractional differential equations with \( p \)-Laplacian

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Abstract
In this paper, we study the existence of positive solutions to boundary value problem for fractional differential equation. By using the Leggett-Williams fixed point theorem in a cone, the existence of three positive solutions for nonlinear singular boundary value problems is obtained.

Keywords: Cone, \( p \)-Laplacian operator, Multi point boundary value problem, Fixed point theorem

Mathematics Subject Classification: 47H10, 26A33, 34A08

1 Introduction
Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order, the fractional calculus may be considered an old and yet novel topic.

Recently, fractional differential equations have been of great interest. This is because of both the intensive development of the theory of fractional calculus itself and its applications in various sciences, such as physics, mechanics, chemistry, engineering, etc. For example, for fractional initial value problems, the existence and multiplicity of solutions were discussed in [1, 2, 3, 4].

Now, our purpose in this paper is to show the existence and multiplicity of positive solutions for the boundary value problem of fractional \( p \)-Laplacian equation with the following form, by using the Leggett-Williams fixed point theorem,

\[
\begin{align*}
D_0^\beta \phi_p(D_0^\alpha u)(t) &= f(t, u(t)), \quad t \in (0, 1), \\
D_0^\alpha u(0) &= D_0^\alpha u(1) = 0, \\
u(0) &= 0, \quad u(1) - \sum_{i=1}^{m-2} a_i u(\xi_i) = \lambda,
\end{align*}
\]

where \( 1 < \alpha, \beta \leq 2, 2 < \alpha + \beta \leq 4, D_0^\alpha \) is the Riemann-Liouville fractional derivative of order \( \alpha \), \( m > 2 \) are integers, \( \lambda > 0 \) is a parameter.

The following conditions will be used in this paper:

\( \text{(H1)} \) \( p > 1, \phi_p(s) = |s|^{p-2}s \) is a \( p \)-laplacian operator. Obviously, \( \phi_p \) is invertible and \( \phi_p^{-1} = \phi_q \), where \( q > 1 \) is a constant such that \( \frac{1}{q} + \frac{1}{p} = 1 \).

\( \text{(H2)} \) \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, a_i > 0 \) for \( i = 1, 2, \ldots, m - 2 \) and \( \sum_{i=1}^{m-2} a_i \xi_i^{\alpha-1} < 1 \).

\( \text{(H3)} \) \( f : [0, 1] \times [0, +\infty) \to [0, +\infty) \) is a given continuous function.

The rest of the article is organized as follows: in Section 2, we present some preliminaries that will be used in Section 3. The main result will be given in Section 3.
2 Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory.

**Definition 2.1.** Let $X$ be a real Banach space. A non-empty closed set $P \subset X$ is called a cone of $X$ if it satisfies the following conditions:

1. $x \in P, \mu \geq 0$ implies $\mu x \in P$,
2. $x \in P, -x \in P$ implies $x = 0$.

**Definition 2.2.** The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L_1(\mathbb{R}^+)$ is defined as

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

**Definition 2.3.** The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$$D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where the function $f(t)$ have absolutely continuous derivatives up to order $(n - 1)$.

3 Main result

In this section, we assume that $\gamma \in (0, \frac{1}{4})$ and $\delta = 1 - \gamma$.

For convenience, we introduce the following notations. Define

$$g(s) = \begin{cases} 
\frac{s^{\alpha-1} - (1-s)^{\alpha-1} - (\delta-s)^{\alpha-1}}{s^{\alpha-1} - (1-s)^{\alpha-1}} & s \in (0, m_1] \\
\frac{1}{\gamma} & s \in [m_1, 1)
\end{cases}$$

with $\gamma < m_1 < \delta$, and

$$\eta = \min_{\gamma \leq t \leq \delta} g(t),$$

$$\sigma = \min\{\eta, \gamma^{\alpha-1}\},$$

and let

$$M = \int_0^1 G(s, s) \phi_q \left( \int_0^1 H(s, r) dr \right) ds + \frac{1}{(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s) \phi_q \left( \int_0^1 H(s, r) dr \right) ds,$$

$$m = \min_{\gamma \leq t \leq \delta} \left\{ \int_\gamma^\delta G(t, s) \phi_q \left( \int_\gamma^\delta H(s, r) dr \right) ds + \frac{\delta^{\alpha-1}}{(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_\gamma^\delta G(\xi_i, s) \phi_q \left( \int_\gamma^\delta H(s, r) dr \right) ds \right\}.$$

The basic space used in this paper is a real Banach space $E = C([0,1], \mathbb{R})$ with the norm $\|u\| = \max_{t \in [0,1]} |u(t)|$.

Then, choose a cone $K \subset E$, by
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It is obvious that $K$ is a cone.

Define an operator $T : E \rightarrow E$ by

\[
(Tu)(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,r)f(r,u(r))dr\right)ds \\
+ \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q\left(\int_0^1 H(s,r)f(r,u(r))dr\right)ds + \frac{\lambda t^{\alpha-1}}{1-\Delta}.
\]

Lemma 3.1. The operator $T : K \rightarrow K$ is well-defined, i.e. $T(K) \subseteq K$.

It is clear that the existence of a positive solution for the system (13) is equivalent to the existence of a nontrivial fixed point of $T$ in $K$.

Finally, we define the nonnegative continuous concave functional on $K$ by

\[
\alpha(u) = \min_{\gamma \leq t \leq \delta} (u(t)).
\]

It is obvious that, for each $u \in K$, $\alpha(u) \leq \|u\|$.

Now, we can state our main result.

Theorem 3.2. Assume that (H1), (H2) and (H3) hold. In additional, assume there exist nonnegative numbers $a, b, c$ such that $0 < a < b \leq \frac{c}{3}$, and $f(t, u(t))$ satisfy the following conditions:

(H4) $f(t, u(t)) \leq \phi_b\left(\frac{t^{\alpha-1}}{(1-\Delta)}\right)$, for all $(t, u) \in [0,1] \times [0, c]$,

(H5) $f(t, u(t)) \leq \phi_b\left(\frac{t^{\alpha-1}}{(1-\Delta)}\right)$, for all $(t, u) \in [0,1] \times [0, a]$,

(H6) $f(t, u(t)) > \phi_b\left(\frac{t^{\alpha-1}}{(1-\Delta)}\right)$, for all $(t, u) \in [\gamma, \delta] \times [b, \frac{3c}{\sigma}]$,

then, for

\[
0 < \lambda < \frac{c(1-\Delta)}{2},
\]

the system (13) has at least three positive solutions $u_1, u_2, u_3$ such that $\|u_1\| < a, b < \min_{\gamma \leq t \leq \delta}(u_2(t))$, and $\|u_3\| > a$, with $\min_{\gamma \leq t \leq \delta}(u_3(t)) < b$.

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Solving inverse heat conduction problem by using genetic algorithm

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Abstract
In this paper a numerical approach combining the use of the least squares method and the genetic algorithm is proposed for the determination of temperature in an inverse parabolic problem. Results show that an excellent estimation can be obtained within a couple of minutes CPU time at pentium IV-2.4 GHz PC.

Keywords: Inverse heat conduction problem, Genetic algorithm, The least squares method.

Mathematics Subject Classification: 65M32, 35K05.

1 Introduction
Inverse problems are encountered in many branches of engineering and science. In one particular branch, heat transfer, the inverse problem can be used to such conditions as temperature or surface heat flux, or can be used to determine important thermal properties such as the thermal conductivity or heat capacity of solids. One example of the inverse heat conduction problem (IHCP) is the estimation of the heating history experienced by a shuttle or missile reentering the earth’s atmosphere from space. The heat flux at the heated surface is needed[1]. To estimate the surface heat flux history, it is necessary to have a mathematical model of the heat transfer process. For example, it is assumed that the section of the skin is of a single material, homogeneous and isotropic, and that it closely approximates a flat plate. Then a possible mathematical model for the temperature in the plate is a one dimensional inverse heat conduction problem, [1].

2 The inverse heat conduction problem
In this section, we consider the following IHCP in the dimensionless form

\[ T_t(x, t) = T_{xx}(x, t), \quad 0 < x < 1, \quad 0 < t < t_M \]  
\[ T(x, 0) = f(x), \quad 0 \leq x \leq 1, \]  
\[ T(0, t) = p(t), \quad 0 \leq t \leq t_M, \]  
\[ T(1, t) = q(t), \quad 0 \leq t \leq t_M, \]  
\[ T(a, t) = s(t), \quad 0 \leq t \leq t_M. \]  

and the overspecified condition

\[ T(a, t) = s(t), \quad 0 \leq t \leq t_M. \]  

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where \( f(x) \) is a continuous known function, \( p(t) \) and \( q(t) \) are infinitely differentiable known functions and \( t_{f1} \) represents the final time of interest for the time evolution of the problem, while the function \( q(t) \) is unknown which remains to be determined from some interior temperature measurements. The problem can be solve in least-square sense and a cost function can be define as a sum of squared differences between measured temperatures and calculated values of \( T \) by considering guesses estimated values for \( q(t) \):

\[
f(G) = \sum_{j=1}^{m} (T_j - s_j)^2,
\]

where \( T_j, j = 1, 2, 3, \ldots, m \), is calculated values from solving direct heat conduction by considering guess \( q(t) \) and \( s_j = s(t_j), j = 1, 2, 3, \ldots, m \), is measured temperatures. To find optimal solution \( q(t) \), Eq. (2) should be minimum.

### 3 A real-valued genetic algorithm (RVGA) for solving IHCP

Genetic algorithms, primarily developed by Holland [2], have been successfully applied to various optimization problems. It is essentially a searching method based on the Darwinian principles of biological evolution. Genetic algorithm is a stochastic optimization algorithm which employs the operation of biological evolution. In this work a RVGA is considered for solving IHCP. For determining \( q(t) \), we consider chromosomes as \( G_i = g_{i,1}, g_{i,2}, g_{i,3}, \ldots, g_{i,m}, i = 1, 2, 3, \ldots, n \), that each chromosome estimate \( q(t) \) at \( t_j, j = 1, 2, 3, \ldots, m \), Then solve direct heat conduction problem by the cranck-nicelson method for each chromosome and select the best chromosome by consider Eq. (2). For determining \( q(t) \), we interpolate m-points of the best chromosome. In this work, we added a additional part to algorithm. This part copies the best chromosome to population by a probability. Numerical results show that addition this part improved performance of algorithm. The application of the present method for determining the solution of problem (1) can be divided into the following steps:

- **Step 1.** Generate randomly an initial population of chromosomes.
- **Step 2.** Evaluate the fitness of each chromosome in the population.
- **Step 3.** Choose by tournament selection pairs of chromosome for combination. To generate new chromosomes, choose \( \beta_i \in [-0.25, 1.25], i = 1, 2, 3, \ldots, m \), randomly for \( m \) genes, then parents combine together as follow:
  - New chromosome 1 = \{ \( g_{p1,1} \times \beta_1 + (1 - \beta_1) \times g_{p2,1}, \ldots, g_{p1,m} \times \beta_m + (1 - \beta_m) \times g_{p2,m} \) \}
  - New chromosome 2 = \{ \( g_{p2,1} \times \beta_1 + (1 - \beta_1) \times g_{p1,1}, \ldots, g_{p2,m} \times \beta_m + (1 - \beta_m) \times g_{p1,m} \) \}
  where \( p1 \) is first parent and \( p2 \) is second parent.
- **Step 4.** Apply mutation on new chromosomes.
- **Step 5.** Put new chromosomes into population and copy the best individual by \( \alpha \in (0, 1) \) probability.
- **Step 6.** Repeat step 2 to step 5, until finding acceptable fitness.

### 4 Main Result

We are going to demonstrate numerically, some of results for the unknown function \( q(t) \) in the IHCP (1). The propose of this section is to illustrate the applicability of the present method described in section 3 for solving IHCP. As expected the IHCP is ill-posed and therefore it is necessary to investigate the stability of the present method by giving a test problem.
In this example, for $0 < x < 1$, $0 < t < 1$, we consider the following IHCP:

\begin{align}
T_t(x,t) &= T_{xx}(x,t), \quad \text{(3a)} \\
T(x,0) &= \sin(x), \quad \text{(3b)} \\
T(0,t) &= 0, \quad \text{(3c)} \\
T(1,t) &= q(t), \quad \text{(3d)}
\end{align}

and the overspecified condition

\[ T(0.5,t_j) = s(t_j), \quad j = 0, 1, 2, \ldots, 19, \quad (3e) \]

The experimental data $s(t_j)$ (measured temperatures at $0 < t < 1$) are obtained from the exact solution of the direct problem. In this example the exact $U(x,t)$ and $q(t)$ are $e^{-t}\sin(x)$ and $e^{-t}\sin(1)$ respectively. In this section, a population of 80 chromosomes of 20 genes ($t = 0.05, 0.1, 0.15, \ldots, 0.95$) is used as the initial guess to obtain numerical results of RVGA.

Fig. 1 shows exact and numeric $q(t)$, for three generations 100, 1000 and 10000.

Fig. 2 shows the difference between exact and numeric $q(t)$, for three generations 100, 1000 and 10000.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Exact and numeric $q(t)$, for different number of generation.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Exact and numeric $q(t)$, for different number of generation.}
\end{figure}

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Estimation of unknown boundary functions in an IHCP with mollification method

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Abstract

In this paper we develop a numerical scheme based on mollification method and space marching scheme for solving an inverse heat conduction problem with two unknown boundary conditions. The temperature and heat flux measurements in an interior point are considered as overspecified data with the presence of noise. Convergence and stability of the solution for the proposed method are analyzed and a numerical example considered to support this method.

Keywords: Inverse heat conduction problem, marching scheme, mollification method.

Mathematics Subject Classification: 65M06, 65M12, 65M32

1 Introduction

Inverse heat conduction problem (IHCP) arises in the modeling and control of processes with heat propagation in thermophysics and mechanics of continuous media. Some kinds of non-destructive evaluation techniques to determine the interior heat flows in an inaccessible domain from exterior scattered data are indispensable for industrial and engineering applications. Studies of the complicated phenomena are particularly challenging for quantitative analysis and modeling. Consider a linear inverse heat conduction problem as follows

\[
\begin{align*}
  u_t(x,t) - a^2 u_{xx}(x,t) &= 0, \quad 0 \leq x \leq 1, \quad t > 0, \\
  u(x^*, t) &= \alpha(t), \quad u_x(x^*, t) = \beta(t), \\
  u(x, 0) &= f(x), \\
  u(0, t) &= p(t), \quad u(1, t) = q(t),
\end{align*}
\]

where all function are considered to be $L_2$ functions. The direct problem regards to this problem is to determine $u(x, t)$ from initial and boundary conditions.

2 Inverse Problem Formulation and Marching Scheme

In the problem (1)-(4), we consider that the boundary conditions namely $p(t)$ and $q(t)$ are unknown functions. With respect to this assumption, we deal with an inverse heat conductor problem. Let $\alpha(t)$, $\beta(t)$ and $f(x)$ are only known approximately as $\alpha^\varepsilon(t)$, $\beta^\varepsilon(t)$ and $f^\varepsilon(x)$ such that $\|\alpha(t) - \alpha^\varepsilon(t)\|_{\infty} \leq \varepsilon$, $\|\beta(t) - \beta^\varepsilon(t)\|_{\infty} \leq \varepsilon$ and $\|f(t) - f^\varepsilon(t)\|_{\infty} \leq \varepsilon$.

Because of the presence of the noise in the problem's data, we first stabilize the problem using the mollification method (see [2]). The regularized problem is formulated as follows. Determine
For the left side, let \( |\delta|_{-\infty} = \min(\delta_0, \delta_0^*) \). Applying Theorem 3 in [2] yields

\[
|D_t(Q_{i,n})| \leq \frac{C}{|\delta|_{-\infty}} |Q_{i,n}|
\]  

(15)
Using Taylor series, we obtain some useful equations satisfied by the mollified solution

\begin{equation}
|Q_{i+1,n}| \leq |Q_{i,n}| + \frac{h}{a^2} |W_{i,n}| \leq (1 + hC_1) \max\{|Q_{i,n}|, |W_{i,n}|\}
\end{equation}

where \(C_1 = \frac{h}{a^2}\). Similarly using (2), (8) and (9) yields

\begin{equation}
|W_{i+1,n}| \leq \left(1 + h \frac{C}{\delta - \infty}\right) \max\{|Q_{i,n}|, |W_{i,n}|\}, \quad |U_{i+1,n}| \leq (1 + h) \max\{|U_{i,n}|, |Q_{i,n}|\}.
\end{equation}

Let \(C_\delta = \max\{1, C_1, \frac{C}{\delta - \infty}\}\) and \(M_1 = M - M^\ast\). From (16) and (17) one may obtain

\[
\max\{|U_M|, |Q_M|, |W_M|\} \leq (1 + hC_\delta)M_1 \max\{|U_M^\ast|, |Q_M^\ast|, |W_M^\ast|\}.
\]

\[\square\]

**Theorem 3.2** (Formal convergence). For fixed \(\delta\) as \(h, k\) and \(\varepsilon\) tend to zero, the discrete mollified solution converges to the mollified exact solution restricted to the grid points.

**Proof.** First we check forward problem. The backward problem is analyzed in similar manner. From the definitions of discrete error functions, it follows that

\[
\Delta U_{i,n} = U_{i,n} - v(ih,nk), \quad \Delta Q_{i,n} = Q_{i,n} - v_x(ih,nk), \quad \Delta W_{i,n} = W_{i,n} - v_t(ih,nk).
\]

Using Taylor series, we obtain some useful equations satisfied by the mollified solution \(v\), namely,

\begin{align}
v((i + 1)h, nk) &= v(ih, nk) + hv_x(ih, nk) + O(h^2), \\
v_x((i + 1)h, nk) &= v_x(ih, nk) + \frac{h}{a^2} v_t(ih, nk) + O(h^2), \\
v_t((i + 1)h, nk) &= v_t(ih, nk) + h \left(\frac{d}{dt} v_x(ih, nk)\right) + O(h^2).
\end{align}

Now from equalities (18)-(20), we have

\[
|\Delta U_{i+1,n}| \leq |\Delta U_{i,n}| + h|\Delta Q_{i,n}| + O(h^2),
\]

\[
|\Delta Q_{i+1,n}| \leq |\Delta Q_{i,n}| + \frac{h}{a^2} |\Delta W_{i,n}| + O(h^2)
\]

\[
|\Delta W_{i+1,n}| \leq |\Delta W_{i,n}| + h \left(C\frac{|\Delta Q_{i,n}| + k}{\delta - \infty} + C_\delta k^2\right) + O(h^2).
\]

Suppose

\[
\Delta_i = \max\{|\Delta U_{i,n}|, |\Delta W_{i,n}|, |\Delta Q_{i,n}|\}, \quad C_0 = \max\left\{1, \frac{1}{a^2}, \frac{C}{\delta - \infty}\right\}, \quad C_1 = \frac{ck}{\delta - \infty} + C_\delta k^2.
\]

Then we obtain \(\Delta_{i+1} \leq (1 + hC_0)(\Delta_i + C_1) + O(h^2)\) and after \(L\) iterations \(\Delta_L \leq \exp(C_0)(\Delta_0 + C_1)\). Moreover from

\[
|\Delta U_{M^*,n}| \leq C(\varepsilon + k), \quad |\Delta Q_{M^*,n}| \leq C(\varepsilon + k),
\]

\[
|\Delta W_{M^*,n}| \leq \frac{C}{\delta M^*}(\varepsilon + k) + C_\delta k^2,
\]

we see that when \(\varepsilon, h, k\) tend to 0, \(\Delta_0\) and \(C_1\) tend to 0. Consequently \((\Delta_0 + C_1)\) tends to 0 and so does \(\Delta_L\).\[\square\]
Example 3.3. In the problem (1)-(4), put $x^* = 0.9$, $a = 1$, $\alpha(t) = \frac{243}{100}t + \frac{3362592558541097}{2251799813685248}\exp(4t) + \frac{3}{t^2} + \frac{561}{1000}$, $\beta(t) = \frac{47}{5}t - \frac{5409647096636263}{1125899906842624}\exp(4t) + \frac{729}{1000}$. 

The errors between exact and approximate solution are measured by the relative weighted $l_2$-norm [2]. The exact solution for $u(x, t)$ may be derived as $u(x, t) = xe^{-4t}(\sin 2x + \cos 2x) + 3(t^2 + tx^2 + \frac{1}{12}x^4)$. Table 1 shows the comparison between exact and numerical solutions and the relative $l_2$ errors.

<table>
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<th>$u_x$</th>
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<td>0.029082</td>
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Table 1: Relative $l_2$ error norms

References


Stability for a competitive Lotka-Volterra system with delays based on LMI optimization approach

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Abstract
In this paper, the stability of a two-species competitive Lotka-Volterra system with discrete delays has been studied. Using an LMI (linear matrix inequality) optimization approach and constructing appropriate a Lyapunov functional, a stability criterion is obtained for the asymptotic and exponential stability of the positive equilibrium of a Lotka-Volterra system with delays.

Keywords: Lotka-Volterra system, Competitive system, Delay, Exponential stability, Asymptotic stability

Mathematics Subject Classification: 34D20, 34D23, 34A34

1 Introduction
The competitive Lotka-Volterra system is an important population model and has been considered by many authors. In this paper, the following two-species competitive Lotka-Volterra type system with discrete delays is considered:

\[
\begin{aligned}
\frac{dx(t)}{dt} &= x(t)[b_1 - a_{11}x(t - \tau_{11}) - a_{12}y(t - \tau_{12})], \\
\frac{dy(t)}{dt} &= y(t)[b_2 - a_{12}x(t - \tau_{21}) - a_{22}y(t - \tau_{22})].
\end{aligned}
\]  

(1)

where \(x(t)\) and \(y(t)\) stand for densities of the populations at time \(t\), respectively, \(b_i, a_{ij}\) are all positive constants, and the delays \(\tau_{ij}\) are positive.

The initial condition of system (1) is given by

\[
\begin{aligned}
x(s) &= \phi_1(s) > 0 \quad -\tau \leq s < 0, \quad \phi_1(s) > 0, \\
y(s) &= \phi_2(s) > 0 \quad -\tau \leq s < 0, \quad \phi_2(s) > 0.
\end{aligned}
\]  

(2)

where \(\tau = \max \tau_{ij}, i,j=1,2\).

Definition 1.1. The equilibrium point \(x^*, y^*\) of (1) is said to be globally exponentially stable (GES), if there exist two positive constants \(\alpha > 0\) and \(k > 0\) such that for any \(\phi_1(s) \geq 0\), and \(\phi_2(s) \geq 0, s \in \mathbb{C}[\tau, 0]\), we have

\[
|x(t) - x^*| + |y(t) - y^*| \leq \alpha (\|\phi_1 - x^*\| + \|\phi_2 - y^*\|) e^{-kt}
\]  

(3)

Lemma 1.2. Under the condition \(\frac{a_{11}}{b_1} > \frac{a_{12}}{b_2} > \frac{a_{21}}{b_1} > \frac{a_{22}}{b_2}\), all positive solutions \(z(t) = (x(t), y(t))\) of system (1) have unique positive equilibrium \(z^* = (x^*, y^*)\), where:

\[
\begin{aligned}
x^* &= \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}}, \\
y^* &= \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}.
\end{aligned}
\]  

(4)

(5)
From (4), the global stability of a positive equilibrium of (1) holds when the intraspecific competition dominates the interspecific interactions. For Lotka-Volterra systems with delays, stability is delineated in two ways: systems that contain delay independent terms which dominate other intraspecific and interspecific interaction effects with and without delays, called a “no-pure-delay-type”, and systems with delay feedback, called “pure-delay-type”. For no-pure-delay-type systems, one can use the no-delay terms to control the delay terms. For the pure-delay-type systems, analysis of the global stability is more difficult. For the competitive Lotka-Volterra pure-delay-type systems, the analysis on the local stability is also difficult if the characteristic equation of the corresponding linearization of the system is employed. In this paper, we study local asymptotic and global exponential stability of the positive equilibrium of pure-delay-type Lotka-Volterra competitive systems. By means of Lyapunov functionals and Using an LMI optimization approach, we obtain sufficient conditions for both local and global stability of the positive equilibrium of the system (1). Through the article, * represents the elements below the main diagonal of a symmetric matrix.

2 Main Results

Putting:

\[ u(t) = x(t) - x^*, \quad v(t) = y(t) - y^* \]  \hspace{1cm} (6)

system (1) is transformed to the system, with an equilibrium point at \( Z^* = 0 \),

\[ \begin{cases} \frac{du(t)}{dt} = [-a_{11}x^*u(t - \tau_{11}) - a_{12}x^*v(t - \tau_{12})], \\ \frac{dv(t)}{dt} = [-a_{21}y^*u(t - \tau_{21}) - a_{22}y^*v(t - \tau_{22})]. \end{cases} \]  \hspace{1cm} (7)

Eq. (7) can be rewritten in the following form:

\[ \begin{cases} \frac{d}{dt} \left( \frac{u(t)}{x^*} \right) - a_{11}\int_{t-\tau_{11}}^t u(s)ds = -a_{11}u(t) - a_{12}v(t), \\ \frac{d}{dt} \left( \frac{v(t)}{y^*} \right) - a_{22}\int_{t-\tau_{22}}^t v(s)ds = -a_{11}u(t) - a_{12}v(t). \end{cases} \]  \hspace{1cm} (8)

Then, using the Lyapunov method and the LMI optimization technique, we have the following theorem for the local stability of system (1).

**Theorem 2.1.** For given delays \( \tau_{ij} \), if there exist positive scalars \( \varepsilon_1, \varepsilon_2, p_1, p_2, q_1 \) and \( q_2 \) satisfying the following LMI:

\[ \begin{pmatrix} -2x^*a_{11}p_1 + x^*\varepsilon_1q_1 & -y^*a_{12}p_2 + y^*\varepsilon_2q_2 & y^*a_{12}p_1 \end{pmatrix} \begin{pmatrix} -a_{11} & -a_{12} & a_{21} \\ -a_{12} & a_{22} & a_{22} \\ a_{21} & a_{22} & 0 \end{pmatrix} \begin{pmatrix} x^* \varepsilon_1q_1 & y^*\varepsilon_2q_2 & y^*a_{12}p_1 \end{pmatrix} \begin{pmatrix} -2y^*a_{12}p_2 + y^*\varepsilon_2q_2 \end{pmatrix} < 0. \]  \hspace{1cm} (9)

then the positive equilibrium \( Z^* \) of (1) is locally asymptotically stable. Now in the following theorem, with introduction of a new LMI for checking the global stability, we improve and extend earlier results.

**Theorem 2.2.** For given delays \( \tau_{ij} \), if there exist positive constants \( k, r, p_1, p_2, q_1 \) and \( q_2 \) satisfying the following LMI:

\[ \begin{pmatrix} \hat{H}_1 & \left( -y^*a_{12}(p_{1+r}) \right) \\ -y^*a_{12}(p_{1+r}) & \hat{H}_2 \end{pmatrix} \begin{pmatrix} a_{11}x^* - 2ka_{11}p_1 & a_{21}a_{22}x^*p_2 \\ a_{12}a_{11}p_1 & y^*a_{12}a_{11}p_1 \end{pmatrix} \begin{pmatrix} x^* - 2k\tau_{11}p_1 & \left( a_{22}y^* - 2ka_{22}p_2 \right) \end{pmatrix} < 0. \]  \hspace{1cm} (10)
where

\[
\begin{align*}
\tilde{H}_1 &= 2p_1(k - x^*a_{11}) + x^{*2}\tau_{11}q_1 + r(2k + a^{*2}_{11} + x^{*2}e^{2k\tau_{11}}), \\
\tilde{H}_2 &= 2p_2(k - y^*a_{22}) + y^{*2}\tau_{22}q_2 + r(2k + a^{*2}_{22} + y^{*2}e^{2k\tau_{22}}),
\end{align*}
\]

then the positive equilibrium \( Z^* \) of (1) is globally exponentially stable.

**Remark 2.3.** Taking \( r = 0, k = 0 \) in the proof above, we can easily obtain the local asymptotic stability. Obviously this improves and extends earlier results.

**Remark 2.4.** The LMI optimization problems (9) and (10) in Theorem 2.1 and 2.2 are to determine whether the problem is feasible or not. They are called the feasibility problems. The solutions of the problem can be found by solving the eigenvalue problem in \( \varepsilon_1, \varepsilon_2, p_1, p_2, q_1 \) and \( q_2 \) and \( k, r, p_1, p_2, q_1 \) and \( q_2 \) which is a convex optimization problem. Various efficient convex optimization algorithms can be used to check the feasibility of LMI.

**Remark 2.5.** In this paper, based on Lyapunov method and the LMI optimization approach, a series of sufficient criteria on asymptotic stability have been established for the equilibrium of the two-species competitive Lotka-Volterra system with discrete delays. Then extending earlier LMI can prove exponential stability of the two-species competitive Lotka-Volterra system with discrete delays.

**References**


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Capturing outlines of persian fonts with Bezier cubic approximation

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Abstract
In this paper an algorithm for outline capture of digital character image, stored as bitmap, is presented. This method is well suited for characters of non-Roman languages like Persian, Arabic, Japanese, etc. In most of the desktop publishing systems, the shape of the characters are stored in the computer memory in terms of their outlines, and the outline are expressed as cubic Bezier curves. The process of capturing outline includes steps: detection of boundary, finding corner point and break points and fitting the curve.

Keywords: Bezier curve, Curve fitting, Digital character image, Corner points, Break points

Mathematics Subject Classification: 65Y99, 68U07

1 Introduction
Fonts are essential part of any computer system. Two fundamental approaches for storing fonts in computers are bitmap and outline. In bitmap fonts each character is stored as an array of pixels. Outline fonts describe the character outline with a combination of control points and curve. Outline representation has many advantages over bitmap such as scaling, shearing, translation, rotation, and clipping. Therefor most of the contemporary desktop publishing systems are based on outline fonts.

This paper proposes an algorithm to obtain the outline of bitmap characters. The methodology, in this paper, mainly differs from the traditional approaches in various ways. Some times only corner points are not sufficient to fit the curve and few additional point are needed to achieve a best fit. This paper, in addition to corner points, identifies these additional point called break point. So, our set of significant points consists of corner points and break points. Segmentation is done at significant point and cubic Bezier is use for curve fitting. The least square method is used for the achievement of the best fit. In the case when fitted curve is not to desired shape, break points enhance the correctness of the best fit.

The organization of the paper is made as follows. Section 2 discusses about extraction of boundary. Details of corner detection process are given in section 3. The core of algorithm, i.e. curve fitting process is elaborated in section 4.

2 Extracting of boundary
Bitmap image of a character can be obtained by creating a bitmap character on some program like paint. Alternatively an image drawn on paper can scan and store it as bitmap. Figure 1(right) shows the bitmap image of a character. Boundary of digitized image is extracted by using some
boundary detection algorithm. We have used the algorithm in [1]. The algorithm returns a number of boundary points \( p_i = (x_i, y_i), i = 1, \ldots, N \). Figure 1(left) show boundary of Bitmap image.

3 Detecting corner points

After finding out boundary points, the next step is the detection of corner points. The corner points are those points which partition the outline into various segment. For the detection of corners, in this paper, we adopted the technique used in [2]. This algorithm gives a curvature based corner detector that detects both fine and coarse features accurately at low computational cost. Figure 2(right), shows detected corner points obtained from boundary of figure 1.

4 Curve fitting with cubic Bezier curve

We divide the boundary points of each piece in to groups called segments and fit cubic Bezier curve to each segment. The division is based on corner point. The Bezier form of cubic polynomial curve has four control points \( P_0, P_1, P_2 \) and \( P_3 \). The Bezier curve interpolates the two end control points \( P_0 \) and \( P_3 \) and approximates the two intermediate points \( P_1 \) and \( P_2 \). The two end control points are the two corner points of the curve segment. But we have to find the two intermediate control points of cubic Bezier curve to fit the curve. Mathematically, cubic Bezier curve can by written as follows

\[
Q(t) = \sum_{k=0}^{3} P_k B_k(t) = (1 - t)^3 P_0 + 3t(1 - t)^2 P_1 + 3t^2(1 - t) P_2 + t^3 P_3, \tag{1}
\]

where \( 0 \leq t \leq 1 \), \( P_0 = p_u, P_3 = p_w \). For \( k \)th segment whose boundary points are \( p_u, \ldots, p_w \).

Let \( P_i = (P_{x_i}, P_{y_i}) \) and \( Q(t) = (Q_x(t), Q_y(t)) \) then Eq. (1) can be expressed as follows:

\[
Q_x(t) = (1 - t)^3 P_{x_0} + 3t(1 - t)^2 P_{x_1} + 3t^2(1 - t) P_{x_2} + t^3 P_{x_3}. \tag{2}
\]

\[
Q_y(t) = (1 - t)^3 P_{y_0} + 3t(1 - t)^2 P_{y_1} + 3t^2(1 - t) P_{y_2} + t^3 P_{y_3}. \tag{3}
\]

We use chord-length parameterization to estimate the parametric value \( t \) associated with each point \( p_i \) as follows:

\[
t_i = \begin{cases} 
0 & \text{if } i = u; \\
\frac{p_i p_{i+1}}{p_u p_{u+1}} + \frac{|p_{i+1} p_{i+2}| + \ldots + |p_{i-1} p_i|}{|p_u p_{u+1}| + |p_{u+1} p_{u+2}| + \ldots + |p_{w-1} p_w|}, & \text{if } u + 1 \leq i \leq w - 1; \\
1 & \text{if } i = w.
\end{cases} \tag{4}
\]

Our goal is to approximate the boundary of the original image by a parametric curve in an optimal way. To achieve this goal we have to find the values of \( P_1 \) and \( P_2 \) that minimize the distance between boundary and parametric curve. For the purpose, we use the least square method. That is we define the sum of squared distances from the boundary to the parametric curve. Mathematically, we can write:

\[
S = \sum_{i=u}^{w} |Q_i(t) - p_i|^2 = \sum_{i=u}^{w} |Q_{x_i}(t) - p_{x_i}|^2 + \sum_{i=u}^{w} |Q_{y_i}(t) - p_{y_i}|^2. \tag{5}
\]
Our goal is to minimize $S$. We find partial derivatives of (5) with respect to $P_1$ and $P_2$ and equate them to zero. The solution will give us value of $P_1$ and $P_2$ that approximate the boundary by a parametric curve in best way for given values of $t$. Using these control points and $t$ value, we fit the cubic Bezier to the segment. The demonstration of fitted cubic Bezier curve(−)over boundary(...) is shown in Figure. 2(left). We estimate the accuracy of our fit, for the segment under process, by computing squared distance between each of its point $p_i$ on boundary and its corresponding $Q(t_i)$ on parametric curve.

$$
 d_i^2 = |p_i - Q(t_i)| = |p_{x_i} - Q_x(t_i)|^2 + |p_{y_i} - Q_y(t_i)|^2.
$$

Among all the computed distances computed by (6) we find maximum squared distance $d_{max}^2 = Max(d_{u_1}^2, d_{u_2}^2,..., d_{u_n}^2)$. If $d_{max}^2$ exceeds the predefined error tolerance limit $d_{tolerance}^2$ then the segment is broken into two segments at the point of maximum distance and the point corresponding to maximum distance is added to list of significant points. Figure 3 show the fitted Bezier curve. Corner points are shown by (O) and break points are shown by (□).

Figure 1: detected boundary(left), Bitmap image(right).

Figure 2: Fitted cubic Bezier over boundary(left), Corner points(right).
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Figure 3: Fitted Bezier with significant points.

References


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An analytical solution for fractional reaction-diffusion equation

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Abstract
In this paper a fractional reaction-diffusion equation is considered. The Caputo derivative is applied as the time derivative and Riesz-Feller fractional derivative is used as space derivative. An analytical solution of this equation is proposed. For solving this equation with appropriate boundary and initial conditions, a solution is obtained using the Laplace and Fourier transforms in a compact and closed form in terms of H-functions. This work is based on the paper of H.J. Haubod et al. [1] which is applied for a new fractional reaction-diffusion equation. The validity of proposed solution is proved.

Keywords: Reaction-diffusion equations, Fractional calculus, H-function, Riesz-Feller fractional derivative, Caputo derivative.

Mathematics Subject Classification: 35R11

1 Introduction
Fractional calculus grows out of the fractional derivative definitions and the calculus of integrals. In recent decades fractional differential equation has found a particular place in scientific research. Specially in recent years, fractional reaction-diffusion models are studied due to their usefulness and importance in many areas of science and engineering. There are some papers, in this domain for instance, Gafiychuk et al. [2] studied a couple fractional reaction-diffusion. They investigated nonlinear oscillations [3] and stability domain in fractional reaction-diffusion systems. Haubold et al. [1] gave a solution for an special fractional reaction-diffusion. In this paper following the latter work an analytical solution for a fractional reaction-diffusion equation is introduced. The organization of the paper is as follows. In sections 2, the main ideas are presented and the main proposition is given. Section 4 is specified to present an example. In section 4 conclusion is given.

2 The basic concept
The Riemann-Liouville fractional integral of order $\nu$ is defined by

$$\frac{1}{\Gamma(\nu)} \int_{0}^{t} (t-w)^{\nu-1} u(x,w) dw, \quad \text{Re}(\nu) > 0,$$

and the Riemann-Liouville fractional derivatives of order $\alpha > 0$ is defined as

$$\frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_{0}^{t} \frac{f(x,\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \quad n = [\alpha]+1, \quad (n \in N, t > 0).$$

1250
when \([\alpha]\) means the integral part of the number \(\alpha \in \mathbb{R}\).

Then the Laplace transform of the Riemann-Liouville fractional derivative is obtained as follows

\[
L\left\{ \mathcal{D}_t^\alpha u(x,t); s \right\} = s^\alpha u(x,s) - \sum_{r=1}^{\alpha} s^{\alpha-r} \mathcal{D}_t^{\alpha-r} u(x,t)|_{t=0}, \quad n - 1 \leq \alpha \leq n
\]

The Fourier transforms form of Riesz-Feller space fractional derivative of order \(\alpha \in [0,2]\) and skewness of \(\theta\) is given by

\[
\mathcal{F}\left\{ \mathcal{D}_t^\alpha f(x); k \right\} = -\phi_\alpha(k) f^*(k),
\]

that is defined in terms of \(f^*(k)\) which denotes the Fourier transform of \(f(t)\), where

\[
f^*(k) = \int_{-\infty}^{\infty} \exp(-ikt)f(t)dt, \quad \phi_\alpha(k) = |k|^\alpha \exp[i(\sin k)\frac{\theta\pi}{2}], \quad 0 < \alpha \leq 2, \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}.
\]

Consider the following fractional reaction-diffusion equation. This equation is solved by Haubold et al. \cite{1} based on the following theorem.

**Theorem 2.1.** Consider the unified fractional reaction-diffusion model associated with the Riemann-Liouville fractional derivative \(\mathcal{D}_t^\alpha\) and the Riesz-Feller space fractional derivative \(\mathcal{D}_x^\beta\) of order \(\alpha\) and asymmetry \(\theta\) by

\[
\begin{align*}
\mathcal{D}_t^\beta u(x,t) & = \eta \mathcal{D}_x^\alpha u(x,t) + \phi(x,t), \quad \alpha, \theta, \beta \in \mathbb{R}. \quad (1) \\
0 < \alpha \leq 2, \quad 1 < \beta \leq 2, \quad \eta, t > 0, \quad 1 < \alpha \leq 2, \quad x \in \mathbb{R}. \\
\mathcal{D}_t^{\beta-1} u(x,t) & = f(x), \quad \mathcal{D}_t^{\beta-2} u(x,0) = g(x), \quad x \in \mathbb{R}, \quad \lim_{x \to \pm \infty} N(x,t) = 0 \quad t > 0 \quad (2)
\end{align*}
\]

Here \(\mathcal{D}_t^{\beta-1} u(x,0)\) means the Riemann-Liouville fractional derivative of \(u(x,t)\) with respect to \(t\) of order \(\beta - 1\) evaluated at \(t = 0\). Similarly \(\mathcal{D}_t^{\beta-2} u(x,0)\) is the Riemann-Liouville fractional partial derivative of \(u(x,t)\) with respect to \(t\) of order \(\beta - 2\) evaluated at \(t = 0\). In (2.1), \(\eta\) is a diffusion constant and \(\phi(x,t)\) is a nonlinear function. Then for the solution of (2.1), subject to the constraints (2.2), the following formula is held

\[
N(x,t) = \frac{\beta-1}{2\pi} \int_{-\infty}^{\infty} f^*(k) E_{\beta,\beta}(-\eta t^\beta \psi_\alpha(k)) \exp(-ikx)dk
+ \frac{t^{\beta-2}}{2\pi} \int_{-\infty}^{\infty} t g^*(k) E_{\beta,\beta-1}(-\eta t^{\beta-1} \psi_\alpha(k)) \exp(ikx)dk
+ \frac{1}{2\pi} \int_{0}^{t} \int_{-\infty}^{\infty} \phi^*(k,t-k) E_{\beta,\beta}(-\eta t^{\beta-1} \psi_\alpha(k)) \exp(-ikx)dk \xi d\xi
\]

where \(E_{\alpha,\beta}(z)\) is the applied Mittag-Leffler function, defined by the series

\[
E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta) \in \mathbb{C}, \quad Re(\alpha) > 0, \quad Re(\beta) > 0
\]

**Proof:** See \cite{1}.

This Theorem can be generalized for a particular equation.

**Proposition 2.2.** Consider the unified fractional reaction-diffusion model associated with the Riemann-Liouville fractional derivative \(\mathcal{D}_t^\alpha\) and the Riesz-Feller fractional derivative \(\mathcal{D}_x^\beta\) of order \(\alpha\) and asymmetry \(\theta\) defined by

\[
\mathcal{D}_t^\beta N(x,t) = \eta \mathcal{D}_x^\alpha N(x,t) + \phi(x,t), \quad t > 0, \quad |\theta| \leq \min(\alpha, 2 - \alpha), \quad 0 < \beta \leq 1,
\]
and initial and boundary conditions
\[ \eta D_t^{\beta-1} N(x, 0) = f(x), \quad x \in \mathbb{R}, \quad \lim_{x \to \pm\infty} N(x, t) = 0, \quad t > 0. \]

Then the following formula is held.
\[ N(x, t) = \frac{t^{\beta-1}}{2\pi} \int_{-\infty}^{\infty} f^*(k) E_{\beta,\beta}(-\eta t^\beta \psi_0^0(k)) \exp(-ikx) dk \]
\[ + \frac{1}{2\pi} \int_0^t \left( \int_{-\infty}^{\infty} \phi^*(k, t-k) E_{\beta,\beta}(-\eta t^\beta \psi_0^0(k)) \exp(-ikx) dk \right) \xi d\xi \]

**Proof:** A demonstration can be done by using the similar arguments of Theorem 2.1.

### 3 Application and discussion

A special example is given in the following paragraph and is solved with closed form of solution given in [1], and the theorem (3.1) given in this paper.

**Example** Consider the following partial differential equation
\[ \frac{\partial}{\partial t} N(x, t) = \eta \frac{\partial^2}{\partial x^2} N(x, t) \]
with initial and boundary conditions.
\[ N(x, t=0) = \delta(x), \quad \lim_{x \to \pm\infty} N(x, t) = 0 \]
where \( \delta(t) \) is the Dirac function. By using the Proposition 2.2, it is deduced:
\[ N(x, t) = \frac{1}{2|x|} H^{1,0}_{1,1} \left[ \frac{|x|}{\eta^2 t^2} \right] \left( \alpha, \frac{1}{2} \right) \left( 1, 1 \right) = \left( 4\pi \eta t \right)^{-\frac{3}{2}} \exp \left[ \frac{-|x|^2}{4\eta t} \right] \]

### 4 Conclusion

An important fractional reaction-diffusion is solved by proposition 2.2 using the theorem 2.1, in [3,2]. That has been given in this paper. One can investigate similar solutions for the more complicated fractional reaction-diffusion equation in future works.

**References**


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Preconditioned Jacobi iterative method for $Z$-matrix linear systems

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Abstract

In this paper, we discuss the adaptive Jacobi iterative method which uses $P = I + S(\alpha) + K(\beta)$ as a preconditioner. We present some comparison theorems, which show the rate of convergence of the new method is faster than the basic method. Numerical example show the effectiveness of our algorithm.

Keywords: $Z$-matrix, preconditioner, Jacobi method

1 Introduction

We consider iterative methods for solving a linear system

$$Ax = b,$$  
(1)

where $A$ is an $n \times n$ matrix, $x$ and $b$ are $n$-dimensional vectors.

If we write $A = M - N$ with a nonsingular matrix $M$, then the basic iterative techniques for (13) is defined by

$$Mx_{k+1} = Nx_k + b, \quad k = 0, 1, 2, ...$$  
(2)

Let $A = MN^{-1}$, and $c = M^{-1}b$. Then (2) can be also written as

$$x_{k+1} = Tx_k + c, \quad k = 0, 1, 2, ...$$  
(3)

Let $A = -L + D - U$, in which $D$ is the diagonal of $A$, $-L$ its strict lower part, and $-U$ its strict upper part. Since $A$ is a $Z$-matrix, then the diagonal entries of $A$ are positive. Taking $M = D$, and $N = L + U$ in (2) or (1.3) yield the classic Jacobi method. The matrix $D^{-1}(L + U)$ is called the Jacobi iteration matrix.

We now transform the original system (13) into the preconditioner form

$$PAx = Pb, \quad P \in \mathbb{R}^{n \times n}.$$  
(4)

Then the corresponding basic iterative scheme is

$$x_{k+1} = M_p^{-1}N_px_k + M_p^{-1}b, \quad k = 0, 1, 2, ...,$$

where $M_p$ and $N_p$ realize the splitting of $PA$ into

$$PA = M_p - N_p,$$

and $M_p$ is a nonsingular matrix, the nonsingular matrix $P$ is called a preconditioner. The above iteration method used in system $PAx = Pb$ is called the preconditioner iterative method.
Definition 1.1. ([2]). A real $n \times n$ matrix $A$ is said to be Z-matrix, if it satisfies:
1) $a_{ii} > 0$, $i=1,2,\ldots,n$,
2) $a_{ij} \leq 0$, $i=1,2,\ldots,n$, $i \neq j$.

Definition 1.2. ([2]). A real $n \times n$ matrix $A$ is said to be diagonally dominant (strictly diagonally dominant) Z-matrix, if $\sum_{j=1}^{n} a_{ij} \geq 0$ ($\sum_{j=1}^{n} a_{ij} > 0$).

Lemma 1.3. ([2]). Let $A$ be an $n \times n$ nonnegative matrix, then
1) $\rho(A)$, the spectral radius of $A$, is an eigenvalue.
2) $A$ has a nonnegative eigenvector corresponding to $\rho(A)$.
3) $A$ has exactly one nonnegative eigenvector, and this eigenvector is positive.

Lemma 1.4. ([2]). Let $A$ be an $n \times n$ nonnegative matrix, then
1) If $\alpha x \leq Ax$ for some nonnegative vector $x \neq 0$, then $\alpha \leq \rho(A)$.
2) If $Ax \leq \beta x$ for some positive vector $x$, then $\rho(A) \leq \beta$.

2 New Preconditioner Iterative Method and Convergence Analysis

The Jacobi method is frustratingly slow. How ever, the rate of convergence of the Jacobi iteration can, in certain cases, be improved by considering a preconditioning matrix $P$, i.e. We transform the original system (13) into the preconditioner form

$$PAx = Pb,$$

where

$$P = I + S_\alpha^\beta$$

and $S_\beta^\alpha = S(\alpha) + K(\beta)$, where

$$S(\alpha) = \begin{pmatrix}
0 & -\alpha_1 a_{12} & 0 & \ldots & 0 \\
0 & 0 & -\alpha_2 a_{23} & \ddots & \vdots \\
0 & 0 & 0 & \ddots & 0 \\
\vdots & \vdots & \vdots & \ddots & -\alpha_{n-1} a_{n-1,n} \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix},$$

$$K(\beta) = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
-\beta_2 a_{2,1} & 0 & \ddots & \ddots & 0 \\
0 & -\beta_3 a_{3,2} & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & -\beta_{n-1} a_{n,n-1} & 0
\end{pmatrix}.$$
where

\[ D - D_0 = \]

\[
\text{diag}\{a_{11} - \alpha_1 a_{12} a_{21}, a_{22} - \beta_2 a_{21} a_{12} - \alpha_2 a_{23} a_{32}, \ldots, a_{n-1,n-1} - \beta_{n-1} a_{n-1,n-2} a_{n-2,n-1} - \alpha_{n-1} a_{n-1,n} - a_{nn} - \beta_n a_{n,n-1} a_{n-1,n}\}.
\]

If

\[
\begin{aligned}
& \alpha_1 a_{12} a_{21} \neq a_{11}; \\
& \beta_i a_{i,i-1} a_{i-1,i} + \alpha_i a_{i,i+1} a_{i+1,i} \neq a_{ii}, \quad i=2,\ldots,n-1; \\
& \beta_n a_{n,n-1} a_{n-1,n} \neq a_{nn},
\end{aligned}
\]

then matrix \((D - D_0)^{-1}\) exists and the Jacobi iteration matrix \(T^0_\beta\) for \(A^0_\beta\) is defined by

\[
T^0_\beta = (D - D_0)^{-1}(L + L_0 + U + U_0 - S^0_\beta D).
\]

Let

\[
\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}, \quad \beta = \{\beta_1, \beta_2, \ldots, \beta_n\}
\]

and

\[
S = \begin{pmatrix}
0 & -a_{12} & 0 & \cdots & 0 \\
0 & 0 & -a_{23} & \ddots & \vdots \\
0 & 0 & 0 & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \cdots & -a_{n-1,n} \\
\end{pmatrix},
\]

\[
K = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
-\alpha_1 & 0 & \cdots & 0 & 0 \\
0 & -\alpha_2 & \cdots & 0 & 0 \\
0 & 0 & \cdots & \ddots & \ddots \\
0 & 0 & \cdots & \ddots & -\alpha_{n-1,n}
\end{pmatrix},
\]

then \(S^0_\beta = S(\alpha) + K(\beta) = \alpha S + \beta K\). If \(\alpha\) and \(\beta\) are identity matrix, then

\[
\tilde{A} = (I + S + K)A = \tilde{D} - \tilde{L} - \tilde{U}, \quad \tilde{T} = \tilde{D}^{-1}(\tilde{L} + \tilde{U}).
\]

**Theorem 2.1.** Let \(A \in \mathbb{R}^{n \times n}\) \((n \geq 3)\) be a strictly diagonally dominant Z- matrix. If for

\[ 0 < \alpha_i \leq 1, \quad i = 1, 2, \ldots, n - 1, \quad \text{and} \quad 0 < \beta_i \leq 1, \quad \text{strictly diagonally dominant Z-matrix, and} \]

\[ \rho(T^0_\beta) < 1. \]

1) \(a_{12} a_{21} < a_{11}\),
2) \(a_{i,i-1} a_{i-1,i} + a_{i,i+1} a_{i+1,i} < a_{ii}, \quad i = 2, 3, \ldots, n - 1,\)
3) \(a_{n,n-1} a_{n-1,n} < a_{nn},\)
4) \(0 < a_{ii} \leq 1, \quad i = 1, 2, \ldots, n.\)

**Theorem 2.2.** Let \(A = D - L - U\) be an \(n \times n\) Z- matrix \((n \geq 3)\), and matrix \(A\) satisfies the following conditions:

1) \(a_{12} a_{21} < a_{11}\),
2) \(a_{i,i-1} a_{i-1,i} + a_{i,i+1} a_{i+1,i} < a_{ii}, \quad i = 2, 3, \ldots, n - 1,\)
3) \(a_{n,n-1} a_{n-1,n} < a_{nn},\)
4) \(0 < a_{ii} \leq 1, \quad i = 1, 2, \ldots, n.\)

If \(\rho(T) < 1\), then \(\rho(T^0_\beta) < \rho(T)\).

**Example 2.3.** Let

\[
A = \begin{pmatrix}
1 & -0.4 & -0.3 & 0 \\
-0.5 & 1 & -0.2 & -0.1 \\
-0.2 & -0.7 & 1 & 0 \\
-0.1 & 0 & -0.8 & 1
\end{pmatrix}, \quad b = \begin{pmatrix}
0.3 \\
0.2 \\
0.1 \\
0.1
\end{pmatrix}.
\]

It is obviously that \(A\) is a strictly diagonally dominant Z-matrix. Now the classical Jacobi iterative method and the preconditioned Jacobi iterative method has considered for solving the linear system.
of equation (13). The stopping criterion \( \|x^{(k)} - x^*\|_\infty < 10^{-4} \) was used in the computations, where \( x^{(k)} \) is the \( k \)-th iteration for each of the methods and \( x^* \) is exact solution of (13). The number of iterations for convergence needed for the classical Jacobi iterative method and the preconditioned Jacobi iterative method are shown in Figure 1.

![Figure 1: The results obtained for Example 2.3](image1.png)

Thus it can be seen that the preconditioner Jacobi method proposed in this paper has a faster convergence rate than the classical Jacobi iterative method. Let \( x^{(0)} = (0, 0, 0)^T \), \( \alpha = (1, 0.7, 0.7, 0) \), and \( \beta = (0, 0.7, 0.9, 1) \).

References


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Solutions of twelfth-order boundary value problems using polynomial spline off step points

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Abstract
Polynomial spline is used to solve twelfth-order linear boundary value problems. Boundary formulas are developed. We compare our results with the results produced by Non-polynomial spline method. However, it is observed that our approach produce better numerical solutions in the sense that max|\epsilon_i| is a minimum.

Keywords: Twelfth-order boundary-value problem, Polynomial spline functions, Boundary value formulae, Numerical results.

Mathematics Subject Classification: 65L10

1 Introduction

The solutions of twelfth-order boundary value problems are not very much found in the numerical analysis literature. These problems generally arise in the mathematical modelling of viscoelastic flows [1, 2]. The conditions for existence and uniqueness of solution of such boundary value problems are explained by theorems presented in Agarwal [3]. Siddiqi and Twizell [4-7] presented the solutions of sixth-, eighth-, tenth- and twelfth-order boundary value problems using the sixth, eighth, tenth and twelfth degree spline, respectively. Siddiqi and Ghazla [8] Solutions of 12th order boundary value problems using non-polynomial spline technique. In this paper, we used polynomial spline approximation in off step points to develop a family of new numerical methods to smooth approximations to the solution of twelfth-order differential equation. In this manuscript, the following twelfth-order boundary value problem is considered:

\[ y^{(12)}(x) + f(x)y(x) = g(x), \quad x \in [a,b], \]  

With boundary conditions

\[ y(a) = A_0, y^{(1)}(a) = A_1, y^{(2)}(a) = A_2, y^{(3)}(a) = A_3, y^{(4)}(a) = A_4, y^{(5)}(a) = A_5, \]

\[ y(b) = B_0, y^{(1)}(b) = B_1, y^{(2)}(b) = B_2, y^{(3)}(b) = B_3, y^{(4)}(b) = B_4, y^{(5)}(b) = B_5 \]  

Where \( A_i, B_i \) for \( i = 0, 1, 2, 3, 4, 5 \) are finite real constants and the functions \( f(x) \) and \( g(x) \) are continuous on \([a,b] \).

To develop the spline approximation to the twelfth-order boundary-value problem (1)-(2), the interval \([a,b]\) is divided into \( n \) equal subintervals using the grid \( x_{i-\frac{1}{2}} = a + (i - \frac{1}{2})h, i = 1, ..., n, \)

where \( h = \frac{b-a}{n} \). Consider the following polynomial twelfth spline \( S_i(x) \) is each subinterval \([x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], i = 0, 1, ..., n-1, x_0 = a, x_n = b, \)

\[ S_i(x) = a_i(x-x_i)^{12} + b_i(x-x_i)^{11} + c_i(x-x_i)^{10} + d_i(x-x_i)^9 + e_i(x-x_i)^8 + f_i^*(x-x_i)^7 + \]
\[ g_i^*(x-x_i)^6 + k_i(x-x_i)^5 + c_i(x-x_i)^4 + f_i(x-x_i)^3 + g_i(x-x_i)^2 + r_i(x-x_i) + v_i \] 

(3)

Where \( a_i, b_i, c_i, d_i, e_i, f_i, g_i, k_i, r_i, v_i \) are real finite constants. The spline \( S \) is defined in terms of its 1th and 2th derivatives and we denote these values at knots as:

\[ S_i(x_{i-\frac{1}{2}}) = y_{i-\frac{1}{2}}, \quad S_i^{(2)}(x_{i-\frac{1}{2}}) = m_{i-\frac{1}{2}}, \quad S_i^{(4)}(x_{i-\frac{1}{2}}) = M_{i-\frac{1}{2}}, \quad S_i^{(6)}(x_{i-\frac{1}{2}}) = W_{i-\frac{1}{2}}, \]

\[ S_i^{(8)}(x_{i-\frac{1}{2}}) = u_{i-\frac{1}{2}}, \quad S_i^{(10)}(x_{i-\frac{1}{2}}) = n_{i-\frac{1}{2}}, \quad S_i^{(12)}(x_{i-\frac{1}{2}}) = L_{i-\frac{1}{2}}, \]

\[ S_i^{(8)}(x_{i+\frac{1}{2}}) = y_{i+\frac{1}{2}}, \quad S_i^{(2)}(x_{i+\frac{1}{2}}) = m_{i+\frac{1}{2}}, \quad S_i^{(4)}(x_{i+\frac{1}{2}}) = M_{i+\frac{1}{2}}, \quad S_i^{(6)}(x_{i+\frac{1}{2}}) = W_{i+\frac{1}{2}}, \]

\[ S_i^{(8)}(x_{i+\frac{1}{2}}) = u_{i+\frac{1}{2}}, \quad S_i^{(10)}(x_{i+\frac{1}{2}}) = n_{i+\frac{1}{2}}, \quad S_i^{(12)}(x_{i+\frac{1}{2}}) = L_{i+\frac{1}{2}}, \]

for \( i = 1, 2, ..., n. \) 

(4)

Assuming \( y(x) \) to be the exact solution of the boundary value problem (1) and \( y_i \) be an approximate solution obtained by the spline \( S(x_i) \), after lengthy calculations we obtain the following spline relations.

\[ y_{i-\frac{1}{2}} - 12y_{i-\frac{3}{2}} + 66y_{i-\frac{5}{2}} - 220y_{i-\frac{7}{2}} + 49y_{i-\frac{9}{2}} - 792y_{i-\frac{11}{2}} + 924y_{i-\frac{13}{2}} - 792y_{i-\frac{15}{2}} + 49y_{i-\frac{17}{2}} \]

\[-220y_{i+\frac{1}{2}} + 66y_{i+\frac{3}{2}} - 12y_{i+\frac{5}{2}} + y_{i+\frac{7}{2}} = \frac{h^{12}}{362880} \left[ 902L_{i-\frac{1}{2}} + 4098L_{i-\frac{3}{2}} + 10000L_{i-\frac{5}{2}} + 10000L_{i-\frac{7}{2}} + 15000L_{i-\frac{9}{2}} + 262880L_{i-\frac{11}{2}} + 15000L_{i+\frac{1}{2}} + 10000L_{i+\frac{3}{2}} + 10000L_{i+\frac{5}{2}} + 902L_{i+\frac{7}{2}} \right], \quad i = 7, 8, ..., n - 7. \]

(5)

2 Development of the boundary formulas

To obtain unique solution we need 12 more equations to be associated with (5) so that we need to consider the following boundary conditions. In order to obtain the twelfth-order boundary formula we define the following identity:

\[ u_{00}y_0 + \sum_{k=0}^{8} a_k^*y_{k+\frac{1}{2}} + c^*y_{0}^{(1)} + d^*h_{y_{0}^{(2)}} + e^*h_{y_{0}^{(3)}} + f^*h_{y_{0}^{(4)}} + g^*h_{y_{0}^{(5)}} = h^{12} \sum_{k=0}^{9} b_k^{(12)} \]

\[ u_{00}y_0 + \sum_{k=0}^{9} a_k^*y_{k+\frac{1}{2}} + c^*y_{0}^{(1)} + d^*h_{y_{0}^{(2)}} + e^*h_{y_{0}^{(3)}} + f^*h_{y_{0}^{(4)}} + g^*h_{y_{0}^{(5)}} = h^{12} \sum_{k=0}^{10} b_k^{(12)} \]

\[ u_{00}y_0 + \sum_{k=0}^{10} a_k^*y_{k+\frac{1}{2}} + c^*y_{0}^{(1)} + d^*h_{y_{0}^{(2)}} + e^*h_{y_{0}^{(3)}} + f^*h_{y_{0}^{(4)}} + g^*h_{y_{0}^{(5)}} = h^{12} \sum_{k=0}^{11} b_k^{(12)} \]

\[ u_{00}y_0 + \sum_{k=0}^{11} a_k^*y_{k+\frac{1}{2}} + c^*y_{0}^{(1)} + d^*h_{y_{0}^{(2)}} + e^*h_{y_{0}^{(3)}} + f^*h_{y_{0}^{(4)}} + g^*h_{y_{0}^{(5)}} = h^{12} \sum_{k=0}^{12} b_k^{(12)} \]

\[ u_{00}y_0 + \sum_{k=0}^{12} a_k^*y_{k+\frac{1}{2}} + c^*y_{0}^{(1)} + d^*h_{y_{0}^{(2)}} + e^*h_{y_{0}^{(3)}} + f^*h_{y_{0}^{(4)}} + g^*h_{y_{0}^{(5)}} = h^{12} \sum_{k=0}^{13} b_k^{(12)} \]

\[ u_{00}y_0 + \sum_{k=0}^{13} a_k^*y_{k+\frac{1}{2}} + c^*y_{0}^{(1)} + d^*h_{y_{0}^{(2)}} + e^*h_{y_{0}^{(3)}} + f^*h_{y_{0}^{(4)}} + g^*h_{y_{0}^{(5)}} = h^{12} \sum_{k=0}^{14} b_k^{(12)} \]

...
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Existence, multiplicity and nonexistence results of positive solutions for $m$-point nonlinear fractional differential equation on half axis

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Abstract
This paper shows sufficient conditions for the existence of multiplicity also nonexistence results of positive solutions to the $m$-point nonlinear fractional eigenvalue problem on an infinite interval. At the end we prove that the set of positive solutions is compact.

Keywords: fractional derivative, fixed point theorem, positive solution, infinite interval.

Mathematics Subject Classification: 34B10, 34B15, 34B18

1 Introduction
In this paper we consider the following BVP

$$D_0^+ u(t) + \lambda a(t)f(t, u(t)) = 0, \quad t \in (0, \infty), \quad \alpha \in (2, 3)$$

$$u(0) + u'(0) = 0, \quad \lim_{t \to +\infty} D_0^{\alpha-1} u(t) = \sum_{i=1}^{m-2} \beta_i u' (\xi_i)$$

where $D_0^+$ is representation of fractional Riemann-Liouville derivative of order $\alpha > 0$ and $\lambda$ is a positive parameter. Let the following conditions hold:

$(H_1)$ $f \in C ( (0, \infty) \times [0, \infty), [0, \infty))$, $f(t, 0) \neq 0$ on any subinterval of $(0, +\infty)$, also when $u$ is bounded $f(t, (1 + t^{\alpha-1}) u)$ is bounded on $[0, +\infty)$.

$(H_2)$ $a \in C ( (0, \infty), [0, \infty))$ and $a(t) \neq 0$ on $(0, \infty)$ and $0 < \int_0^\infty a(s) ds < \infty$.

$(H_3)$ $0 < \sum_{i=1}^{m-2} (\alpha - 1) \beta_i \xi_i^{\alpha-2} < \Gamma (\alpha)$.

Definition 1.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to \mathbb{R}$ is given by integral

$$I_0^\alpha u(t) = \frac{1}{\Gamma (\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$
**Definition 1.2.** [1] The Riemann-Liouville fractional derivative of order \( \alpha > 0 \) for a function \( u : (0, \infty) \rightarrow \mathbb{R} \) is defined by

\[
D_0^\alpha u(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n-\alpha-1} u(s) ds, \quad (n = [\alpha] + 1).
\]

**Lemma 1.3.** Let \( h \in C[0, \infty) \) such that \( 0 < \int_0^{+\infty} a(s) ds < +\infty \), then fractional boundary value problem

\[
D_0^\alpha u(t) + h(t) = 0, \quad t \in (0, \infty), \quad \alpha \in (2, 3)
\]

\[
u(0) + u'(0) = 0, \quad \lim_{t \to +\infty} D_0^{\alpha - 1} u(t) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i)
\]

has a unique solution

\[
u(t) = \int_0^{+\infty} G(t, s) h(s) ds,
\]

where

\[G(t, s) = H_1(t, s) + H_2(t, s)\]

with

\[
H_1(t, s) = \frac{1}{\Gamma(\alpha)} \left\{ \begin{array}{ll}
t^{\alpha - 1} - (t-s)^{\alpha - 1} & ; \ 0 \leq s \leq t < +\infty \\
(t-s)^{\alpha - 1} & ; \ 0 \leq t < s < +\infty
\end{array} \right.,
\]

\[
H_2(t, s) = \frac{\sum_{i=1}^{m-2} \beta_i t^{\alpha - 1}}{\Gamma(\alpha) - \sum_{i=1}^{m-2} (\alpha - 1) \beta_i s^{\alpha - 2}} \left| \frac{\partial H_1(t, s)}{\partial t} \right|_{t=\xi_i},
\]

note that the function \( G(t, s) \) is so called Green's function of boundary value problem (1.3),(1.4).

**Lemma 1.4.** Let \( k > 1 \) is fix and \( G(t, s) \) is defined by (1.6)-(1.8),then

\[
\min_{\frac{1}{k} \leq \xi \leq k} \frac{G(t, s)}{1 + t^{\alpha - 1}} \geq \lambda(k) \sup_{0 \leq t < +\infty} \frac{G(t, s)}{1 + t^{\alpha - 1}},
\]

\[
\lambda(k) = \min \left\{ \frac{k^{2(\alpha - 1)}}{k^{\alpha - 1} (1 + k^{\alpha - 1})}, \frac{1}{1 + k^{\alpha - 1}} \right\}.
\]

**Remark 1.5.** We introduce the following Banach space

\[B = \left\{ u \in C[0, +\infty) \left| \| u \| < +\infty \right. \right\}\]

such that equipped with the norm

\[\| u \| = \sup_{t \in [0, +\infty)} \frac{|u(t)|}{1 + t^{\alpha - 1}}\]

also we define the cone \( P \subset B \) by

\[P = \left\{ u \in B \left| \ u(t) \geq 0, \ \min_{t \in \left[ \frac{1}{k}, k \right]} \frac{u(t)}{1 + t^{\alpha - 1}} \geq \lambda(k) \| u \| \right. \right\}\]

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Lemma 1.6. Let conditions (H1), (H2), (H3) be satisfied and we define the integral operator

\[ Tu(t) = \lambda \int_0^t G(t,s)a(s)f(s,u(s))ds. \] (9)

So \( TP \subset P \).

Lemma 1.7. If conditions (H1), (H2), (H3) hold, then operator \( T : P \rightarrow P \) is completely continuous.

Theorem 1.8. [2] Let \( X \) be a real Banach space and \( P \subset X \) be a cone in \( X \). Assume \( \Omega_1, \Omega_2 \) are two open bounded subsets of \( X \) with \( 0 \in \Omega_1, \Omega_1 \subset \Omega_2 \) and \( T : P \cap (\Omega_2 \setminus \Omega_1) \rightarrow P \) be a completely continuous operator such that

(i) \( \| Tu \| \leq \| u \|, u \in P \cap \partial \Omega_1 \) and \( \| Tu \| \geq \| u \|, u \in P \cap \partial \Omega_2 \), or

(ii) \( \| Tu \| \leq \| u \|, u \in P \cap \partial \Omega_2 \) and \( \| Tu \| \geq \| u \|, u \in P \cap \partial \Omega_1 \).

Then \( T \) has a fixed point in \( P \cap (\Omega_2 \setminus \Omega_1) \).

2 Main Results

The following theorems rely on Theorem 1.8 which has two possibilities (i) and (ii). We have to prove any case may occur.

Remark 2.1. We introduce the following notation

\[ f_0 = \lim_{u \to -0} \min_{t \in \left[ \frac{1}{k}, k \right]} \left( 1 + t^{\alpha-1} \right) \frac{f(t,u)}{u}, \quad f_{\infty} = \lim_{u \to +\infty} \min_{t \in \left[ \frac{1}{k}, k \right]} \left( 1 + t^{\alpha-1} \right) \frac{f(t,u)}{u}, \quad t \in \left[ \frac{1}{k}, k \right] \]

\[ f^0 = \lim_{u \to -0} \sup_{t \in \left[ 0, +\infty \right)} \left( 1 + t^{\alpha-1} \right) \frac{f(t,u)}{u}, \quad f^{\infty} = \lim_{u \to +\infty} \sup_{t \in \left[ 0, +\infty \right)} \left( 1 + t^{\alpha-1} \right) \frac{f(t,u)}{u}, \quad t \in (0, \infty) \]

\[ A = \left( L \int_0^{+\infty} a(s)ds \right)^{-1}, \quad B = \left( \lambda^2(k) \int_{1/k}^k a(s)ds \right)^{-1}. \]

Theorem 2.2. Let conditions (H1), (H2), (H3) hold. Then boundary value problem (1.1), (1.2) has at least one positive solution on \( P \) when either

(C1) For every \( \lambda \in \left( \frac{P}{f_0}, \frac{A}{f_{\infty}} \right) \) such that \( f_0, f_{\infty} \in (0, \infty) \) with \( \lambda(k)f_0 > f_{\infty} \) or

(C2) For every \( \lambda \in \left( \frac{P}{f_{\infty}}, \frac{A}{f^0} \right) \) such that \( f_{\infty}, f^0 \in (0, \infty) \) with \( \lambda(k)f_{\infty} > f^0 \).

Theorem 2.3. Let conditions (H1) - (H4) hold. Assume that there exist positive constants \( R_2 > R_1 \), such that

\[ \frac{BR_1}{\min_{t \in [1/k, k]} f(t, \lambda(k)R_1)} \leq \lambda \leq \frac{AR_2}{\sup_{t \in (0, +\infty)} f(t, R_2)} \] (10)

then the boundary value problem (1.1), (1.2) has at least two positive solutions \( v_1, v_2 \) such that

\[ R_1 \leq \| v_1 \| \leq R_2, \quad \lim_{u \to -\infty} T^nu_0 = v_1, \quad u_0 = R_2, \quad t \in (0, +\infty) \]

and

\[ R_1 \leq \| v_2 \| \leq R_2, \quad \lim_{u \to -\infty} T^nu_0 = v_2, \quad u_0 = R_1, \quad t \in (0, +\infty) \]
Theorem 2.4. Let conditions \((H_1)-(H_3)\) hold. If \(f^0, f^\infty < \infty\), then there exist a positive constant \(\lambda_0\), such that for every \(0 < \lambda < \lambda_0\), the boundary value problem (1.1),(1.2) has no positive solution.

Theorem 2.5. Let conditions \((H_1)-(H_3)\) hold. If \(f_0, f_\infty\), then there exist a positive constant \(\lambda_0\), such that for every \(\lambda > \lambda_0\), the boundary value problem (1.1),(1.2) has no positive solution.

Theorem 2.6. Assume conditions \((H_1)-(H_3)\) hold. If

\[ f_0, f_\infty \in (0, +\infty) , \quad f_0 \lambda(k) > f_\infty , \quad \lambda \in \left( \frac{B}{f_0}, \frac{A}{f_\infty} \right). \] (11)

Then the set of positive solutions for boundary value problem (1.1),(1.2) is nonempty and compact.

References


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Trace formula for second order differential equation with one turning point from older of one

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Abstract
In this paper we consider the Sturm-Liouville problem with Dirichlet boundary condition. Then for first time we study trace formula for Sturm-Liouville problem with one turning point from older of one.

Keywords: turning point, trace formula, Residue Theorem, Sturm-Liouville problem

Mathematics Subject Classification: 34E20, 34B24, 34E05

1 Introduction
In a finite-dimensional space, an operator has a finite trace. However, in an infinite-dimensional space, in general, ordinary differential operators do not have a finite trace (the sum of all eigenvalues). Gelfand and Levitan [4] firstly obtained a trace formula for a self-adjoint Sturm-Liouville differential equation. In this paper for first time we study trace formula for Sturm-Liouville problem with one turning point from older of one.

2 asymptotic expansion of $\frac{\Delta(\lambda)}{\Delta_0(\lambda)}$

We consider the equation
\[ L(y) = y'' + [\lambda^2 r_1(x) + q(x)]y = 0, \] (1)
Where $0 \leq x \leq \pi$, $0 < r_1 < M$ and we assume $q(x)$ is bounded and differentiable. Now we consider the comparing equation
\[ U''(x) + xU(x) = 0 \] (2)
where satisfying the initial conditions
\[ U_1(0) = 1, \quad U_2(0) = 0 \]
\[ U'_1(0) = 0, \quad U'_2(0) = 1 \], (3)
that in the [1] two linearly independent solutions of the equation (2) been taken in the form
\[ U_1(x) = (3)^{-\frac{1}{2}} \Gamma(\frac{3}{4}) \sqrt{x} J_{\frac{3}{4}} \left( \frac{2}{3} x^{\frac{3}{2}} \right) \]
\[ U_2(x) = (3)^{\frac{1}{2}} \Gamma(\frac{3}{4}) \sqrt{x} J_{\frac{3}{4}} \left( \frac{2}{3} x^{\frac{3}{2}} \right) \], (4)
We shall seek a solution of the equation (1) in the form 
\[ y = A(x)U_1[\phi(x)] + B(x)U_2[\phi(x)], \]
and Substituting this expression in (1) We vanish in the expression for \( L(y) \) terms of a higher order in \( \lambda \), then we conclude the solutions \( \phi(x) = \lambda^{\frac{3}{2}}w(x) \), where \( w(x) \) does not depend on \( \lambda \) and \( w(x)w''(x) = x^2r(x) \).

\[
w(x) = \left\{ \frac{3}{2} \int_0^x \frac{\sqrt{r(x)}dx}{\sqrt{w(x)}} \right\}^{\frac{3}{2}}, \quad A(x) = \frac{\text{const}}{\sqrt{w(x)}}, \quad B(x) = \frac{\text{const}}{w'(x)}.
\]

for \( 0 \leq x \leq \pi \) with the condition \( r_1(x) > 0 \). we have \( |r_1(0)|^{\frac{3}{4}} \neq 0 \), and for a special value of \( q(x) \), namely we put \( q_0(x) = -\sqrt{\frac{\pi}{w(x)}} \frac{d^2}{dx^2} \left( \frac{1}{\sqrt{w(x)}} \right) \). \( L(y) \) vanishes identically, and the solution of the equation (1) will be

\[
y(x) = C_1\alpha_1(x) + C_2\alpha_2(x), \quad \alpha_i(x) = \frac{1}{\sqrt{w(x)}}U_i[\lambda^{\frac{3}{2}}w(x)], \quad i = 1, 2. \tag{5}
\]

In the general case we apply the method of successive approximations. Representing the equation (1) in the form

\[
y''(x) + [\lambda^2x^2r_1(x) + q_0(x)] y(x) = -f(x)y,
\]

where \( f(x) = q(x) - q_0(x) \). and putting

\[
y_1(x) = \frac{U_1[\lambda^{\frac{3}{2}}w(x)]}{\sqrt{w(x)}} + u_1(x), \quad y_2(x) = \frac{U_2[\lambda^{\frac{3}{2}}w(x)]}{\sqrt{w(x)}} + u_2(x)
\]

We get for \( u_1 \) (likewise \( u_2 \)) from [1] the integral equation

\[
u_1(x) = \lambda^{-\frac{3}{2}} \int_0^x k(x, \zeta)U_1[\lambda^{\frac{3}{2}}w(\zeta)] \frac{d\zeta}{\sqrt{w'(\zeta)}} + \lambda^{-\frac{3}{2}} \int_0^x k(x, \zeta)u_1(\zeta) d\zeta. \tag{7}\]

and analogous equation for \( u_2(x) \). The kernel \( k(x, \zeta) \) is given by

\[
k(x, \zeta) = \frac{f(\zeta)}{\sqrt{w'(x)w'(\zeta)}} \{U_1[\lambda^{\frac{3}{2}}w(x)]U_2[\lambda^{\frac{3}{2}}w(\zeta)] - U_2[\lambda^{\frac{3}{2}}w(x)]U_1[\lambda^{\frac{3}{2}}w(\zeta)]\}.
\]

Now we let

\[
y''(x) + [\lambda^2x^2r_1(x) + q_0(x)] y(x) = 0, \tag{8}\]

we know that \( \alpha_1(x) \) and \( \alpha_2(x) \) are linearly independent from (8), Then we have characteristic equation for (8) in the form

\[
\Delta_0(\lambda) = (w'(0)w'((\pi)))^{-\frac{1}{2}}(3)^{-\frac{1}{2}}(\Gamma(\frac{4}{3})\lambda^{\frac{3}{2}}\lambda^{\frac{3}{2}}(\pi))^{\frac{4}{3}}(2)^{\frac{2}{3}}(\lambda(w(\pi)))^{\frac{5}{2}},
\]

since we get asymptotic expansion of \( J_{\frac{4}{3}}(\frac{2}{3}\lambda(w(\pi))^\frac{5}{2}) \) from [3] in the form

\[
J_{\frac{4}{3}}(\frac{2}{3}\lambda(w(\pi))^\frac{5}{2}) = \cos(\frac{2\pi}{9})\sqrt{\frac{2\pi}{3}}(\frac{2}{3})^{-\frac{1}{2}}(w(\pi))^{-\frac{3}{2}}(\omega(0)w'((\pi)))^{-\frac{3}{2}}(\Gamma(\frac{4}{3})\lambda(w(\pi)))^{\frac{5}{2}}.
\]

hence

\[
\Delta_0(\lambda) = c_1(\frac{4}{3})(\omega(\pi))^{\frac{5}{2}}(\omega(\pi))^{-\frac{3}{2}} + O(\lambda^{-\frac{12}{7}}),
\]

where \( c_1 = \cos(\frac{2\pi}{9})\sqrt{\frac{2\pi}{3}}(\frac{2}{3})^{-\frac{1}{2}}(w(\pi))^{-\frac{3}{2}}(\omega(0)w'((\pi)))^{-\frac{3}{2}}(\Gamma(\frac{4}{3})\lambda(w(\pi)))^{\frac{5}{2}}. \) zeros of \( \Delta_0(\lambda) \) are the same of \( J_{\frac{4}{3}}(\frac{2}{3}\lambda(w(\pi))^\frac{5}{2}) \) and since for the order of bessel function we have \( -\frac{1}{2} < \frac{1}{4} < \frac{1}{2}, \) then by [3]
zeros of $J_2\left(\frac{\pi}{2}\lambda(w(\pi))^2\right)$ are real values. Now since $y_1(x)$ and $y_2(x)$ are solutions of the equation (6), then from [1] we get characteristic equation of (6) in the form

$$\Delta(\lambda) = c_2\lambda^{-\frac{3}{2}} + c_3(w'(0))^{-\frac{1}{2}}\lambda^{-\frac{3}{2}}\int_0^\pi (w(\zeta))^{-\frac{1}{2}} \frac{f(\zeta)}{w'(\zeta)} d\zeta + O(\lambda^{-\frac{3}{2}}),$$

where $c_2 = (3)^{-\frac{1}{2}}\Gamma^2(\frac{i}{3})\Gamma(\frac{3}{2})(w(\pi))^{\frac{1}{2}}(w'(\pi))^{-\frac{1}{2}}(\cos(\frac{2\pi}{3})\sqrt{\frac{2}{3}w(\pi)}) \lambda^{-\frac{3}{4}}$. Then we have

$$\frac{\Delta(\lambda)}{\Delta_0(\lambda)} = 1 + c_3\lambda^{-1}\int_0^\pi (w(\zeta))^{-\frac{1}{2}} \frac{f(\zeta)}{w'(\zeta)} d\zeta + O(\lambda^{-2}), \quad c_3 = c_2(w'(0))^{-\frac{1}{2}}.$$

### 3 Trace formula

Let $\Gamma_{N_0}$ be the counterclockwise square contours $ABCD$ as in Fig. 1, integer $N_0 = 0, 1, 2, \cdots \to \infty$, with

\[
\begin{align*}
A &= (N_0 + \frac{1}{2})\pi(1 - i) \quad &B &= (N_0 + \frac{1}{2})\pi(1 + i) \\
C &= (N_0 + \frac{1}{2})\pi(-1 + i) \quad &D &= (N_0 + \frac{1}{2})\pi(-1 - i).
\end{align*}
\]

To obtain trace formula, for all sufficiently large $N_0$, the number $\lambda_n$ which are the zeros of the function $\Delta(\lambda)$, with $|n| \leq N_0$ are inside $\Gamma_{N_0}$ and the number $\lambda_n$, with $|n| > N_0$ are outside $\Gamma_{N_0}$, zeros of function $\Delta_0(\lambda)$, do not lie on the contour $\Gamma_{N_0}$. Now from Residue Theorem we have

$$\sum_{n=0}^{N_0} (\lambda_n + \lambda_{-n}) = -\frac{1}{2\pi i} \int_{\Gamma_{N_0}} \lambda \left[ \frac{\Delta'(\lambda)}{\Delta(\lambda)} - \frac{\Delta'(\lambda)}{\Delta_0(\lambda)} \right] d\lambda$$

$$= \frac{1}{2\pi i} \int_{\Gamma_{N_0}} \lambda d\log \frac{\Delta(\lambda)}{\Delta_0(\lambda)} = -\frac{1}{2\pi i} \int_{\Gamma_{N_0}} \log \frac{\Delta(\lambda)}{\Delta_0(\lambda)} d\lambda.$$

Expanding $\log \frac{\Delta(\lambda)}{\Delta_0(\lambda)}$ by the Maclaurin formula we have

$$\sum_{n=0}^{N_0} (\lambda_n + \lambda_{-n}) = -\frac{1}{2\pi i} \int_{\Gamma_{N_0}} c_3\lambda^{-1} \int_0^\pi (w(\zeta))^{-\frac{1}{2}} \frac{f(\zeta)}{w'(\zeta)} d\zeta + O(\lambda^{-2})$$

and for large $N_0$

$$\left|\int_{\Gamma_{N_0}} O\left(\frac{1}{N_0}\right) d\lambda \right| = O\left(\frac{1}{N_0}\right),$$

Then we get

$$\sum_{n=0}^{N_0} (\lambda_n + \lambda_{-n}) = c_3 \int_0^\pi (w(\zeta))^{-\frac{1}{2}} \frac{f(\zeta)}{w'(\zeta)} d\zeta + O\left(\frac{1}{N_0}\right).$$
Passing to the limit as $N_0 \to \infty$, we find that
\[
\sum_{n=0}^{N_0} (\lambda_n + \lambda_{-n}) \sim c_3 \int_0^{\pi} (w(\zeta))^{-\frac{1}{2}} \frac{f(\zeta)}{w(\zeta)} \, d\zeta.
\]

References


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A Duhamel integral approach to solve an inverse problem for the wave equation

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Abstract

In this paper, we will first study a numerical method of the solution of a one-dimensional Inverse Problem for a linear wave equation with initial and boundary conditions. Then a numerical method consisting of Tikhonov regularization to the matrix form of Duhamel’s principle for solving this inverse problem is presented. The stability and accuracy of the scheme presented is evaluated by comparison with the Singular Value Decomposition method. Some numerical experiments confirm the utility of this algorithm as the results are in good agreement with the exact data.

Keywords: Inverse wave problem, Existence and uniqueness, Stability, Tikhonov regularization method, SVD method.

Mathematics Subject Classification: 35R30

1 Introduction

Inverse problems are encountered in many branches of engineering and science. In one particular branch, hyperbolic and parabolic initial and boundary value problems in one dimension have been studied by several authors. Zhongyan Lin and R. P. Gilbert presented a numerical algorithm for solving an undetermined coefficient problem for an inverse wave equation. In this paper, a numerical method consisting of Tikhonov regularization to the matrix form of Duhamel’s principle for solving a one-dimensional inverse problem, for a linear wave equation with initial and boundary conditions, using measurement data containing noise is presented. The measurements ensure that the inverse problem has a unique solution, but this solution is unstable hence the problem is ill-posed. This instability is overcome using the zeroth-, first-, and second-order Tikhonov regularization method with the gcv criterion for the choice of the regularization parameter. The stability and accuracy of the scheme presented is evaluated by comparison with the Singular Value Decomposition method (SVD). As well, the existence and uniqueness of the solution is also derived.
2 Mathematical formulation

In this section, we consider the following Inverse wave equation in the dimensionless form

\[ T_{tt}(x,t) - T_{xx}(x,t) = F(x,t), \quad 0 < x < 1, \quad 0 < t < t_M \]  
(1a)

\[ T(x,0) = f(x), \quad 0 \leq x \leq 1, \]  
(1b)

\[ T_t(x,0) = h(x), \quad 0 \leq x \leq 1, \]  
(1c)

\[ T(0,t) = q(t), \quad 0 \leq t \leq t_M, \]  
(1d)

\[ T(1,t) = p(t), \quad 0 \leq t \leq t_M, \]  
(1e)

and the overspecified condition

\[ T(a,t) = g(t), \quad 0 \leq t \leq t_M, \]  
(1f)

where \( 0 < a < 1 \) is a fixed point, \( f(x) \) and \( h(x) \) are two continuous known functions, \( p(t) \) is an infinitely differentiable known function and \( t_M \) represents the final time of interest for the time evolution of the problem, while \( q(t) \) is unknown which remains to be determined from some interior measurements.

3 Overview of the Method

The solution of the problem (1) can be written as follows,

\[ u(x,t) = \sum_{i=1}^{4} u_i(x,t), \]  
(2)

where \( u_i(x,t) \), for \( i = 1, 2, 3, 4 \), satisfy the following problem:

\[ \frac{\partial^2 u_i}{\partial t^2}(x,t) = \frac{\partial^2 u_i}{\partial x^2}(x,t), \quad i = 1, 2, 3, 4, \quad 0 < x < 1, \quad 0 < t < t_M \]  
(3a)

\[ u_i(0,t) = \begin{cases} q(t), & i = 1 \\ 0, & \text{otherwise} \end{cases}, \quad 0 \leq t \leq t_M, \]  
(3b)

\[ u_i(1,t) = \begin{cases} p(t), & i = 2 \\ 0, & \text{otherwise} \end{cases}, \quad 0 \leq t \leq t_M, \]  
(3c)

\[ u_i(x,0) = \begin{cases} f(x), & i = 3 \\ 0, & \text{otherwise} \end{cases}, \quad 0 \leq x \leq 1, \]  
(3d)

\[ \frac{\partial u_i}{\partial t}(x,0) = \begin{cases} g(x), & i = 4 \\ 0, & \text{otherwise} \end{cases}, \quad 0 \leq x \leq 1. \]  
(3e)

In function estimation as in parameter estimation a detailed examination of the sensitivity coefficient can provide considerable insight in to the estimation problem. These coefficient can show possible areas of difficulty and also lead to improved experimental design. The sensitivity coefficient is defined as the first derivative of a dependent variable, such as \( u \), with respect to an unknown parameter, such as \( q(t) \) component. If the sensitivity coefficients are either small or correlated with one another, the estimation problem is difficult and very sensitive to measurement errors. For this problem, the sensitivity coefficient are defined by

\[ X_{iM}(x_j, t_i) \equiv \frac{\partial u_i(x_j, t_i)}{\partial q_M} = \begin{cases} \Delta \phi_{i-j}, & i \geq j \\ 0, & i < j \end{cases}, \]  
(4)

for \( j = 1, 2, \ldots, J, \quad i = 1, 2, \ldots, n \), and \( M = 1, 2, \ldots, n \). Note that the number of times \( t_i \) equals the number of \( q \) components.
In the linear problem (3) for \( i = 1 \), the relation between \( q(t) \) and \( u_1(x,t) \) can be expressed analytically by the Duhamel’s integral as follows [1]

\[
  u_1(x,t) = \int_0^t q(s) \frac{\partial \phi}{\partial t}(x,t-s) \, ds + u_1(x,0),
\]

where \( u_1(x,0) \) is the initial condition for problem (3) for \( i = 1 \). Equation (5) can be approximated at time \( t_M \), by the following equation

\[
  (u_1)_M = \sum_{n=1}^{M} q_n \Delta \phi_{M-n},
\]

where \( q_n \) represents the measured \( u(0,t) \) at time \( t_n \) and \( \Delta \phi_j = \phi_{j+1} - \phi_j \). Note that \( \Delta \phi_{j-k} = \frac{\partial q(t)}{\partial q(x)} \), therefore it represents the sensitivity coefficient measured at time \( t_j \) with respect to \( q_k \).

By writing the equation (6) for \( M = 1,2,\ldots \), points, we obtain the following matrix equation

\[
  u_1 = X q
\]

where \( q = [q_1, \ldots, q_M]^T \), \( q_j = q(t_j) \), \( j = 1,2,\ldots,M \) and

\[
  X = \begin{pmatrix}
  \Delta \phi_0 & 0 & \cdots & 0 & 0 & 0 \\
  \Delta \phi_1 & \Delta \phi_0 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  \Delta \phi_{M-2} & \Delta \phi_{M-3} & \cdots & \Delta \phi_1 & \Delta \phi_0 & 0 \\
  \Delta \phi_{M-1} & \Delta \phi_{M-2} & \cdots & \Delta \phi_1 & \Delta \phi_0 & \Delta \phi_0
  \end{pmatrix}.
\]

By solving the direct problem (3), for \( i = 2,3,4 \), and using the overspecified condition (1f) and the equation (2), we have

\[
  u^*(a,t) = \zeta(t) - u_2(a,t) - u_3(a,t) - u_4(a,t) = u_1(a,t).
\]

Considering the Duhamel’s theorem, for \( M = 1,2,\ldots \), we obtain the following equation

\[
  u^* = X q,
\]

where \( u^* = [u_{i1}, \ldots, u_{iM}]^T \), \( u_{ij}^* = u^*(a,t_j) \), \( j = 1,2,\ldots,M \).

The matrix \( X \) is ill-conditioned. On the other hand, as \( \zeta \) is affected by measurement errors, the estimate of \( q \) by (9) will be unstable so that the Tikhonov regularization method must be used to control this measurement errors.

### 4 Numerical Results and Discussion

In this example, let us consider the following one-dimensional inverse problem, for estimating unknown boundary condition \( q(t) \) when \( a = 0.05 \)

\[
  T_{tt}(x,t) - T_{xx}(x,t) = 6t - 6x,
\]

\[
  0 < x < 1, \quad 0 < t < t_M
\]

(10a)

\[
  T(x,0) = x^2,
\]

\[
  0 \leq x \leq 1,
\]

(10b)

\[
  T_t(x,0) = 0,
\]

\[
  0 \leq t \leq t_M,
\]

(10c)

\[
  T(0,t) = q(t),
\]

\[
  0 \leq t \leq t_M,
\]

(10d)

\[
  T(1,t) = 1 + t^3,
\]

\[
  0 \leq t \leq t_M,
\]

(10e)
and the overspecified condition

\[ T(0.05, t) = (0.05)^3 + t^3, \quad 0 \leq t \leq t_M. \]  

The exact solution of this problem is

\[ T(x, t) = x^3 + t^3, \quad q(t) = t^3, \quad 0 \leq x \leq 1, 0 < t < t_M, \]

Table 1 shows the comparison between the exact solution and approximate solution result from our method by Tikhonov regularization 0th, 1st and 2nd and SVD regularization with noisy data. Finally, we compare two methods with computation total error.

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<th>Tikhonov 0th</th>
<th>Tikhonov 1st</th>
<th>Tikhonov 2nd</th>
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<td>0.368060</td>
<td>0.019777</td>
</tr>
</tbody>
</table>

\[ S = 8.0992e - 002 \quad 1.2300e - 002 \quad 7.944e - 003 \quad 8.0225e - 002 \]

Table 1. The comparison between exact and Tikhonov and SVD solutions for \( q(t) \) with noisy data.

References


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Neural network for solving PDE problems

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Abstract
In this paper, a method based on feed-forward neural networks is used to solve a class of biharmonic problems. The trial solution of proposed biharmonic equation is written as a sum of two parts. The first part satisfies the boundary conditions and contains no adjustable parameters. The second part is constructed so as not to affect the boundary conditions. Second part involves a feed-forward networks that trained, and containing adjustable parameters (the weights).

Keywords: Neural Networks, Partial Differential Equation, Biharmonic Problems, Feed-Forward Neural Networks, multilayer perceptron.

Mathematics Subject Classification: 65M06, 92B05, 35K15

1 Introduction

A series of problems in many scientific fields can be modelled with the use of differential equations such as problems in physics [7,8,9,10,11], chemistry [12,13,14], biology [15,16], economics [17], etc. Many methods have been developed for solving differential equations both solving ODEs both PDEs both biharmonic problems. Some of them produce a solution in the form of an array that contains the value of the solution at a selected group of points. Some of them use basis-functions to represent the solution in analytic form and transform the original problem usually in a system of linear equations. Most of the previous works in solving differential equations using neural networks is restricted to the case of solving the linear systems of algebraic equations which result from the discretization of the domain. The solution of a linear system of equations is mapped onto the architecture of a Hopfield neural network. Another approach to the solution of ordinary differential equations is based on the fact that certain types of splines, for instance linear B-splines, can be derived by the superposition of piecewise linear activation functions [2, 3]. The minimization of the network’s energy function provides the solution to the system of equations [4, 5, 6]. The solution of a differential equation using linear B-splines as basis functions, can be obtained by solving a system of linear or non-linear equations in order to determine the parameters of the splines. Such a solution form is mapped directly on the architecture of a feedforward neural network by replacing each spline with the sum of piecewise linear activation functions that correspond to the hidden units. (In [1] view the problem from a different angle, that present a model for solving ODEs and Poisson PDEs using feed-forward neural network.)

We present a general method for solving biharmonic problems, that relies on the function approximation capabilities of feedforward neural networks and results in the construction of a solution written in a differentiable, closed analytic form. This form employs a feedforward neural network as the basic approximation element, whose parameters (weights and biases) are adjusted to minimize
an appropriate error function. To train the network we employ optimization techniques, which in turn require the computation of the gradient of the error with respect to the network parameters. In the proposed approach the model function is expressed as the sum of two terms: the first term satisfies the boundary conditions and contains no adjustable parameters. The second term involves a feedforward neural network to be trained so as to satisfy the differential equation. Since it is known that a multilayer perceptron with one hidden layer can approximate any function to arbitrary accuracy, it is reasonable to consider this type of network architecture as a candidate model for treating differential equations.

2 Description of the method

2.1 Description Neural Networks

The term neural network was traditionally used to refer to a network or circuit of biological neurons. The modern usage of the term often refers to artificial neural networks, which are composed of artificial neurons or nodes. Thus the term has two distinct usages:

1) Biological neural networks are made up of real biological neurons that are connected or functionally related in a nervous system. In the field of neuroscience, they are often identified as groups of neurons that perform a specific physiological function in laboratory analysis.

2) Artificial neural networks are composed of interconnecting artificial neurons (programming constructs that mimic the properties of biological neurons).

Artificial neural networks may either be used to gain an understanding of biological neural networks, or for solving artificial intelligence problems without necessarily creating a model of a real biological system. The real, biological nervous system is highly complex: artificial neural network algorithms attempt to abstract this complexity and focus on what may hypothetically matter most from an information processing point of view. Good performance (e.g. as measured by good predictive ability, low generalization error), or performance mimicking animal or human error patterns, can then be used as one source of evidence towards supporting the hypothesis that the abstraction really captured something important from the point of view of information processing in the brain. Another incentive for these abstractions is to reduce the amount of computation required to simulate artificial neural networks, so as to allow one to experiment with larger networks and train them on larger data sets.

In next section we report on the solution of a number of model problems. In all cases we used a multilayer perceptron having one hidden layer with 10 hidden units and one linear output unit. The sigmoid activation of each hidden unit is \( \sigma_{z_i} = \frac{1}{1+e^{-z_i}} \). Consider a multilayer perceptron with \( n \) input units, one hidden layer with \( H \) sigmoid units and a linear output unit. The extension to the case of more than one hidden layers can be obtained accordingly. For a given input vectors \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) the output of the network is:

\[
N = \sum_{i=1}^{H} v_i \sigma_{z_i},
\]

where

\[
z_i = \sum_{j=1}^{n} (w_{ij} x_j + w_{ij} y_j + u_i).
\]

and \( w_{ij} \) denotes the weight from the input unit \( j \) to the hidden unit \( i \), \( v_i \) denotes the weight from the hidden unit \( i \) to the output, \( u_i \) denotes the bias of hidden unit \( i \) and \( \sigma_{z_i} \) is the sigmoid transfer function.
2.2 Description Solution

We treat here two dimensional problems only. However it is straightforward to extend the method to more dimensions. Consider the Biharmonic problems as follow

\[ u_{xxxx} + P(x,y)u_{xxyy} + u_{yyyy} = f(x,y). \]  
\(0 \leq x \leq l\) and \(0 \leq y \leq l\) with follow wind boundary condition (BCs)

\[
\begin{align*}
  u(0,y) &= f_1(y), 
  u_x(0,y) = g_1(y), \\
  u(x,0) &= h_1(x), 
  u_y(x,0) = k_1(x), \\
  u(l,y) &= f_2(y), 
  u_x(l,y) = g_2(y), \\
  u(x,l) &= h_2(x), 
  u_y(x,l) = k_2(x).
\end{align*}
\]

The trial solution is written as:

\[ u_T(x,y) = D(x,y) + F(x,y, N(x,y,p)). \]

where \(N(x,y,p)\) is a single-output feedforward neural network with parameters \(p\) and \(n\) input units with the input vectors \(x\) and \(y\).

The term \(D(x,y)\) contains no adjustable parameters and satisfies the boundary conditions. The second term \(F\) is constructed so as not to contribute to the BCs, since \(u_T(x,y)\) must also satisfy them. This term employs a neural network whose weights and biases are to be adjusted in order to deal with the minimization problem. Note at this point that the problem has been reduced from the original constrained optimization problem to an unconstrained one (which is much easier to handle) due to the choice of the form of the trial solution that satisfies by construction the BCs.

The equation (3) is then transformed into the following system of equations:

\[ u_T(x,y) = D(x,y) + x^2(1-x)^2y^2(1-y)^2N(x,y,p). \]

where \(D(x,y)\) is chosen so as to satisfy the BC, namely:

\[
D(x,y) = \{f_1(y) - B(0,y)\}(1 + \frac{2x}{l})(\frac{x-l}{l})^2 + \{f_2(y) - B(1,y)\}(3 - \frac{2x}{l})(\frac{x}{l})^2 + \]

\[
\{g_1(y) - \frac{\delta B}{\delta x}(0,y)\}x(\frac{x-l}{l})^2 + \{g_2(y) - \frac{\delta B}{\delta x}(1,y)\}(x-l)(\frac{x}{l})^2 + \]

\[
\{h_1(x) - A(x,0)\}(1 + \frac{2y}{l})(\frac{y-l}{l})^2 + \{h_2(x) - A(x,1)\}(3 - \frac{2y}{l})(\frac{y}{l})^2 + \]

\[
\{k_1(x) - \frac{\delta A}{\delta y}(x,0)\}y(\frac{y-l}{l})^2 + \{k_2(x) - \frac{\delta A}{\delta y}(x,1)\}(y-l)(\frac{y}{l})^2.
\]

where:

\[
A(x,y) = \{f_1(y)\}(1 + \frac{2x}{l})(\frac{x-l}{l})^2 + \{f_2(y)\}(3 - \frac{2x}{l})(\frac{x}{l})^2 + \]

\[
\{g_1(y)\}x(\frac{x-l}{l})^2 + \{g_2(y)\}(x-l)(\frac{x}{l})^2.
\]

and

\[
B(x,y) = \{h_1(x)\}(1 + \frac{2y}{l})(\frac{y-l}{l})^2 + \{h_2(x)\}(3 - \frac{2y}{l})(\frac{y}{l})^2 + \]

\[
\{k_1(x)\}y(\frac{y-l}{l})^2 + \{k_2(x)\}(y-l)(\frac{y}{l})^2.
\]
Note that the second term of the trial solution does not affect the boundary conditions since it vanishes at the part of the boundary where BCs are imposed and its gradient component normal to the boundary vanishes at the part of the boundary where Neumann BCs are imposed. In proposed PDE problems the error to be minimized is given by:

\[
E(p) = \sum_i (u_{xxxx} + P(x,y)u_{xxyy} + u_{yyyy} - f(x,y))^2.
\]

where \(0 \leq x \leq l\) and \(0 \leq y \leq l\).

### 2.3 Example

\[
u_{xxxx} + 2u_{xxyy} + u_{yyyy} = \frac{24xy}{(1 + x + y)^3} - \frac{8y}{(1 + x + y)^2} - \frac{8x}{(1 + x + y)^2}.
\]

with:

\[
\begin{align*}
u(0, y) &= 0, \quad u_x(0, y) = y \ln(1 + y), \\
u(x, 0) &= 0, \quad u_y(x, 0) = x \ln(1 + x), \\
u(1, y) &= y \ln(2 + y), \quad u_x(1, y) = y \ln(2 + y) + \frac{y}{2 + y}, \\
u(x, 1) &= x \ln(2 + x), \quad u_y(x, 1) = x \ln(2 + x) + \frac{x}{2 + x}.
\end{align*}
\]

and \(0 \leq x \leq 1, \ 0 \leq y \leq 1\). The analytic solution is

\[
u(x, y) = xy \ln(1 + x + y).
\]

and is displayed in Figure 1a. According to the equation (6), the trial neural form of the solution is taken to be:

\[
u_T(x, y) = D(x, y) + x^2(1 - x)^2 y^2(1 - y)^2 N(x, y, p),
\]

where

\[
D(x, y) = \{-B(0, y)(1 + 2x)(x - 1)^2 + \{y \ln(2 + y) - B(1, y)\}(3 - 2x)(x)^2 + \{y \ln(1 + y) - \frac{\delta B}{\delta x}(0, y)\}x(x - 1)^2 + \{y \ln(2 + y) + \frac{y}{2 + y} - \frac{\delta B}{\delta x}(1, y)\}(x - 1)(x)^2 + \{-A(x, 0)\}(1 + 2y)(y - 1)^2 + \{x \ln(2 + x) - A(x, 1)\}(3 - 2y)(y)^2 + \{x \ln(1 + x) - \frac{\delta A}{\delta y}(x, 0)\}y(y - 1)^2 + \{x \ln(2 + x) + \frac{x}{2 + x} - \frac{\delta A}{\delta y}(x, 1)\}(y - 1)(y)^2, \\
A(x, y) &= \{y \ln(2 + y)\}(3 - 2x)(x)^2 + \{y \ln(1 + y)\}x(x - 1)^2 + \{y \ln(2 + y) + \frac{y}{2 + y}\}(x - 1)(x)^2.
\]

and

\[
B(x, y) = \{x \ln(2 + x)\}(3 - 2y)(y)^2 + \{x \ln(1 + x)\}y(y - 1)^2 + \{x \ln(2 + x) + \frac{x}{2 + x}\}(y - 1)(y)^2.
\]

The network was trained using a grid of 10 equidistant points in \([0,1]\). Figure 2a displays \(u_T(x, y)\). Table 1 displays the deviation trial solution from the exact solution at 20 points.
Table 1: The comparison between exact and trial solutions.

<table>
<thead>
<tr>
<th>(x, y)</th>
<th>u\textsubscript{e}(x, y)</th>
<th>u\textsubscript{T}(x, y)</th>
<th>Δu(x, y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>1.099</td>
<td>1.088</td>
<td>0.011</td>
</tr>
<tr>
<td>(0.9, 0.95)</td>
<td>0.8955</td>
<td>0.8923</td>
<td>0.00325</td>
</tr>
<tr>
<td>(1, 0.85)</td>
<td>0.8902</td>
<td>0.8752</td>
<td>0.0150</td>
</tr>
<tr>
<td>(0.9, 0.8 )</td>
<td>0.7151</td>
<td>0.7116</td>
<td>0.0035</td>
</tr>
<tr>
<td>(0.75, 0.85)</td>
<td>0.6091</td>
<td>0.5844</td>
<td>0.0247</td>
</tr>
<tr>
<td>(0.65, 0.95)</td>
<td>0.59</td>
<td>0.593</td>
<td>0.003</td>
</tr>
<tr>
<td>(0.55, 0.95)</td>
<td>0.4788</td>
<td>0.4826</td>
<td>0.0038</td>
</tr>
<tr>
<td>(0.7, 0.75)</td>
<td>0.4704</td>
<td>0.4661</td>
<td>0.0043</td>
</tr>
<tr>
<td>(0.85, 0.6 )</td>
<td>0.456</td>
<td>0.4445</td>
<td>0.0115</td>
</tr>
<tr>
<td>(1, 0.4)</td>
<td>0.3302</td>
<td>0.3401</td>
<td>0.0101</td>
</tr>
<tr>
<td>(0.7, 0.5)</td>
<td>0.276</td>
<td>0.2739</td>
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<tr>
<td>(0.5, 0.6)</td>
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<td>0.221</td>
<td>0.0016</td>
</tr>
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<td>0.0061</td>
</tr>
<tr>
<td>(0.2, 0.8)</td>
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<td>0.127</td>
<td>0.0161</td>
</tr>
<tr>
<td>(0.1, 0.9)</td>
<td>0.06238</td>
<td>0.0754</td>
<td>0.01302</td>
</tr>
<tr>
<td>(0, 0.7)</td>
<td>0</td>
<td>0.04379</td>
<td>0.04379</td>
</tr>
<tr>
<td>(0.5, 0.3)</td>
<td>0.8817</td>
<td>0.08699</td>
<td>0.00118</td>
</tr>
<tr>
<td>(0.85, 0)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(0.6, 0)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(0.75, 0.05)</td>
<td>0.2204</td>
<td>0.04624</td>
<td>0.02420</td>
</tr>
</tbody>
</table>
3 Conclusion

In this work, the feed-forward neural networks are used to solve an important class of partial differential equations, named biharmonic equations with respect to the appropriate boundary conditions. There are some advantages using this procedure. For example it is not require to discretize the proposed problem and does not deal with the solution of system of equations.

References


Five points non-equispace finite difference method for solving fourth order ODE

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Abstract
In this paper a method for solving ordinary differential equations (ODE’s) with boundary or initial conditions is presented. For this purpose a five points finite difference formula for calculating the derivatives are used. The order of convergence of these formulas are introduced. The results show the efficiency and performance of the method.

Keywords: Interpolation, Taylor series, Initial value problem (IVP), Boundary value problem (BVP)

Mathematics Subject Classification: 63I10

1 Introduction

Finite difference method (FDM) is an important tool for solving boundary value problems. The conventional FDM uses the equispaced finite difference derivatives. But these methods are not suitable when there is necessity for using the irregular mesh points, which can be happened in the case of singular points or multiple points boundary conditions or other situations that non-uniform meshes is considered. The non-equispaced finite difference method are applied by different authors for instance in Poplau et al.[2] apply non-equispaced three points finite difference formula for first and second order derivatives to solve the Poisson’s equations, and Izadian et al. [1] have used this method to solve elliptic PDE’s.

In this paper the five points non-equispaced finite difference formulas for approximation of derivatives up to fourth order are presented and the order of these approximate derivatives are deduced. The paper is organized as follows: In section 2 the description of method, and related propositions are given. In section 3, the numerical results are shown. Finally in section 4, the comments and conclusion are presented.

2 Description of method and the approximation error

Consider the following ordinary differential equations

\[ y^{(4)} = f(x, y, y', y'', y'''), \quad x \in [a, b], \quad (1) \]

subject to the following 3 points linear boundary conditions:

\[ \alpha_i y^{(i)}(a) + \beta_i y^{(i)}(c) + \gamma_i y^{(i)}(b) = \delta_i, \quad i = 0, 1, 2, 3 \quad (2) \]
(α₁, β₁, γ₁, δ₁) ∈ ℝ⁴, and (αᵢ, βᵢ, γᵢ) ≠ 0, i ∈ [a, b]. Note that for αᵢ = 1, βᵢ = γᵢ = 0, i = 0, 1, 2, 3, the conditions (2.2) reduce to initial conditions.

Assume the following partition of [a, b],
\[ a = x_1 < x_2 < \ldots < x_m = c < x_{m+1} < \ldots < x_n = b, \]
\[ h_i = x_{i+1} - x_i, \quad i = 1, 2, \ldots, n. \]
and consider
\[ y_i = y(x_i) = f(x_i), \quad i = 1, 2, \ldots, n. \]
Then, the approximate derivatives of y of order \( l \) ≤ 4 on point \( x_i \), \( i = 1, 2, \ldots, n \), can be defined by
\[
D_{i,k}^{(l)} f = \alpha_{i1}^{(l,k)} f_1 + \alpha_{i2}^{(l,k)} f_2 + \ldots + \alpha_{i5}^{(l,k)} f_5 + E_{i,k}^{(l,k)}, \quad (3)
\]
where \( k \) is the number of point \( x_i \), in ordered set \((x_1, x_2, \ldots, x_5)\)

**Lemma 2.1.** If \( f \in C^5[a, b] \), and \( \alpha_{i,j}^{(l,k)} \), are the coefficients of approximate derivatives of order \( l \) with five points given in (2.3). Then
\[ \alpha_{i,j}^{(l)} = O(K^l), \quad l = 1, 2, 3, 4, \quad K = \frac{1}{H}, \]
where
\[ H = \max\{h_1, h_2, h_3, h_4\}. \]

**Proof:** Can be accomplished by using the formulas of \( \alpha_{i,j}^{(l,k)} \).

**2.1 Proposition**

If \( f \in C^5(a, b) \) and \( x_1, x_2, x_3, x_4, \) and \( x_5 \) are points of \([a, b]\) with
\[ a \leq x_1 < x_2 < x_3 < x_4 < x_5 \leq b \]
and
\[ h_i = x_{i+1} - x_i, \quad i = 1, 2, 3, 4. \]
If
\[
D_{i,k}^{(l)} f = \alpha_{i1}^{(l,k)} f(x_1) + \alpha_{i2}^{(l,k)} f(x_2) + \alpha_{i3}^{(l,k)} f(x_3) + \alpha_{i4}^{(l,k)} f(x_4) + \alpha_{i5}^{(l,k)} f(x_5),
\]
\[ l = 1, 2, 3, 4, \quad i, k = 1, 2, 3, 4, 5, \]
are numerical derivatives of order \( l \)th on point \( x_i, i = 1, 2, 3, 4, 5 \), Then
\[ E_{i,k}^{(l)} = f^{(l)}(x_i) - D_{i,k}^{(l)} = O(H^{5-l}), \quad l = 1, 2, 3, 4, \quad i = 1, 2, 3, 4, 5, \]
where
\[ H = \max_{1 \leq i \leq 4} h_i. \]

**Proof.** See [1].

Next by replacing these approximate derivatives in (2.1) and (2.2) in node points, to determine the approximates values of \( y(x) \) on mesh points \( x_i \) and related points in (2.2) a system of linear or non-linear equations is obtained that can be solved by Newton’s method.

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3 Numerical results

In this section the proposed generalized finite difference method is used to solve a ODE 4th order. The numerical and analytical solution, are compared.

3.1 Example

Consider the following ODE with initial condition

\[ y^{(4)} = 5 - \frac{1}{x}y^{(3)} - \frac{2}{x^2}y'' - \frac{6}{x^3}y' - \frac{24}{x^4}y, \quad x \in [0, 10], \]

\[ y(1) = \frac{1}{24}, \quad y'(1) = \frac{1}{6}, \quad y''(1) = \frac{1}{2}, \quad y'''(1) = 1, \]

the exact solution is \( y(x) = \frac{x^4}{24} \) and \( N = 50 \). The numerical and exact results are shown in Table 3.1.

<table>
<thead>
<tr>
<th>x</th>
<th>( y^*(x) )</th>
<th>y(x)</th>
<th>e(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.041666666666</td>
<td>0.041666666666</td>
<td>0</td>
</tr>
<tr>
<td>2.98</td>
<td>3.285896669</td>
<td>3.285896663</td>
<td>0.2363 \times 10^{-12}</td>
</tr>
<tr>
<td>6.22</td>
<td>62.3663450664</td>
<td>62.3663456066</td>
<td>0.175077 \times 10^{-10}</td>
</tr>
<tr>
<td>8.73</td>
<td>243.12772640169</td>
<td>243.1272640166</td>
<td>-0.14964 \times 10^{-9}</td>
</tr>
<tr>
<td>10</td>
<td>416.6666666666</td>
<td>416.66666667</td>
<td>-0.61 \times 10^{-9}</td>
</tr>
</tbody>
</table>

The cpu time is: 71.66s, an maximum error on mesh points is \( 10^{-11} \).

4 Conclusion

The presented method are suitable for the ODE with singularity, and multiple point-boundary conditions. This method can be used for numerical solution of ODE with adaptive meshes. The numerical results show the performance and efficiency of proposed method.

References


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Restarting Arnoldi schemes for solving large eigenvalue problems

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Abstract
Arnoldi’s methods is an orthogonal projection onto \( K_k \) for general non-Hermitian matrices which reduces a dense matrix into Hessenberg form. Due to the drawbacks of Arnoldi method for large eigenvalue problems, we consider variants of corresponding restarting schemes. These schemes use an initial vector along with a polynomial which is constructed to damp unwanted components for the starting vector. Through several numerical examples, we show that the restarting method outperforms some well-known existing methods.

Keywords: Arnoldi method, Implicit restarting, Polynomial restarting, Eigenvalues.

Mathematics Subject Classification: 65F15, 65G05

1 Introduction

The algebraic eigenvalue problem \( Ax = \lambda x \) is a fundamental to scientific computation. There are many numerical methods for large scale eigenvalue problems. The Arnoldi method is one of the most important methods, which reduce a non-Hermitian matrix to Hessenberg form in which the eigenvalues of Hessenberg matrix are approximations of main matrix eigenvalues and they are called Ritz values.

If \( A \) is non-Hermitian, then the orthogonal tridiagonalization \( A = VHV^* \) does not exist in general. The Arnoldi approach involves the column-by-column generation of an orthogonal \( V \) such that \( H = V^*AV \) becomes the upper Hessenberg form. Finding initial vector is very important, therefore there are several ways to choose these vectors which are based on restarting procedure and iterative. Each step applies the Arnoldi algorithm with an improved initial vector.

Definition 1.1. Let \( A \) be a \( n \times n \) matrix where \( n \) is a large number. We define a Krylov subspace the following form:
\[
K_k := \text{span}\{v, Av, A^2v, \ldots, A^{k-1}v\}.
\]

1.1 Restarting the Arnoldi process, polynomial and implicit schemes

The Arnoldi factorization is viewed as a truncated reduction of an \( n \times n \) matrix \( A \) to upper Hessenberg form. After \( k \) steps of the factorization one has
\[
AV_k = V_k H_k + f_k e_k^* 
\]
where \( V_k \in \mathbb{C}^{n \times n} \) has orthonormal columns, \( f_k \) is a new vector that is orthogonalized with respect to the previous basis vectors, \( V^* f_k = 0 \) and \( H = V_k^* AV_k \) is a \( k \times k \) upper Hessenberg matrix.
with nonnegative subdiagonal elements. This is called a \( k \)-step Arnoldi factorization of \( A \). If \( A \) is Hermitian, then \( H_k \) becomes real, symmetric and tridiagonal which is called a \( k \)-step Lanczos factorization of \( A \). The columns of \( V_k \) are referred to as the Arnoldi vectors or Lanczos vectors, respectively [4].

The main drawback of the Arnoldi method is that the number of steps required to calculate eigenvalues of interest within a specified accuracy can not be predetermined. In fact the number of steps depend on the initial vector and not all vectors are numerically orthogonal. To remedy these disadvantages, we use the restarting process.

**Lemma 1.2.** If \( v = \sum_{j=1}^{k} q_j \gamma_j \) where \( Aq_j = q_j \gamma_j \), and
\[
AV = VH + fe_k^T
\]

be a \( k \)-step Arnoldi factorization with unreduced \( H \), then \( f = 0 \) and \( \sigma(H) = \{\lambda_1, \lambda_2, \cdots, \lambda_k\} \). For proof we refer to [2].

**Lemma 1.3.** \( f_k = 0 \) if and only if \( v_1 = Q_k y \), where \( AQ_k = Q_k R_k \) is a partial Schur decomposition of \( A \) with \( R_k \) non-derogatory. Moreover, the Ritz values of \( A \) with respect to \( K_k \) are eigenvalues of \( A \), and are given by the diagonal elements of \( R_k \).

For proof we refer to [2].

To start the process, we replace \( \psi(A)v_1 \) by \( v_1 \) in polynomial restarting, where \( \psi \) is a polynomial constructed to damp unwanted components from the starting vector. If \( v_1 = \sum_{j=1}^{n} q_j \gamma_j \) where \( Aq_j = q_j \lambda_j \), then
\[
v_1^+ = \psi(A)v_1 = \sum_{j=1}^{n} q_j \gamma_j \psi(\lambda_j).
\]

A straightforward way to implement polynomial restarting is to explicitly construct the starting vector \( v_1^+ \) by applying \( \psi(A) \) through a sequence of matrix-vector products. However, there is an alternative implementation that provides a more efficient and numerically stable formulation. This approach, called implicit restarting, uses a sequence of implicitly shifted \( QR \) steps to an \( m \)-step Arnoldi factorization to obtain truncated form of the implicitly shifted \( QR \)-iteration.

An Arnoldi factorization of length \( m = k + p \),
\[
AV_m = V_m H_m + f_m e_m^T
\]
is compressed to a factorization of length \( k \) which stores the eigen-information of interest.

If \( \mu_j, j = 1, 2, \cdots, p \) be roots of polynomial \( \psi \), this polynomial can be determined with the following form:
\[
\psi_p(t) = (t - \mu_1)(t - \mu_2) \cdots (t - \mu_p)
\]

where \( \mu_j, j = 1, 2, \cdots, p \) are the unwanted Ritz values.

\( QR \) steps are used to apply \( p \) linear polynomial factors \( A - \mu_j I \) implicitly to the starting vector. Then we have
\[
AV_m^+ = V_m^+ H_m^+ + f_m^T e_m^T Q
\]

where \( V_m^+ = VQ, H_m^+ = Q^* H_p Q \) and \( Q = Q_1 Q_2 \cdots Q_p \). Each \( Q_j \) is the orthogonal matrix associated with implicit application of the shift \( \mu_j \) and it’s Hessenberg [3]. It turns out that the first \( k - 1 \) entries of \( e_k + p Q \) are zero, so that a new \( k \)-step Arnoldi factorization can be obtained by equating the first \( k \) columns on each side:
\[
AV_k^+ = V_k^+ H_k^+ + f_k^T e_k^T.
\]
2 Main Result

By means of a numerical test, we show that the implicit restarting Arnoldi method outperforms the Arnoldi method to approximate eigenvalues of a large matrix.

2.0.1 Test case

We consider $A$ as a $500 \times 500$ random matrix with $A(1, 1) = 1.5$. The eigenvalues are approximately uniformly distributed in the unit disk except for the outlier $\lambda = 1.4852$. Figures 1 and 2, respectively, show the Arnoldi and the implicit restarting lemniscates at step $k = 10$. By comparing two figures, it turns out that the Ritz value corresponding to the Hessenberg matrix obtained by the implicit restarting method, is fairly close to the extreme eigenvalue of $A$, whereas the Ritz value corresponding to $H$ obtained by the Arnoldi method, is far too close to the same eigenvalue of $A$.

In the figures the small dots are the eigenvalues of $A$, and the large dots are the Ritz values.

Figure 1: Ritz values by Arnoldi method

Figure 2: Ritz values by implicit restarting method

References


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Solving high-order nonlinear Volterra integro-differential equations by using block-pulse functions

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Abstract
In this paper, an effective numerical method is introduced for treatment of nonlinear high-order Volterra integro-differential equations. Here, we use the so-called block-pulse functions (BPFs). First, we present the block-pulse operational matrix of integration, then by using this matrix, the nonlinear high-order Volterra integro-differential equation reduces to an algebraic system.

Keywords: Nonlinear equations, Volterra integro-differential equations, Block-pulse functions, Operational matrix

Mathematics Subject Classification: 45G10, 45D05

1 Introduction
Concept of the block-pulse functions was first introduced to electrical engineers by Harmuth. Several researchers (Gopalsami and Deekshatulu, 1997 [1]; Sannuti, 1977 [2]; Chen and Tsay, 1977 [3] and G.P. Rao 1983 [4]) discussed the block pulse functions and their operational matrix. The aim of this work is to present a numerical method for approximating the solution of BPF of the form:

\[ u^r(x) + \sum_{k=1}^{r} A_{r-k}u^{(r-k)}(x) = g(x) + \lambda \int_{0}^{x} k(x, t, F(u(x)))dt, \] (1)

with initial conditions

\[ u(0) = u_0, \] (2)

where the parameters \( \lambda \) and functions \( g(x), k(x, t, F(u(x))) \) are known and belong to \( L^2[0,1] \)and \( u(x) \) is unknown function. In this work, we consider that, the nonlinear function has the following form

\[ F(u(t)) = (u(t))^p, \]

where \( p \) is a positive integer.

2 Properties of block-pulse functions (BPFs)
An \( m \)-set of BPFs is defined as follows:

\[ \phi_i(t) = \begin{cases} 1 & ih \leq t \leq (i + 1)h \\ 0 & \text{otherwise} \end{cases}, \] (3)

where \( i = 1, 2, \ldots, m \) with positive integer values for \( m \) and \( h = \frac{T}{m} \). There are some properties for BPFs; e.g. disjointness, orthogonality, and completeness.
Completeness

For every \( f \in L^2[0,1] \) when \( m \) goes to infinity, Parseval identity holds:

\[
\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \| \phi_i(t) \|^2, \tag{4}
\]

where

\[
f_i = \frac{1}{m} \int_0^1 f(t)\phi_i(t)dt.
\]

The set of BPFs may be written as a \( m \)-vector \( \Phi(t) \):

\[
\Phi_m(t) = [\phi_0(t), \ldots, \phi_{m-1}(t)]^T, \tag{5}
\]

where \( t \in [0,1) \). From the representation and disjointness property, it follows that:

\[
\Phi(t)\Phi^T(t) = \begin{bmatrix}
\phi_0(t) & 0 & \cdots & 0 \\
0 & \phi_1(t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \phi_{m-1}(t,x)
\end{bmatrix}, \tag{6}
\]

\[
\Phi^T(t)\Phi(t) = 1 \tag{7}
\]

\[
\Phi(t)\Phi^T(t) = \tilde{V}\Phi(t), \tag{8}
\]

where \( V \) is an \( m \)-vector and \( \tilde{V} = \text{diag}(V) \). Moreover, it can be concluded that for every \( m \times m \) matrix \( A \):

\[
\Phi^T(t)A\Phi(t) = \tilde{A}^T\Phi(t) \tag{9}
\]

where \( \tilde{A} \) is an \( m \)-vector with elements equal to the diagonal entries of matrix \( A \).

2.1 Operational matrix of integration

BPF integration property expressed by an operational equation as

\[
\int_0^t \Phi(\tau)d\tau = \Upsilon \Phi(t), \tag{10}
\]

where \( \Upsilon_{m \times m} \) is called operational matrix of integration which can be represented by:

\[
\Upsilon = \frac{1}{2} \begin{pmatrix}
1 & 2 & 2 & \cdots & 2 \\
0 & 1 & 2 & \cdots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}. \tag{11}
\]

3 Applying the method

In this section, we calculate \( U^{k-\tau}(x) \) with using of \( U^\tau(x) \)

\[
U^\tau(x) = \sum_{i=0}^{m-1} u_i\Phi(x) = U^\tau \Phi_m(x), \tag{12}
\]

by integrating (12) from 0 to \( t \) and using (10) we get

\[
U^{t-\tau}(x) = U^\tau \Phi_m(x) + U^{t-1}_0(x) \tag{13}
\]
The \( k \)th integration of (12) yields
\[
U^{r-k}(x) = U^r \Phi_m(t) + Z_k \Phi_m(x),
\]
where
\[
Z_k = \frac{1}{h} \sum_{i=1}^{k} \frac{1}{i!(k-i+1)!} U_0^{r-i} Y_{k-i}^T
\]
is an \( n \times m \) constant matrix

### 3.1 BPFs expansion

A function \( f(t) \in L^2([0,1]) \) may be expanded by the BPFs as:
\[
f(t) \simeq \sum_{i=0}^{m-1} f_i \phi_i(t) = F^T \Phi(t) = \Phi^T(t)F,
\]
where \( F \) is a \( m \)-vector given by \( F = [f_0, \ldots, f_{m-1}]^T \) and \( \Phi(x) \) is defined in (5). The block-pulse coefficients \( f_i \) are obtained as
\[
f_i = \frac{1}{h} \int_{x_0}^{x_1} f(t) \text{dt}
\]
such that error between \( f(t) \), and its block-pulse expansion (16) in the region of \( t \in [0,1] \) is minimal.

Now, we solve the nonlinear Volterra high-order integro-differential equations of form (1) by using BPFs. As we show before, we can write
\[
U(x) = U^T \Phi_m(x),
\]
\[
g(x) = G^T \Phi_m(x),
\]
\[
U'(x) = U^T \Phi_m(x),
\]
\[
[u(x)]^p = \Phi^T(x) \Lambda,
\]
\[
k(x,t) = \Phi^T(x) K \Phi(t),
\]
where the \( m \)-vectors \( U, G, \Lambda \), and matrix \( K \) are BPFs coefficients of \( u(x), g(x), [u(t)]^p \), and \( K(x,t) \) respectively. \( \Lambda \) is a column vector whose elements are \( p \)th power of the elements of the vector \( U \).

To approximate the equation (1) and (18) we get
\[
U^T \Phi_m(x) + \sum_{k=1}^{r} A_{r-k} (U^T \Phi_m(x) + Z_k) \Phi_m(x) = G^T \Phi_m(x) + \Phi_m^T(x) K \int_0^x \Phi(t) \Phi^T(t) \Lambda
\]
\[
= G^T \Phi_m(x) + \Phi_m^T(x) K \Lambda \int_0^x \Phi(t) \text{dt}
\]
\[
= G^T \Phi_m(x) + \Phi_m^T(x) K \Lambda \Phi_m(x)
\]
if we put \( A = K \Lambda \) then it can be written from equation (9),
\[
U^T \Phi_m(x) + \sum_{k=1}^{r} A_{r-k} (U^T \Phi_m(x) + Z_k) \Phi_m(x) = G^T \Phi_m(x) + \Lambda \Phi_m(x)
\]

hence, we have
\[
U^T + \sum_{k=1}^{r} A_{r-k} (U^T \Phi_m(x) + Z_k) = G^T + \Lambda \Phi_m(x)
\]
finally, equation (19) can be consider as a system like:

\[ AU = F \]  \hspace{1cm} (20)

where \( A \) and \( F \) are the combination of block-pulse coefficient matrix and \( U \) can be obtained from Newton-Raphson method for solving nonlinear systems.

**References**


Ground state solutions for a semilinear elliptic equation involving concave-convex nonlinearities

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Abstract
This work is devoted to the existence and multiplicity properties of the ground state solutions of the semilinear boundary value problem
\[-\Delta u = \lambda a(x)|u|^{q-2} + b(x)|u|^{2^*-2}, \quad x \in \Omega,\]
\[u = 0, \quad x \in \partial \Omega,\]
where \(\Omega \subset \mathbb{R}^N (N \geq 3)\) is a smooth bounded domain, \(\lambda > 0, 1 \leq q < 2,\) and \(2^* = \frac{2N}{N-2}\) is the critical Sobolev exponent and the weight functions \(a, b\) are satisfying the following conditions:
(A) \(a^+ = \max\{a, 0\} \neq 0\) and \(a \in L^{r_q}(\Omega)\) where \(r_q = \frac{r}{r-q}\) for some \(r \in (q, 2^* - 1),\) with in addition \(a(x) \geq 0\ a.e\ in\ \Omega\ in\ case\ q = 0;\)
(B) \(b^+ = \max\{b, 0\} \neq 0\) and \(b \in C(\overline{\Omega}).\)

Definition 1.1. Suppose that \(H^1_0(\Omega)\) denotes the completion of \(C_0^\infty(\Omega)\) with respect to the norm
\[\| u \| = \| u \|_{H^1_0(\Omega)} = \left( \int_\Omega |\nabla u|^2 dx \right)^{1/2}.\]
The function \(u \in H^1_0(\Omega)\) is said to be a solution of the \(E_q(1)\), if \(u\) satisfies
\[\int_\Omega \left( \nabla u \nabla v - |u|^{2^*-2}uv - \lambda |u|^{q-2}uv \right) dx = 0, \quad \forall v \in H^1_0(\Omega).\]
The energy functional corresponding to \(E_q(1)\) is defined as follows:
\[J_\lambda(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{2^*} \int_\Omega b(x)|u|^{2^*} dx - \frac{\lambda}{q} \int_\Omega a(x)|u|^q dx,\]

1 Introduction
We consider the following semilinear elliptic equation:
\[
\begin{cases}
-\Delta u = \lambda a(x)|u|^{q-2} + b(x)|u|^{2^*-2}, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\]
where \(\Omega \subset \mathbb{R}^N (N \geq 3)\) is a smooth bounded domain, \(\lambda > 0, 1 \leq q < 2,\) and \(2^* = \frac{2N}{N-2}\) is the critical Sobolev exponent and the weight functions \(a, b\) are satisfying the following conditions:
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Definition 1.1. Suppose that \(H^1_0(\Omega)\) denotes the completion of \(C_0^\infty(\Omega)\) with respect to the norm
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and then $J_\lambda$ is well defined on $H^1_0(\Omega)$. The solutions of $E_q(1)$ are the critical points of the functional $J_\lambda$.

As the energy functional $J_\lambda$ is not bounded below on $H^1_0(\Omega)$, it is useful to consider the functional on the Nehari manifold

$$M_\lambda = \{ u \in H^1_0(\Omega) \setminus \{0\} : \langle J'_\lambda(u), u \rangle = 0 \}. $$

We define

$$M^+_\lambda = \{ u \in M_\lambda : \varphi''(1) > 0 \};$$

$$M^0_\lambda = \{ u \in M_\lambda : \varphi''(1) = 0 \};$$

$$M^-_\lambda = \{ u \in M_\lambda : \varphi''(1) < 0 \},$$

such that $\varphi_u(t) = J_\lambda(tu)$ for $t > 0$.

**Lemma 1.2.** The energy functional $J_\lambda$ is coercive and bounded below on $M_\lambda$.

**Lemma 1.3.** Assume that $u_0$ is a local minimizer for $J_\lambda$ on $M_\lambda$ and $u_0 \notin M^0_\lambda$. Then $J'_\lambda(u_0) = 0$ in $H^{-1}(\Omega)$.

**Lemma 1.4.** If $\lambda \in \Lambda$, then $M^0_\lambda = \emptyset$ where $\Lambda = (0, \lambda_0)$ for

$$\lambda_0 := \left( \frac{2 - q}{2^* - q} \right) \left( \frac{2^* - 2}{2^* - q} \right) S^{\frac{4(2 - q)}{4 - 2q}} \| a \|^{-1} S^{-q},$$

where $S_r$ is the best Sobolev constant for the embedding of $H^1_0(\Omega)$ in $L^r(\Omega)$.

**Remark 1.5.** It follows from lemma 1.4 that

$$M_\lambda = M^+_\lambda \cup M^-_\lambda$$

for all $\lambda \in \Lambda$. Furthermore, by lemma 1.2 we may define

$$\alpha_\lambda = \inf_{u \in M_\lambda} J_\lambda(u); \quad \alpha^+_\lambda = \inf_{u \in M^+_\lambda} J_\lambda(u); \quad \alpha^-_\lambda = \inf_{u \in M^-_\lambda} J_\lambda(u).$$

**Proposition 1.6.** If $\lambda \in \Lambda$, then

(i) there exist a minimizing sequence $\{u_n\} \subset M_\lambda$ such that

$$J_\lambda(u_n) = \alpha^-_\lambda + o_n(1),$$

$$J'_\lambda(u_n) = o_n(1).$$

(ii) there exist a minimizing sequence $\{u_n\} \subset M^-_\lambda$ such that

$$J_\lambda(u_n) = \alpha^-_\lambda + o_n(1),$$

$$J'_\lambda(u_n) = o_n(1).$$
Theorem 1.7. If \( \lambda \in \Lambda \), then \( J_\lambda \) has a minimizer \( u_\lambda \) in \( \mathcal{M}_\lambda^+ \) and it satisfies the following.

(i) \( J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+ \).

(ii) \( u_\lambda \) is positive solution of \( E_q(1) \).

(iii) \( \| u_\lambda \| \to 0 \) as \( \lambda \to 0^+ \).

Theorem 1.8. If \( \lambda \in \Lambda \), then \( J_\lambda \) has a minimizer \( U_\lambda \) in \( \mathcal{M}_\lambda^- \) and it satisfies the following.

(i) \( J_\lambda(U_\lambda) = \alpha_\lambda^- \).

(ii) \( U_\lambda \) is positive solution of \( E_q(1) \).

2 Main Result

Our main result is the following.

Theorem 2.1. Assume that the conditions (A) and (B) hold; then there exists an interval \( \Lambda \) such that for \( \lambda \in \Lambda \), equation (1) has at least two positive solutions.

Proof. By theorems 1.7 and 1.8, we have the equation(1) has two positive solutions \( u_\lambda \) and \( U_\lambda \) such that \( u_\lambda \in \mathcal{M}_\lambda^+, U_\lambda \in \mathcal{M}_\lambda^- \). Since \( \mathcal{M}_\lambda^+ \cap \mathcal{M}_\lambda^- = \emptyset \), this implies that \( u_\lambda \) and \( U_\lambda \) are distinct. This completes the proof of theorem. \( \square \)

References


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The solving of parabolic distributed optimal control problems with quartic B-spline collocation method

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Abstract
Quartic B-spline collocation method that solve parabolic distributed optimality systems discretized by quartic B-spline are investigated. Accuracy properties of quartic B-spline approximation are discussed and validated. In this paper, the resulting scheme also shows robustness with respect to changes of the value of $\nu$, the weight of the cost of the control, is sufficiently small.

Keywords: Quartic B-spline collocation method, parabolic distributed optimality systems, optimal control problems.

Mathematics Subject Classification: 49M25, 35K10, 49J20.

1 Introduction
The purpose of this work is to present quartic B-spline solvers for the following optimal control problem

$$\begin{align*}
\min_{u \in L^2(Q)} J(y, u) \\
-\partial_t y + \Delta y &= u, & \text{in } Q = \Omega \times (0, T), \\
y(x, 0) &= y_0(x), & \text{in } \Omega \text{ at } t = 0, \\
y(x, t) &= 0, & \text{on } \Sigma = \partial \Omega \times (0, T),
\end{align*}$$

(1)

where we take $y_0(x) \in H^1_0(\Omega)$. We consider cost functionals of the tracking type given by

$$J(y, u) = \frac{1}{2} ||y - z||^2_{L^2(Q)} + \frac{\nu}{2} ||u||^2_{L^2(Q)}$$

(2)

Here, $\nu > 0$ is the weight of the cost of the control and $z \in L^2(Q)$ denotes the desired state. Then there exists a unique solution $(y^*, u^*) = (y^*(u^*), u^*)$ to the optimal control problem above [1]. Corresponding to our setting we have $y^*(u^*) \in H^{2,1}(Q)$ where $H^{2,1}(Q) = L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap H^1(0, T; L^2(\Omega))$. The solution to (1) is characterized by the following optimality system

$$\begin{align*}
-\partial_t y + \Delta y &= u, & \text{in } Q = \Omega \times (0, T), \\
\partial_t p + \Delta p + (y - z) &= 0, & \text{in } \Omega \times (0, T), \\
\nu u - p &= 0, & \text{in } \Omega \times (0, T).
\end{align*}$$

(3) (4) (5)

with initial condition $y(x, 0) = y_0(x)$ for the state equation (evolving forward in time) and terminal condition $p(x, T) = 0$ for the adjoint equation (evolving backward in time). The fact that $u$ attains the same regularity of $p$ is due to the special form of (5), the optimality condition. We focus on cases where the optimality condition provides a scaler relation between $u$ and $p$. 

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2 Quartic B-spline collocation method

Let $\Omega$ be a uniform partition of an interval $[a,b]$ as follows $a = x_0 < x_1 < \ldots < x_N = b$ where $b = x_{j+1} - x_j$, $j = 0, 1,\ldots, N - 1$. The quartic B-splines are defined upon an increasing set of $N+1$ knots over the problem domain plus 8 additional knots outside the problem domain 8 additional knots are positioned as

$$x - 4 < x - 3 < x - 2 < x - 1 < x_0 \text{ and } x_N < x_{N+1} < x_{N+2} < x_{N+3} < x_{N+4}.$$ 

The set of quartic B-spline $\{Q_{-2}, Q_{-1},\ldots, Q_{N+1}\}$ form a basis over the problem domain $[a,b]$. Let $Q_m(x)$, $m = -2, -1,\ldots, N + 1$,

$$Q_m(x) = \frac{1}{h^4} \begin{cases} (x - x_{m-2})^4, & x \in [x_{m-2}, x_{m-1}], \\ (x - x_{m-2})^4 - 5(x - x_{m-1})^4, & x \in [x_{m-1}, x_m], \\ (x - x_{m-2})^4 - 5(x - x_{m-1})^4 + 10(x - x_m)^4, & x \in [x_m, x_{m+1}], \\ (x_{m+3} - x)^4 - 5(x_{m+2} - x)^4, & x \in [x_{m+1}, x_{m+2}], \\ (x_{m+3} - x)^4, & x \in [x_{m+2}, x_{m+3}], \\ 0, & \text{otherwise,} \end{cases}$$ (6)

be quartic B-splines, which vanish outside interval $[x_{m-2}, x_{m+3}]$. Each quartic B-spline cover five elements so that an element is covered by five quartic B-splines. Now the solutions of the problem is considered as follows:

$$y(x,t) = \sum_{m=-2}^{N+1} c_m(t)Q_m(x)$$ (7)

$$p(x,t) = \sum_{m=-2}^{N+1} d_m(t)Q_m(x)$$ (8)

Where $c_m$, $d_m$, $m = -2,\ldots, N + 1$ are unknown time dependent quantities to be determined from boundary and initial conditions [2]. The values of $Q_m(x)$ and its derivatives $Q'_m(x)$, $Q''_m(x)$ and $Q'''_m(x)$ at the knots are given in Table 1.

The optimality system (3)-(5) together with initial, boundary and terminal conditions are first discretized in collocation points $x_i, i = 0,\ldots, N$. Now, the resulting matrix system of the putting is reduced as $AX = B$, where $A$ is a matrix of $(3N + 7)\times (6N + 24)$ and $B$ is a vector of $(3N + 7)$. We solved the obtained system by means of a home-made program which is based on singular value decomposition (SVD) method and get the coefficients $c_{-2},\ldots, c_{N+1}$ and $d_{-2},\ldots, d_{N+1}$, and then $y(x,t)$ and $p(x,t)$.

$$\begin{bmatrix} x & x_{i-2} & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} \\ Q_i & 0 & 1 & 11 & 11 & 1 & 0 \\ Q_i' & 0 & -4 & -12 & 12 & 4 & 0 \\ Q_i'' & 0 & \frac{15}{2} & \frac{15}{2} & \frac{15}{2} & \frac{15}{2} & \frac{15}{2} & 0 \\ Q_i''' & 0 & \frac{15}{2} & \frac{15}{2} & \frac{15}{2} & \frac{15}{2} & \frac{15}{2} & 0 \end{bmatrix}$$

Table 1: The values of $Q_i, Q_i', Q_i'', Q_i'''$.  

3 An illustrative example

In this section, the numerical results of an illustrative example is present. The simulation is conducted on Matlab 7.
Example: In problem (1), we take $\nu = 10^{-6}$, $T = 1$ and $x \in [0,1]$, where the problem has the exact solution as

$$
\begin{align*}
    z &= \nu(2(t-1)^3x(x-1) - 6t^2(t-1)^2 - 4t(1-t)) + 12(t-1)^2x(x-1) + 3t^2(2t-2)x(x-1) + 4t(1-t)^2 + 6t^2(1-t)^2x(x-1), \\
    y &= t^2(1-t)^3x(x-1), \\
    p &= \frac{1}{\nu}2(t-1)^2x(x-1) - 2t^2(1-t)^2 + 3t^2(t-1)x(x-1), \\
    u &= \frac{p}{\nu}2(t-1)^3x(x-1) - 2t^2(1-t)^3 + 3t^2(t-1)x(x-1).
\end{align*}
$$

With $N = 80$ and by a simple calculation, we find the coefficients $c_{-2}, c_{-1}, \ldots, c_{N+1}$ and $d_{-2}, d_{-1}, \ldots, d_{N+1}$. Putting in (7) and (8) we find the approximate values of $y(x,t)$ and $p(x,t)$ on $[0,1] \times [0,1]$. Table 2 shows the $L_2$ and $L_\infty$ norm errors of $y(x,t)$ and $p(x,t)$ for different values of $t$ on $[0,1]$. Then, Figures 1, 2 have been plotted for the exact and approximate solutions for different values of $t$ on $[0,1]$ to taking $N = 80$, $\nu = 10^{-6}$ and $x \in [0,1]$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$L_2$ - error($y$)</th>
<th>$L_\infty$ - error($y$)</th>
<th>$L_2$ - error($p$)</th>
<th>$L_\infty$ - error($p$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2.6556 \times 10^{-12}$</td>
<td>$6.2541 \times 10^{-14}$</td>
<td>$1.3367 \times 10^{-19}$</td>
<td>$5.4542 \times 10^{-20}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$6.193 \times 10^{-14}$</td>
<td>$1.5248 \times 10^{-15}$</td>
<td>$3.4710 \times 10^{-11}$</td>
<td>$1.4002 \times 10^{-11}$</td>
</tr>
<tr>
<td>0.7</td>
<td>$1.1865 \times 10^{-14}$</td>
<td>$2.7332 \times 10^{-16}$</td>
<td>$6.1213 \times 10^{-12}$</td>
<td>$2.5095 \times 10^{-12}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$4.4513 \times 10^{-7}$</td>
<td>$1.0255 \times 10^{-8}$</td>
<td>$2.3813 \times 10^{-14}$</td>
<td>$9.3984 \times 10^{-15}$</td>
</tr>
</tbody>
</table>

Table 2: The errors of $y(x,t)$ and $p(x,t)$.

Figure 1: Comparisons between exact and approximate solutions of $y(x,t)$ with $t = 0, 0.1, 0.5, 0.7, 0.9$ and $t = 1$. 


Figure 2: Comparisons between exact and approximate solutions of \( p(x,t) \) with \( t = 0, t = 0.3, t = 0.5, t = 0.7, \) and \( t = 1 \).

References


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Existence theorem for the distributed order fractional hybrid differential equations

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Abstract
In this article, we develop the fractional hybrid differential equations of distributed order (DOFHDEs) involving the Riemann-Liouville differential operator of order $0 < q < 1$ with respect to a nonnegative density function. Furthermore, an existence theorem for (DOFHDEs) is proved under mixed the Lipschitz and the Caratheodory conditions.

Keywords: Fractional hybrid differential equations, Fractional derivative, Distributed order.

Mathematics Subject Classification: 26A33

1 Introduction
In recent years, quadratic perturbations of nonlinear differential equations have attracted much attention to researchers. We call such differential equations hybrid differential equations. Details of perturbations for a differential equations are given in Dhage [2]. Dhage and Lakshmikantham [3] established the existence uniqueness results and some fundamental differential inequalities for the following first order hybrid differential equation

$$\frac{d}{dt} \left[ \frac{x(t)}{f(t, x(t))} \right] = g(t, x(t)), \quad t \in J,$$

$$x(t_0) = x_0 \quad x_0 \in \mathbb{R},$$

where $J = [t_0, t_0 + a]$ is a bounded interval in $\mathbb{R}$ for some $t_0$ and $a \in \mathbb{R}$ with $a > 0$. Also, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R})$, such that $C(J \times \mathbb{R}, \mathbb{R})$ denote the class of continuous functions $f : J \times \mathbb{R} \to \mathbb{R}$ and $C(J \times \mathbb{R})$ is called the Caratheodory class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ which are Lebesgue integrable when is bounded by a Lebesgue integrable function on $J$. Moreover,

(i) the map $t \mapsto g(t, x)$ is measurable for each $x \in \mathbb{R},$

(ii) the map $x \mapsto g(t, x)$ is continuous for each $t \in J$.

Later, Zhao et al. [6] develop the following fractional hybrid differential equations involving the Riemann-Liouville differential operators of order $0 < q < 1$

$$D^q_0 \left[ \frac{x(t)}{f(t, x(t))} \right] = g(t, x(t)), \quad t \in J,$$

$$x(0) = 0,$$
where $J = [0, T]$ is bounded in $\mathbb{R}$ for some $T \in \mathbb{R}$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R})$. They established the existence, uniqueness results and some fundamental fractional differential inequalities to prove the existence of extremal solutions of equation (3) and (4). They considered necessary tools under mixed the Lipschitz and the Caratheodory conditions to prove the comparison principle.

Now, with respect to the idea of fractional derivative of distributed order which was stated by Caputo [5, 4], we develop the one class of distributed order fractional hybrid differential equations (DOFHDEs) with respect to a nonnegative density function.

**Definition 1.1.** The distributed order fractional hybrid differential equation (DOFHDEs), involving the Riemann-Liouville differential operator of order $0 < q < 1$ with respect to the nonnegative density function $b(q) > 0$ is defined as follows:

$$\int_0^1 b(q)D^q [\frac{x(t)}{f(t, x(t))}] dq = g(t, x(t)), \quad t \in J, \quad \int_0^1 b(q) dq = 1,$$

$$x(0) = 0,$$

where $J = [0, T]$ is a bounded in $\mathbb{R}$ for some $T \in \mathbb{R}$. Also $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R})$.

In order to find the solution of the DOFHDE (5) and (6) we find a function $x \in C(J, \mathbb{R})$ such that

1. the function $t \mapsto \frac{x(t)}{f(t, x(t))}$ is continuous for each $t \in \mathbb{R}$,
2. $x$ satisfies the equations in (5) and (6).

If we consider

$$b(q) = b_1 \delta(q - q_1) + b_2 \delta(q - q_2) + \ldots + b_n \delta(q - q_n), \quad b_i > 0, \quad 0 < q_i < 1, \quad i = 1, ..., n$$

(7)

Where $\delta$ is the Dirac delta function, then we get

$$b_1 D^{q_1}[Y(t)] + b_2 D^{q_2}[Y(t)] + \ldots + b_n D^{q_n}[Y(t)] = g(t, x(t))$$

(8)

such that $Y(t) = \frac{x(t)}{f(t, x(t))}$.

**2 Main Results**

Now, for expressing the existence theorem of (5) and (6) we state a lemma in Banach algebra and some hypotheses as follows

**Lemma 2.1.** ([1]). Let $S$ be a non-empty, closed convex and bounded subset of the Banach algebra $X$ and let $A : X \rightarrow X$ and $B : S \rightarrow X$ are two operators such that

(a) $A$ is Lipschitz constant $\alpha$,
(b) $B$ is completely continuous,
(c) $x = AxBy \Rightarrow x \in S$ for all $y \in S$, and
(d) $\alpha M < 1$, where $M = \| B(S) \| = \sup \| B(x) \| : x \in S$.

Then the operator equation $AxBy = x$ has a solution in $S$.

(A0) The function $x \mapsto \frac{x}{f(t, x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.

(A1) There exists a constant $L > 0$ such that

$$| f(t, x) - f(t, y) | \leq L \cdot | x - y |$$

for all $t \in J$ and $x, y \in \mathbb{R}$.
(A2) There exists a function \( h \in L^1(J, \mathbb{R}) \) such that
\[
| g(t, x) | \leq h(t)
\]
for all \( t \in J \) and \( x \in \mathbb{R} \).

**Lemma 2.2.** Assume that hypotheses (A0) holds, then for any \( h \in L^1(J, \mathbb{R}) \) and \( 0 < q < 1 \), the function \( x \in C(J, \mathbb{R}) \) is a solution of the DOFHDE (5) and (6) if and only if \( x \) satisfies the following equation
\[
x(t) = \frac{f(t, x(t))}{\pi} \int_0^t L\{3\{\frac{1}{B(re^{-i\pi})}; t - \tau\}g(\tau, x(\tau))d\tau,
\]
such that \( 0 \leq \tau \leq t \leq T \) and
\[
B(s) = \int_0^1 b(q)s^q dq.
\]

**Theorem 2.3.** Assume that hypotheses (A0)-(A2) hold. Further, if
\[
\frac{LM \| h \|_{L^1}}{\pi} < 1, \quad M > 0,
\]
then the DOFHDE (5) and (6) has a solution defined on \( J \).

**Proof.** Set \( X = C(J, \mathbb{R}) \) and define a subset \( S \) of \( X \) defined by
\[
S = \{ x \in X \| x \| \leq N \},
\]
where \( N = \frac{F_0M \| h \|_{L^1}}{\pi - LM \| h \|_{L^1}} \) and \( F_0 = \sup_{t \in J} |f(t, 0)| \). Clearly \( S \) is a closed, convex and bounded subset of the Banach space \( X \). By Lemma 2.2, DOFHDE (5) and (6) is equivalent to the nonlinear integral equation (13). We define two operators \( A : X \rightarrow X \) and \( B : S \rightarrow X \) by
\[
Ax(t) = f(t, x(t)) \quad t \in J,
\]
and
\[
Bx(t) = \frac{1}{\pi} \int_0^t L\{3\{\frac{1}{B(re^{-i\pi})}; t - \tau\}g(\tau, x(\tau))d\tau,
\]
then the equation (13) is transformed into the operator equation as
\[
Ax(t)Bx(t) = x(t) \quad t \in J.
\]
It is possible to show that the operators \( A \) and \( B \) satisfy all the conditions of Lemma 2.1 and hence the operator equation \( AxBx = x \) has a solution in \( S \). As a result the DOFHDE (5) and (6) has a solution defined on \( J \). \( \square \)

**References**


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Comparing results of fractional Sturm-Liouville problems using homotopy analysis method and Adomian decomposition method

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Abstract

In this work, we consider the application of the Homotopy analysis method (HAM) to compute the eigenvalues of fractional Sturm-Liouville problems. The auxiliary parameter $\eta$ has two important application. The first one is controlling the convergence of the HAM approximate series solutions and the second one is predicting multiple solutions. The efficiency of this method is illustrated by expressing an example. 

Keywords: Caputos fractional derivative, Fractional Sturm-Liouville problems, Homotopy analysis method

Mathematics Subject Classification: 34B24

1 Introduction


In this paper, we consider the following class of eigenvalue problems of the form

$$D^\alpha[p(x)y'(x)] + \lambda q(x)y(x) = 0, \quad x \in (0, 1), \quad 0 < \alpha \leq 1$$

subject to

$$ay(0) + by'(0) = 0, \quad cy(1) + dy'(1) = 0,$$

where $a, b, c, d \in \mathbb{R}, q(x), p(x) > 0, q(x)$ and $p(x)$ are smooth functions and $D^\alpha$ denotes the fractional differential operator of order $\alpha$. Al-Mdallal [8] applied the Adomian decomposition method for solving fractional Sturm- Liouville problems.
Example 1.1. To illustrate the basic idea of HAM, Consider the regular fractional eigenvalue problem

\[ D^{1/2}u'(x) + \lambda u(x) = 0, \quad x \in (0,1), \]  

with the boundary conditions

\[ u'(0) = 0 \]  

\[ u(1) = 0 \]  

The solution of (1) can be expressed by a set of base functions

\[ \{1, x^{3/2}, x^3, x^{9/2}, \ldots\} \]

in the form

\[ u(x) = \sum_{i=0}^{\infty} d_i x^{\alpha i} \]  

where \( d_i \) are coefficients to be determined. The solution of (1) must be written like (4). Under the rule of solution expression denoted by (4), We choose the linear operator

\[ \mathcal{L}[U(x;p)] = D^{1/2}U'(x;p) \]

From (1), we define the nonlinear operator

\[ \mathcal{N}[U(x;p)] = D^{1/2}U'(x;p) + \lambda U(x;p) \]  

According to boundary conditions (2) and the rule of solution expression (4), The initial approximation should be in the form \( u_0(x) = x^3 - 1 \). The initial approximation should satisfy in (2) and (3). From definition of \( R_m(\vec{u}_{m-1}) \) and (5), we have

\[ R_m(\vec{u}_{m-1}) = D^{1/2}u_{m-1}' + \lambda u_{m-1}, \]

where prime denotes differentiation with respect to the similar variable \( x \). Now, the solution of the \( m \)th order deformation equation, for \( m \geq 1 \) becomes

\[ u_m(x) = \chi_m u_{m-1}(x) + h^\alpha [\mathcal{H}(x)R_m(\vec{u}_{m-1})], \]

with the initial condition

\[ u_m'(0) = 0. \]

According to the rule of solution expression denoted by (4), we understand that \( \mathcal{H}(x) = 1 \). Substituting \( \mathcal{H}(x) \) in above equation and using Caputo’s fractional derivative, we have

\[ \alpha = 3/2 \]

Consequently, the first few terms of the HAM series solution are as follows

\[ u_1(x) = \frac{4\lambda x^{3/2}}{3\sqrt{\pi}}, \]

\[ u_2(x) = \frac{4\lambda + 4\lambda h^2}{3\sqrt{\pi}} x^{3/2} + \frac{1}{6} \lambda^2 h^2 x^3, \]

\[ \vdots \]
And so on, in this manner the rest of components of the HAM series can be obtained. Accordingly, the \(m\)th order approximate solution of the HAM, \(U_m(x)\), is in the form

\[ u(x) \approx U_m(x) = \sum_{i=0}^{m} u_i(x) \]  

(7)

To the \(m\)th order approximate solution (7), which depends on the eigenvalue \(\lambda\) and the auxiliary parameter \(\tilde{h}\), the condition (3) reads

\[ u(1) \approx U_m(1) = 0 \]  

(8)

By using Maple software, we plotted (8), several horizontal plateaus occur, each of which corresponds to an eigenvalue of the fractional Sturm-Liouville problem. As a first illustration of this uniqueness criterion of the HAM, \(\lambda\) has been plotted according to (8) in the \(h\) range \([-1.8, 0]\) and \(\lambda\) range \([1, 25]\) for \(m = 25\), Fig. 1. Three \(\lambda\)−plateaus can be identified in this figure, namely \(\lambda_1 = 2.11027708\) in the range \([-1.5,-0.2]\), \(\lambda_2 = 13.76538223\) in the range \([-1.2,-0.4]\) and \(\lambda_3 = 24.24328676\) in the range \([-0.7,0.4]\) of \(\tilde{h}\). In Table 1, results of Adomian decomposition method and homotopy analysis method has been compared.

## 2 Main Result

In the present paper, the Homotopy Analysis Method has been applied to numerically approximate the eigenvalues of the fractional Sturm-Liouville problems. In the plot of \(\lambda\) as a function of \(h\), several horizontal plateaus appears. Indeed, every such horizontal plateau corresponds to an eigenvalue of the fractional Sturm-Liouville problem. These multiple solutions, i.e. eigenvalues, can be calculated by starting the HAM algorithm with one and the same initial guess and linear operator \(L\). It was observed in this paper that the auxiliary parameter \(h\), which controls the convergence of the HAM approximate series solutions, has another important application. This important application is predicting multiple solutions by considering the number of plateaus which occur in the \(h\)-curve.

![Figure 1: (left) \(\lambda\)−plateau corresponding to \(\lambda_1\); (middle) \(\lambda\)−plateau corresponding to \(\lambda_2\); (right) \(\lambda\)−plateau corresponding to \(\lambda_3\)](image)

<table>
<thead>
<tr>
<th>The eigenvalue</th>
<th>(\lambda_1)</th>
<th>(\lambda_2)</th>
<th>(\lambda_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADM</td>
<td>2.11027708</td>
<td>13.76538223</td>
<td>24.24328676</td>
</tr>
<tr>
<td>HAM</td>
<td>24.24328676</td>
<td>13.76538223</td>
<td>2.11027708</td>
</tr>
</tbody>
</table>

Table 1: comparison between results of Adomian decomposition method and Homotopy analysis method

1300
References


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Comparison numerical solutions of differential algebraic equations by differential quadrature and Pade series

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Abstract

In this paper, numerical solution of linear differential algebraic equations considered that after reduction index using differential quadrature method and Pade series. Therefore, we present the notation and basic definitions and theorems of the Hessenberg forms of the differential algebraic equations that by reduction index transform to a system of equations that solvable with above methods. In addition, we present the properties of the proposed differential quadrature method and Pade series, at last comparison with together. Numerical results demonstrate that the differential quadrature is effective toward to Pade series.

Keywords: Differential algebraic equation, Reduction index, Hessenberg forms, Power series, Differential quadrature method, Pade series.

1 Introduction

The most general form of a DAE is given following where \( \frac{\partial F}{\partial y} \) may be singular.

\[
F(t, y, y') = 0
\]

1.1 DAE in Hessenberg form with index-\( \nu \)

\[
\begin{cases}
x_1' = f_1(x_1, \ldots, x_{\nu-1}, x_{\nu}) \\
\vdots \\
x_{\nu-1}' = f_{\nu-1}(x_{\nu-2}, x_{\nu-1})
\end{cases}
\quad \text{with} \quad \frac{\partial f_\nu}{\partial x_{\nu-1}} \frac{\partial f_{\nu-1}}{\partial x_{\nu-2}} \cdots \frac{\partial f_2}{\partial x_1} \frac{\partial f_1}{\partial x_\nu} \text{ nonsingular}
\]

for all relevant points \((x_1, \cdots, x_\nu) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_\nu} \).

Definition 1.1. Reducing index We consider a linear (or linearized) model problem,

\[
\begin{cases}
X^{(m)} = \sum_{j=0}^{m} A_j X^{(j-1)} + By + q, \\
0 = CX + r
\end{cases}
\]

where \( A_j, B \) and \( C \) are smooth functions of \( t \), \( 0 \leq t \leq t_f \), \( A_j(t) \in \mathbb{R}^{n \times n} \), \( j = 1, \cdots, m \), \( B(t) \in \mathbb{R}^n \), \( C(t) \in \mathbb{R}^{n \times n} \), \( n \geq 2 \), and \( CB \) is nonsingular for each \( t \) (hence the DAE has index \( m + 1 \)). The
inhomogeneities are \( q(t) \in \mathbb{R} \) and \( r(t) \in \mathbb{R} \). The DAE (1.3) will be transformed into an implicit DAE form by representing a simple formulation. For this reason, we put

\[
y = (CB)^{-1}C\left(X^m - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right)
\]

and by substituting (1.3) in (1.3), we obtain an implicit DAE which has index \( m \), as follows,

\[
\sum_{j=0}^{m} E_j X^{(j)} = \dot{q}
\]

where \( E_j(t) \in \mathbb{R}^{n \times n} \), \( j = 1, \ldots, m \), and except \( E_0(t) \), others are singular matrices.

**Theorem 1.2.** Consider problem (1.3), when it has index 2, \( n = 2 \) and \( k = 1 \). This problem is equivalent to index-1 DAE system

\[
E_1 X' + E_0 X = \dot{q}
\]

where \( E_0 = \begin{bmatrix} b_1 a_{21} - b_2 a_{11} & b_1 a_{22} - b_2 a_{12} \\ c_1 & c_2 \end{bmatrix} \), \( E_1 = \begin{bmatrix} b_2 & -b_1 \\ 0 & 0 \end{bmatrix} \), \( \dot{q} = \begin{bmatrix} -b_2 q_1 - b_1 q_2 \\ -r \end{bmatrix} \)

Proof is presented in [6].

By theorems (5.1), the above system can be transformed to the following full rank DAE system, with \( n \) equations and \( n \) unknowns,

\[
E_m X^{(m)} + E_{m-1} X^{(m-1)} + \cdots + E_1 X' + E_0 X = \dot{q}.
\]

Index of (6) is equal to \( m \).

## 2 Differential quadrature method

Differential quadrature method (DQM) is a numerical method for evaluating derivatives of a sufficiently smooth function, proposed by Bellman and Casti in 1971. Suppose function \( f(x) \) is sufficiently smooth on the interval \([a, b]\). On the interval, \( N \) distinct nodes are defined: \( a = x_1 < x_2 < \cdots < x_N = b \) The function values on these nodes are assumed to be \( f_1, f_2, \ldots, f_N \). Based on DQM, the first and second order derivatives on each of these nodes are given by

\[
\frac{df(x_i)}{dx} \approx \sum_{j=1}^{N} a_{ij} f_j = a_i f_N, \quad \frac{d^2 f(x_i)}{dx^2} \approx \sum_{j=1}^{N} b_{ij} f_j = b_i f_N, \quad i = 1, 2, \ldots, N
\]

The coefficients \( a_{ij} \) and \( b_{ij} \) are the weighting coefficients of the first- and second-order derivatives with respect to \( x \), respectively. Using the Lagrange interpolating functions, Shu and Richards [2] gave a convenient and recurrent formula for determining the derivative weighting coefficients as follows:

\[
a_{ij} = \frac{M(x_i)}{(x_i - x_j).M(x_j)} \quad i \neq j, \quad a_{ii} = - \sum_{j=1, i \neq j}^{N} a_{ij} \quad i, j = 1, 2, \ldots, N
\]

also for second derivative weighting coefficients we have

\[
b_{ij} = 2[a_{ii} a_{ij} - \frac{a_{ij}}{(x_j - x_i)}] \quad i \neq j, \quad b_{ii} = - \sum_{j=1, i \neq j}^{N} b_{ij} \quad i, j = 1, 2, \ldots, N
\]
where \( M(x_i) = \prod_{j=1, i \neq j}^N (x_i - x_j) \). When above equations is used, there is no restriction on the choice of the nodes [2]. Consider (1.3) that \( m = 1 \), now by using DQM we have

\[
E_1 a_i X^T_N + E_0 X^T_N = \hat{q}_N
\]  

(10)

After implementation initial conditions last system can be expressed in the following vector matrix form

\[ A k = B \]

Where in this equation \( A \) is an \((2N) \times (2N - 2)\) matrix of known constants, \( k \) is an \((2N - 2) \times 1\) vector of unknown values and \( B \) is an \((2N) \times 1\) vector of known values that this equation can be solve by Least Square method.

### 3 Solution of problem

In this section we present some numerical results to demonstrate the efficiency of DQM toward Pade series for solving the DAEs. These example are chosen such that their exact solutions are known. The numerical computations have been done by the software Matlab edition 7.10 .

**Example:** Consider the equations for \( 0 \leq t \leq 1 \),

\[
\begin{align*}
x_1'(t) &= 10(t - 2)z - 9e^t \\
x_2'(t) &= 9z + \left(\frac{11}{2} - t\right)e^t \\
0 &= (t + 2)x_1 + (t^2 - 4)x_2 - (t^2 + t - 2)e^t
\end{align*}
\]

(11)

This DAE is in pure index-2. From the initial conditions \( x_1(0) = 1, \ x_2(0) = 1 \) we have the exact solution

\[ x_1(t) = e^t, \quad x_2(t) = e^t, \quad z(t) = -\frac{e^t}{2 - t} \]

By using theorem (5.1) we have:

\[
\begin{align*}
9x_1'(t) + 10(2 - t)x_2'(t) &= (29 - 10t)e^t \\
(t + 2)x_1 + (t^2 - 4)x_2 &= (t^2 + t - 2)e^t
\end{align*}
\]

\[ z = \frac{1}{19(t - 2)} x_1' + \frac{1}{19} x_2' + \frac{20 - t}{19(t - 2)} e^t \]

### 4 Conclusion

Both the DQM and Pade series have been applied to compute the solution of DAE. Through example which has exact solution, it was found that in order to obtain accurate numerical results demonstrate DQM is efficient.
Table 1: Numerical solution of $x_1(t)$ and $x_2(t)$ and $z(t)$ by DQM

<table>
<thead>
<tr>
<th>$t$</th>
<th>Exact $x_1(t) = x(t) = x_2(t)$</th>
<th>$z(t)$</th>
<th>$a \approx x(t)$</th>
<th>$r \approx z(t)$</th>
<th>$Er x_1(t) = x_2(t)$</th>
<th>$Er z(t)$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00000</td>
<td>−0.500000</td>
<td>1.00000</td>
<td>−0.500000</td>
<td>0 $E$ − 12</td>
<td>7 $E$ − 12</td>
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<tr>
<td>0.1</td>
<td>1.10517</td>
<td>−0.581668</td>
<td>1.10517</td>
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<td>0 $E$ − 12</td>
<td>1 $E$ − 12</td>
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References


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Optimal partitions for first eigenvalues of Laplace operator

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Abstract

Given a bounded open set $\Omega \subset \mathbb{R}^2$, we aim to approximate partitions of $\Omega$ minimizing the sum of first eigenvalues of Dirichlet Laplacian. To do this, a new idea to approximate the second eigenfunction and second eigenvalue is presented. We use the qualitative properties of the minimization problem to construct a numerical algorithm to approximate optimal configurations. Moreover, we discuss the numerical implementation of the resulting approach and present computational tests confirming the expected asymptotic behavior of optimal partitions with large numbers of partitions.

Keywords: Finite differences scheme, Free boundary problems, Optimal partition, Eigenvalue problems.

Mathematics Subject Classification[2010]: 35P99,65N06,65N25,65N30

1. Introduction and statement of problem

Let $\Omega \subset \mathbb{R}^2$ be a connected, bounded and open domain with regular boundary $\partial \Omega$. Consider Laplace operator $-\Delta$ on $\Omega$ with Dirichlet boundary condition. The eigenvalue problem for Laplace operator is

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases}$$

The eigenvalues of the self adjoint, positive operator $-\Delta$ in $\Omega$ are denoted by $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \cdots$.

**Problem A:** Given a bounded open set $\Omega \subset \mathbb{R}^2$, we are looking for a family of disjoint, open and connected subsets $\{\Omega_i\}_{i=1}^n$ such that

$$\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n \subseteq \Omega, \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j,$$

which minimize the following domain functional

$$I(\Omega_1, \cdots \Omega_n) = \sum_{i=1}^n \lambda_1(\Omega_i).$$

Such partition of $\Omega$ is called optimal partition. The objective of our work is to find numerical approximation for the optimal partition problem for Dirichlet Laplacian eigenvalues.
Note that since the first eigenvalue has variational characterization then the cost functional can be written as energy type as follows.

\[
\text{Minimize } E(u_1, \ldots, u_n) = \int_\Omega \sum_{i=1}^n \frac{1}{2} |\nabla u_i|^2 \, dx,
\]

over the set

\[S = \{(u_1, \ldots, u_n) \in (H_0^1(\Omega))^n : \int_\Omega |u_i|^2 \, dx = 1, u_i \cdot u_j = 0, \text{ for } i \neq j\}.
\]

In this case the optimal partitions are the supports of \(u_i\) i.e,

\[\Omega_i = \{x \in \Omega : u_i(x) > 0\}.
\]

In [2] Caffarelli and Lin obtained regularity results for the optimal partition and estimates for the asymptotic behavior of (1.2) when \(n \to \infty\). They conjectured that for the optimal partition \(\{\Omega_i^*\}_{i=1}^n\), the following holds

\[\sum_{i=1}^n \lambda_1(\Omega_i^*) \approx n^2 \lambda_1(H)/|\Omega|,
\]

where \(H\) is the regular hexagon of area 1 in \(\mathbb{R}^2\). This estimate says that, far from \(\partial \Omega\), a tiling by regular hexagons of area \(\frac{|\Omega|}{n}\) is asymptotically close to the optimal partition.

2. Mathematical Background

In this section, we state some facts which states qualitative properties of both optimal sets \(\Omega_i\) and eigenfunctions associated to \(\lambda_1(\Omega_i)\). These results will be used in our numerical scheme. The first result in this issue is (see [3]):

**Lemma 2.1.** ([3]) There exists \((\Omega_1, \ldots, \Omega_n)\) minimizing the given functional (1). Furthermore, if \(\phi_1, \ldots, \phi_n\) are corresponding eigenfunctions normalized in \(L_2\), then, there exist \(a_i \in \mathbb{R}\) such that the functions \(u_i = a_i \phi_i\) verify in \(\Omega\) the differential inequalities (in distributional sense)

- \(-\Delta u_i \leq \lambda_1(\Omega_i)u_i, \text{ a.e. in } \Omega,
- \(-\Delta (u_i - \sum_{j \neq i} u_j) \geq \lambda_1(\Omega_i)u_i - \sum_{j \neq i} \lambda_1(\Omega_j)u_j.

3. Numerical Algorithms for optimal partitions

3.1. Inverse Power Method for the first eigenvalue and the first eigenfunction

To explain our algorithm, first we briefly explain the numerical approximation to obtain first eigenvalue and first eigenfunction simultaneously. **Algorithm 1:**
1. Set \( k = 0 \), choose an initial guess \( u^0 \) such that \( u^0 = 0 \) on the boundary and \( \| u^0 \|_{L_2(\Omega)} = 1 \). Let
\[
\lambda^0 = \int_{\Omega} |\nabla u^0(x)|^2 \, dx.
\]

2. Given \( u^k \) such that \( \| u^k \|_{L_2} = 1 \), \( u^k = 0 \) on \( \partial \Omega \), set \( \lambda^k = \int_{\Omega} |\nabla u^k(x)|^2 \, dx \), then solve the following Dirichlet problem.
\[
\begin{align*}
-\Delta u &= \lambda^k u & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega.
\end{align*}
\]
(3)

3. Set \( u^{k+1} = \frac{u^k}{\| u^k \|_{L_2}} \) and calculate \( \lambda^{k+1} = \int_{\Omega} |\nabla u^{k+1}(x)|^2 \, dx \).

4. Set \( k = k + 1 \) and go to the first step.

5. Stop if \( |\lambda^{k+1} - \lambda^k| \) is sufficiently small.

3.2. Numerical Algorithm for Optimal partitions for first eigenvalues for \( n \) partitions

Now we turn to the case that the number of partitions is \( n \geq 2 \). We discretize the \( m \)-th differential inequality in second part of Lemma(2.1), by finite difference scheme and letting \( \Delta x = \Delta y = h \), to arrive at
\[
\frac{1}{h^2} \left[ 4u_m(x_i, y_j) - 4u_m(x_i, y_j) \right] - \frac{1}{h^2} \sum_{l \neq m} \left[ 4u_l(x_i, y_j) - 4u_l(x_i, y_j) \right]
\geq \lambda_i(\Omega_m)u_m(x_i, y_j) - \sum_{l \neq m} \lambda_l(\Omega_l)u_l(x_i, y_j), \quad m = 1, \ldots, n.
\]
(4)

We can obtain \( u_m(x_i, y_j) \) from (4) and we impose the following conditions
\[
u_m(x_i, y_j) - u_l(x_i, y_j) = 0 \quad \text{for } m \neq l \text{ and } u_m(x_i, y_j) \geq 0.
\]

1. Set \( k = 0 \), choose \( u^0 \) for \( i = 1, \ldots, n \) such that \( \| u^0 \|_{L_2(\Omega_i)} = 1 \), where \( \Omega_i = \{ x \in \Omega : u_i(x) > 0 \} \).

2. Given \( u^k_m \) with \( \| u^k_m \|_{L_2} = 1 \) then obtain \( \lambda^k_m \). We iterate as
For \( t = 0, 1, \ldots, k \)
For \( m = 1, \ldots, n \)
For \( i = 1, \ldots, n_x \)
For \( j = 1, \ldots, n_y \)
\[
u_m^{(t+1)}(x_i, y_j) = \max \left( \nu_m^{(t)}(x_i, y_j) - \sum_{l \neq m} \alpha_l^{(t)}(x_i, y_j) - \lambda^k(\Omega_m)\frac{h^2}{4} \nu_l^{(t)}(x_i, y_j), 0 \right).
\]
OPTIMAL PARTITIONS FOR FIRST EIGENVALUES OF LAPLACE OPERATOR

(a) Initial guess $u^0$  
(b) Optimal partition for nine cells.

REFERENCES


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A note on fuzzy isoperimetric problem

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Abstract
The purpose of the present paper is to extend the results of one of the problems in [3], named Fuzzy Isoperimetric Problem. In this regard, concerning the concept of the strongly generalized differentiability introduced first by [1], we establish and study a modified version of Theorem 5.1 in [3].

Keywords: Fuzzy variational problems, Fuzzy Euler-Lagrange conditions, Strongly generalized differentiability.

Mathematics Subject Classification: 34A12, 34A99

1 Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper.

Let us denote by $R_F$ the class of fuzzy subsets of the real axis (i.e. $u : R \to [0, 1]$ ) satisfying the following properties:

1. $u$ is normal, i.e., there exists $s_0 \in R$ such that $u(s_0) = 1$,
2. $u$ is a convex fuzzy set (i.e. $u(ts + (1-t)r) \geq \min\{u(s), u(r)\}$, $\forall t \in [0, 1]$, $s, r \in R$),
3. $u$ is upper semicontinuous on $R$,
4. $cl\{s \in R | u(s) > 0\}$ is compact where $cl$ denotes the closure of a subset.

Then $R_F$ is called the space of fuzzy numbers.

The metric structure is given by the Hausdorff distance

$$D : R_F \times R_F \to R_+ \cup \{0\},$$

$$D(u, v) = \sup_{\alpha \in [0, 1]} \max\{|u^l(\alpha) - v^l(\alpha)|, |u^r(\alpha) - v^r(\alpha)|\}.$$ 

($R_F, D$) is a complete metric space.

Definition 1.1. [3] (Partial ordering). Let $u, v \in R_F$. We write $u \preceq v$, if $u^l(\alpha) \leq v^l(\alpha)$ and $u^r(\alpha) \leq v^r(\alpha)$ for all $\alpha \in [0, 1]$. Moreover, $u \simeq v$, if $u \preceq v$ and $u \succeq v$. In the other words, $u \simeq v$, if $u[\alpha] = v[\alpha]$ for all $\alpha \in [0, 1]$.

Definition 1.2. [3] Let $x, y \in R_F$. If there exists $z \in R_F$ such that $x = y + z$ then $z$ is called the $H$-difference of $x, y$ and it is denoted $x \ominus y$.

Definition 1.3. [4] Let $F : I \to R_F$. Fix $x_0 \in I$. We say $F$ is differentiable at $x_0$, if there exists an element $F'(x_0) \in R_F$ such that
We here choose the following deformation of the fuzzy curve
\[ x(t) := x^*(t) + \epsilon \delta(t) \]  
where \( \epsilon \) is a small real number and \( \delta(t) := \sigma \eta(t) + \beta \zeta(t) \). In the last equation \( \sigma, \beta \) are real constants and the arbitrary independent fuzzy functions \( \eta(t), \zeta(t) \) vanish in the fuzzy sense at the endpoints.

\[ (1) \] for all \( h > 0 \) sufficiently close to 0, there exist \( F(x_0 + h) \cap F(x_0), F(x_0) \cap F(x_0 - h) \) and the limits (in the metric \( D \))
\[ \lim_{h \to 0^+} \frac{F(x_0 + h) \cap F(x_0)}{h} = \lim_{h \to 0^-} \frac{F(x_0) \cap F(x_0 - h)}{h} = F'(x_0), \]

\[ (2) \] for all \( h > 0 \) sufficiently close to 0, there exist \( F(x_0 + h) \cap F(x_0), F(x_0) \cap F(x_0 - h) \) and the limits (in the metric \( D \))
\[ \lim_{h \to 0^+} \frac{F(x_0 + h) \cap F(x_0)}{h} = \lim_{h \to 0^-} \frac{F(x_0) \cap F(x_0 - h)}{h} = F'(x_0). \]

**Definition 1.4.** [4] Let \( F : I \to \mathbb{R}_F \). We say \( F \) is \((1)-\)differentiable on \( I \) if \( F \) is differentiable in the sense (1) of Definition (1.3) and its derivative is denoted \( D_1 F \), and similarly for \((2)-\)differentiability we have \( D_2 F \).

The principal properties of defined derivatives are well known and can be found in [4]. We make use of the following Theorem [2].

**Theorem 1.5.** [2] Let \( F : I \to \mathbb{R}_F \) and put \( [F(x)]^\alpha = [f_\alpha(x), f_\alpha^*(x)] \) for each \( \alpha \in [0, 1] \).

(i) If \( F \) is \((1)-\)differentiable then \( f_\alpha \) and \( f_\alpha^* \) are differentiable functions and \( [D_1 F(x)]^\alpha = \left[ \frac{df_\alpha(x)}{dx}, \frac{df_\alpha^*(x)}{dx} \right] \).

(ii) If \( F \) is \((2)-\)differentiable then \( f_\alpha \) and \( f_\alpha^* \) are differentiable functions and we have \( [D_2 F(x)]^\alpha = \left[ \frac{df_\alpha(x)}{dx}, \frac{df_\alpha^*(x)}{dx} \right] \).

## 2 Fuzzy isoperimetric problem

The problem involving minimization of a fuzzy functional while giving a fuzzy integral constraints is called the fuzzy isoperimetric problem and it is stated as follows:

\[
\text{(FVP) \ Minimize } J(x) := \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) dt \\
\text{subject to } I(x) := \int_{t_0}^{t_f} h(x(t), \dot{x}(t), t) dt \approx c \\
\text{with } x(t_0) \approx x_0, x(t_f) \approx x_f,
\]

where \( c \in \mathbb{R} \) is a given fuzzy number.

**Theorem 2.1.** (Extension of Theorem 5.1 [3]) (Fuzzy Euler-Lagrange conditions in general case). Let \( x^* = x^*(t) \) be an admissible fuzzy curve, i.e., it is twice continuously differentiable (in the sense of Definition 3.3 in [3]) fuzzy curve that joints the given endpoints. Then, in order to \( x^* \) gives a relative (local) minimum to the fuzzy functional \( J \) in (FVP), it is necessary that for all \( \alpha \in [0, 1] \)

\[
D_{m,n}^1 \phi_{x,t}^l, r(z) = \frac{d}{dt} D_{m,n}^1 \phi_{x,t}^l, r(z) = 0, \ m, n = 1, 2. \tag{1}
\]

Here, the fuzzy functional \( \phi \) is as follows
\[ \phi(z) = g(\dot{x}^*(t, \alpha), \dot{x}^*(t, \alpha), x^*(t, \alpha), \dot{x}^*(t, \alpha), t, \alpha) + \lambda h(x^*(t, \alpha), \dot{x}^*(t, \alpha), x^*(t, \alpha), \dot{x}^*(t, \alpha), t, \alpha) \] and \( z = (x^t, \dot{x}^t, x^t, \dot{x}^t, t, \alpha) \).

**Proof.** We here choose the following deformation of the fuzzy curve \( x^* \) by taking into consideration an arbitrary twice continuously differentiable fuzzy function \( \delta(t) \) as

\[
x(t) := x^*(t) + \epsilon \delta(t) \tag{2}
\]

where \( \epsilon \) is a small real number and \( \delta(t) := \sigma \eta(t) + \beta \zeta(t) \). In the last equation \( \sigma, \beta \) are real constants and the arbitrary independent fuzzy functions \( \eta(t), \zeta(t) \) vanish in the fuzzy sense at the endpoints.
There are 16 cases. Here we prove the theorem only for a case which \( \frac{\partial h}{\partial x}, \frac{\partial h}{\partial t}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial t}, \frac{\partial y}{\partial x}, \frac{\partial y}{\partial t} \) and \( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial t} \) have (2)-differentiable. Since \( I \) is equal to the fuzzy number \( c \), therefore its increment is identically zero, particularly, the first variation must be zero. That is, for all \( \alpha \in [0,1] \)

\[
\int_{t_0}^{t_f} \left\{ \delta \frac{\partial h}{\partial x}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) + \delta \frac{\partial h}{\partial t}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) \right\} dt = 0, \tag{3}
\]

\[
\int_{t_0}^{t_f} \left\{ \delta \frac{\partial h}{\partial x}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) + \delta \frac{\partial h}{\partial t}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) \right\} dt = 0, \tag{4}
\]

\[
\int_{t_0}^{t_f} \left\{ \delta \frac{\partial h}{\partial x}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) + \delta \frac{\partial h}{\partial t}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) \right\} dt = 0, \tag{5}
\]

\[
\int_{t_0}^{t_f} \left\{ \delta \frac{\partial h}{\partial x}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) + \delta \frac{\partial h}{\partial t}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) \right\} dt = 0. \tag{6}
\]

In the remaining we will ignore the similar arguments and thus we consider only (3).

By integrating by parts the terms involving \( \delta t \) and letting \( \delta t^\tau(t, \alpha) := \sigma \eta^\tau(t, \alpha) + \beta \zeta^\tau(t, \alpha) \), we may find that

\[
\int_{t_0}^{t_f} \left\{ \sigma \eta^\tau + \beta \zeta^\tau \right\} \delta \left( \frac{\partial h}{\partial x}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) - \frac{d}{dt} \left( \frac{\partial h}{\partial x}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) \right) \right) dt = 0, \tag{7}
\]

Observe that \( x^{s\ast} \) is not the minimizer of \( I \), therefore

\[
\frac{\partial h}{\partial x^s}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) - \frac{d}{dt} \left( \frac{\partial h}{\partial x^s}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) \right) \neq 0.
\]

Furthermore, for any \( \eta^\tau, \zeta^\tau \), the constants \( \sigma, \beta \) are related together by (7).

The assumption that \( x^{s\ast} \) is the minimizer of \( J \) guarantees the increment of \( J \) must be non-negative in the fuzzy sense with respect to the deformation given in (2). Consequently, the first variation is zero and after integrating by parts, it holds that

\[
\int_{t_0}^{t_f} \left\{ \sigma \eta^\tau + \beta \zeta^\tau \right\} \delta \left( \frac{\partial g}{\partial x}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) - \frac{d}{dt} \left( \frac{\partial g}{\partial x}(x^{s\ast}, \dot{x}^{s\ast}, x^{s\ast\ast}, \dot{x}^{s\ast\ast}, t, \alpha) \right) \right) dt = 0, \tag{8}
\]

where \( \sigma, \beta \) are those satisfy (7).

If we eliminate \( \sigma, \beta \) between (7) and (8), then we arrive at

\[
\frac{\int_{t_0}^{t_f} \eta^\tau \left( \frac{\partial g}{\partial x}(z) - \frac{d}{dt} \frac{\partial g}{\partial x}(z) \right) dt}{\int_{t_0}^{t_f} \zeta^\tau \left( \frac{\partial g}{\partial x}(z) - \frac{d}{dt} \frac{\partial g}{\partial x}(z) \right) dt} = \frac{\int_{t_0}^{t_f} \zeta^\tau \left( \frac{\partial h}{\partial x}(z) - \frac{d}{dt} \frac{\partial h}{\partial x}(z) \right) dt}{\int_{t_0}^{t_f} \eta^\tau \left( \frac{\partial h}{\partial x}(z) - \frac{d}{dt} \frac{\partial h}{\partial x}(z) \right) dt} \tag{9}
\]

for every independent and twice continuously differentiable functions \( \eta^\tau, \zeta^\tau \).

Introducing the constant \( -\lambda_1^\tau = -\lambda_1^\tau(\alpha) \) which is equal to both sides of the equality in (9) gives that

\[
\int_{t_0}^{t_f} \eta^\tau \left( \frac{\partial g}{\partial x}(z) - \frac{d}{dt} \frac{\partial g}{\partial x}(z) \right) + \lambda_1^\tau \left( \frac{\partial h}{\partial x}(z) - \frac{d}{dt} \frac{\partial h}{\partial x}(z) \right) = 0 \tag{10}
\]

for any admissible \( \eta^\tau = \eta^\tau(t, \alpha) \).

Applying Lemma 4.2 in [3], we derive from (10) that

\[
D_{x,t}^\tau \phi_{\sigma,l}^{\ell,t}(z) - \frac{d}{dt} D_{x,t}^\tau \phi_{\sigma,l}^{\ell,t}(z) = 0. \tag{11}
\]

Now following the scheme of obtaining (11) and adapting it to the case under consideration involving (4), (5) and (6). We may show that

\[
D_{x,t}^\tau \phi_{\sigma,l}^{\ell,t}(z) - \frac{d}{dt} D_{x,t}^\tau \phi_{\sigma,l}^{\ell,t}(z) = 0. \tag{12}
\]
A note on fuzzy isoperimetric problem

\begin{align}
D^{1/2}_2 \phi_{r,r}^{r,r}(z) - \frac{d}{dt} D^{1/2}_2 \phi_{r,r}^{r,r}(z) &= 0, \quad (13) \\
D^{1/2}_2 \phi_{s,s}^{r,r}(z) - \frac{d}{dt} D^{1/2}_2 \phi_{s,s}^{r,r}(z) &= 0. \quad (14)
\end{align}

The other cases obtain similarly.

References

Application of two different methods for finding exact solutions of nonlinear PDE

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Abstract
This paper studies the (3+1)-dimensional MKdV-ZK equation. The exp-function method, modified F-expansion method are used to extract a few exact solutions to this equation. We show these two different methods provide the powerful mathematical tools for solving nonlinear evolution equations in mathematical physics. The proposed methods can be used for many other nonlinear wave equations.

Keywords: Exp-function method, modified F-expansion method, the (3+1)-dimensional MKdV-ZK equation, exact solutions.

Mathematics Subject Classification: 37K10, 35Q51, 35Q55.

1 Introduction
In recent years, the investigation of exact solutions to nonlinear evolution equations plays an important role in the nonlinear physical phenomena. Several powerful methods have been proposed to construct exact solutions for nonlinear partial differential equations, such as homogeneous balance method, Hirota bilinear method, improved generalized F-expansion method [1], Riccati equation method [2], asymptotic methods and so on. By using these methods we obtained many valuable exact solutions for nonlinear evolution equations. Physical structures of exact solutions are important to give more insight into the physical aspects of the nonlinear wave equations. In this paper, two such modern tools of integrability will be utilized to retrieve solutions to the (3+1)-dimensional MKdV-ZK equation although this equation was studied in the past few decade. The goal of the present paper is to apply the exp-function method [3] and the modified F-expansion method to the (3+1)-dimensional MKdV-ZK equation to finding some new exact solutions, and shows the great effectiveness of the introduced methods.

2 EXP-FUNCTION METHOD
The exp-function method is a powerful and utterly simple method for obtaining exact solutions of NLEEs[4]. Also, this method is oriented towards ease of use and capability of computer algebra systems. We consider the (3+1)-dimensional MKdV-ZK equation in the following form

\[ u_t + \alpha u^2 u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0, \]  

(1)
With the aid of Maple and the exp-function method, we obtain the following coefficients.

**Case 1.**

\[
a_0 = a_0, \quad a_1 = a_{-1} = 0, \quad b_{-1} = \frac{k\alpha a_0^2}{24b_1(k^2 + l^2 + p^2)}, \quad k = k, \\
l = l, \quad c = -(k^2 + l^2 + p^2), \quad p = p, \quad b_1 = b_1, \quad b_0 = 0.
\]

**Case 2.**

\[
b_1 = b_1, \quad b_{-1} = b_{-1}, \quad a_0 = 0, \quad a_1 = -\frac{b_1a_{-1}}{b_{-1}}, \quad c = c, \quad k = -\frac{3cb_1}{\alpha a_0^2}, \\
p = p, \quad l = \pm \sqrt{\frac{2c\alpha_1^2\alpha^2 - 36c^2b_1^2 - 4p^2a_1^2\alpha^2}{2\alpha a_0^2}}, \quad a_{-1} = a_{-1}, \quad b_0 = 0.
\]

**Case 3.**

\[
p = p, \quad b_1 = \frac{a_2^2b_0^2 - b_2^2a_0^2}{4a_0^2b_1}, \quad b_{-1} = b_{-1}, \quad a_0 = a_0, \\
b_0 = b_0, \quad k = -\frac{3cb_1}{\alpha a_0^2}, \quad a_1 = \frac{b_1^2a_0^2 - a_2^2b_0^2}{4a_0^2b_1}, \quad a_{-1} = a_{-1}, \\
c = c, \quad l = \pm \sqrt{\frac{2c\alpha_1^2\alpha^2 - p^2a_1^2\alpha^2 - 9e^2b_1^2}{\alpha a_0^2}}.
\]

We obtain the following solutions of eq. (1) for (2), (3) and (4) respectively

\[
u_2(\xi) = \frac{2a_0b_1(k^2 + l^2 + p^2)}{24b_1^2e^4k^2 + 24b_1^2e^2l^2 + 24b_1^2e^4p^2 + k\alpha a_0^2 e^{-\xi}},
\]

Where \(\xi = kx + ly + pz - (k^2 + l^2 + p^2)t\).

\[
u_3(\xi) = \frac{a_{-1}(b_{-1}e^{-\xi} - b_1e^\xi)}{b_{-1}(b_1e^\xi + b_{-1}e^{-\xi})},
\]

Where \(\xi = -(\frac{3cb_1}{\alpha a_0^2})x \pm \sqrt{\frac{2c\alpha_1^2\alpha^2 - 36c^2b_1^2 - 4p^2a_1^2\alpha^2}{2\alpha a_0^2}}y + pz + ct\).

\[
u_4(\xi) = \frac{a_{-1}(4b_2e^{-\xi} - e^\xi b_0^2)}{b_{-1}(e^\xi b_0^2 + 4b_0b_{-1} + 4b_2e^{-\xi})},
\]

Where \(\xi = -(\frac{3cb_1}{\alpha a_0^2})x \pm \sqrt{\frac{2c\alpha_1^2\alpha^2 - p^2a_1^2\alpha^2 - 9e^2b_1^2}{\alpha a_0^2}}y + pz + ct\).

3 MODIFIED F-EXPANSION METHOD

We apply the modified F-expansion method to construct the traveling wave solutions to the eq. (1). First, we obtain the following coefficients
Case 1. When $A = 0$, we have

\[
\begin{align*}
    a_0 &= \frac{a_1 B}{2C}, \quad a_{-1} = 0, \quad c = -\frac{a_1^2 B^2 \alpha k}{12 C^2}, \quad a_1 = a_1, \\
    k &= k, \quad l = \pm \frac{i \sqrt{6 k a a_1^2 + 36 k^2 C^2 + 36 C^2 p^2}}{6 C}, \quad p = p.
\end{align*}
\]

Case 2. When $B = 0$, we have

\[
\begin{align*}
    a_0 &= 0, \quad a_{-1} = -\frac{A a_1}{C}, \quad c = \frac{4 k a a_1^2 A}{3 C}, \quad a_1 = a_1, \\
    k &= k, \quad l = \pm \frac{i \sqrt{6 k a a_1^2 + 36 k^2 C^2 + 36 C^2 p^2}}{6 C}, \quad p = p.
\end{align*}
\]

and

\[
\begin{align*}
    a_0 &= a_{-1} = 0, \quad p = p, \quad c = \frac{k a a_1^2 A}{3 C}, \quad a_1 = a_1, \\
    k &= k, \quad l = \pm i \sqrt{6 k a a_1^2 + 36 k^2 C^2 + 36 C^2 p^2}.
\end{align*}
\]

Where $i = \sqrt{-1}$. Substituting over coefficients into

\[
    u(\xi) = a_{-1} F^{-1}(\xi) + a_0 + a_1 F(\xi),
\]

Then by using Appendix, we obtain several exact solutions of eq. (1) as

\[
\begin{align*}
    u_1(x, y, z, t) &= \frac{1}{2} a_1 \tanh(\frac{1}{2} \xi), \\
    u_2(x, y, z, t) &= \frac{2 \cosh(\xi) a_1}{\sinh(\xi)}, \\
    u_3(x, y, z, t) &= \frac{a_1 (1 + \tanh(\xi))^2}{\tanh(\xi)}.
\end{align*}
\]

Where $\xi = k x \pm (\frac{i \sqrt{6 k a a_1^2 + 36 k^2 C^2 + 36 C^2 p^2}}{6 C}) y - (\frac{a_1^2 B^2 \alpha k}{12 C^2}) t + p z$ and $i = \sqrt{-1}$.

4 Main Result

In this paper, by using the exp-function method and the modified F-expansion method, we have been able to obtain lots of new exact solutions of the (3+1)-dimensional MKdV-ZK equation in a unified way with the aid of Maple.
Appendix

Relations between values of \((A, B, C)\) and corresponding \(F(\xi)\) in Riccati equation

\[
F'(\xi) = A + BF(\xi) + CF^2(\xi), \quad (C \neq 0)
\]

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>(\frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\xi))</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>(\frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}\xi))</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>-1</td>
<td>(\coth(\xi) \pm \csch(\xi), \tanh(\xi) \pm \text{sech}(\xi))</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>1</td>
<td>(\tanh(\xi), \coth(\xi))</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>(\sec(\xi) + \tan(\xi), \csc(\xi) - \cot(\xi))</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>-1</td>
<td>(\csc(\xi) + \cot(\xi), \sec(\xi) - \tan(\xi))</td>
</tr>
<tr>
<td>1(-1)</td>
<td>0</td>
<td>1(-1)</td>
<td>(\tan(\xi) \left(\cot(\xi)\right))</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>(\neq 0)</td>
<td>(\frac{1}{\xi + \lambda})</td>
</tr>
<tr>
<td>arbitrary constant</td>
<td>0</td>
<td>0</td>
<td>(A\xi)</td>
</tr>
<tr>
<td>arbitrary constant</td>
<td>(\neq 0)</td>
<td>0</td>
<td>(\frac{\exp(\lambda\xi) - A}{B})</td>
</tr>
</tbody>
</table>

References


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A generalized boundary element method for problems with infinite domains

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Abstract
In this study, we consider a boundary element method (BEM) to solve a class of PDEs with infinite domains. The proposed method is based on converting the problem to a PDE with a finite domain using a suitable transformation. Consequently, any technique used for interior problems can be employed to deal with the new problem obtained.

Keywords: Boundary element method; Interior problems; Infinite domain; Transformation.

Mathematics Subject Classification: 65N38

1 Introduction

The BEM is a powerful tool to deal with a class of PDEs. The main attractive characteristic of this method is that the domain discretization is avoided and only the boundary is discretized. This advantage reduces the dimensionality of the problem by one. The BEM has been successfully applied to many problems with finite domains [1]. Being a boundary type, makes the method more attractive for PDEs with infinite domains, namely exterior problems (see [2]). However, a few number of works regarding this sort of problems has been reported. The BEM is in disadvantage when dealing with non-homogeneous problems, since some domain integrals appear in the integral equation. There are many methods to convert the domain integral to a boundary integral. One of the most efficient methods is dual reciprocity method (DRM) which is based on approximating a particular solution [3].

In this work, we describe a technique based on transformation to convert an exterior problem to one with finite domain. Therefore, the new problem can be treated by the classical approaches presented for interior problems.

2 Dual reciprocity method

We first describe the formulation of the DRM for Poisson’s equation

\[ \nabla^2 u = b(x, y) \] (1)
We use the following transformation in the polar coordinate

\[ r' = \frac{1}{r} , \quad \theta' = \theta. \]  

in the infinite domain \( \Omega \) bounded by \( \Gamma \) (Fig.1) where \( b \) is a known function.

Let \( \Gamma_\infty \) be a fictitious boundary which is a circle of radius \( R \) centred at the source point \( \xi \), on which the fundamental solution of the Laplace equation acts. Now the DRM can be applied to equation (13) in the domain \( \Omega_1 \) bounded by the real boundary \( \Gamma \) and the fictitious boundary \( \Gamma_\infty \), that is, \( \int_{\Omega_1} u^* \nabla^2 u d\Omega_1 = \int_{\Omega_2} u^* b d\Omega_1 \).

The main point in the DRM is, first, to find an approximation to the particular solution. In order to do so, we approximate the function \( b \) by some interpolation functions \( \phi_j \) as \( b(x, y) = \sum_{j=1}^{N+L} \alpha_j \phi_j \), where \( N \) and \( L \) are the number of the boundary and internal points respectively. Also assume that the functions \( \hat{u}_j \) can be found such that \( \nabla^2 \hat{u}_j = \phi_j \). Then, it follows \( \int_{\Omega_1} (\nabla^2 u) u^* d\Omega_1 = \sum_{j=1}^{N+L} \alpha_j \int_{\Omega_1} (\nabla^2 \hat{u}_j) u^* d\Omega_1 \). Using the Green’s theorem gives

\[
d\xi u_\xi + \int_{\Gamma} u q^* d\Gamma - \int_{\Gamma} u^* q d\Gamma + \int_{\Gamma_\infty} u q^* d\Gamma_\infty - \int_{\Gamma_\infty} u^* q d\Gamma_\infty =
\sum_{j=1}^{N+L} \left\{ \alpha_j \left( d\xi \hat{u}_j \xi + \int_{\Gamma} \hat{u}_j q^* d\Gamma - \int_{\Gamma} u \hat{q}_j d\Gamma + \int_{\Gamma_\infty} \hat{u}_j q^* d\Gamma_\infty - \int_{\Gamma_\infty} u \hat{q}_j d\Gamma_\infty \right) \right\},
\]

which is a typical integral equation for interior problems. The above integral equation can be numerically solved by discretizing the boundary.

As observed, the application of DRM to exterior problems is restricted by the fact that special basis functions can be used to satisfy the regularity condition. To avoid the above difficulty, we present a method to convert equation (13) to a problem with a finite domain.

3 Transformation approach

We use the following transformation in the polar coordinate

\[ r' = \frac{1}{r} , \quad \theta' = \theta. \]
Suppose the internal boundary $\Gamma$ is expressed by $r = H(\theta)$, then using transformation (1.3) the corresponding boundary is given by $r' = \frac{1}{H(\theta)}$. The artificial boundary is taken as a circle with a sufficiently large radius (Fig. 2). The transformation (1.3) maps the boundary $\Gamma_{\infty}$ into the boundary $\Gamma_{\epsilon}$ with radius $\epsilon = \frac{1}{R}$. Also $\Omega_1$ is transformed into $\Omega''$ which includes all the interior points of $\Gamma'$. Since the Laplace operator in the polar system is given by \[ \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) \]
and $r = \frac{1}{r'}$, the Laplace operator in terms of the variable $r'$ is given by (see [2]) \[ \nabla^2 u = r'^4 \nabla''^2 u' \]
where $\nabla''^2$ is the Laplace operator in the new coordinate $(r', \theta')$. Therefore equation (13) in terms of the new variables $r'$ and $\theta'$ takes the form
\[ r'^4 \nabla''^2 u' = b'(r', \theta') , \]
where $b'$ and $u'$ are new functions obtained by the transformation as follows
\[ b'(r', \theta') = b(r(r', \theta'), \theta(r', \theta')) , \quad u'(r', \theta') = u(r(r', \theta'), \theta(r', \theta')) . \]

For the sake of simplicity, we denote the $b'$, $u'$ and $q'$ by $b$, $u$ and $q$. Using the Laplace operator $\nabla''^2$, it can be shown that (see [2]) \[ \nabla''^2 (r'^4 u) = 16r'^2 u + 8r'^3 \frac{\partial u}{\partial r'} + r'^4 \frac{\partial^2 u}{\partial r'^2} . \]
Consequently, equation (1.3) can be expressed as
\[ \nabla''^2 (r'^4 u) = b(r', \theta') + 16r'^2 u + 8r'^3 \frac{\partial u}{\partial r'} , \]
where $\frac{\partial u}{\partial r'}$ can be evaluated by
\[ \frac{\partial u}{\partial r'} = \frac{\partial u}{\partial x'} \times \frac{\partial x'}{\partial r'} + \frac{\partial u}{\partial y'} \times \frac{\partial y'}{\partial r'} , \]
and $x' = r' \cos \theta'$, $y' = r' \sin \theta'$ and $r'^2 = x'^2 + y'^2$.

Substituting (1.3) into (1.3) yields
\[ \nabla''^2 (r'^4 u) = b(r', \theta') + 16r'^2 u + 8r'^3 x' \frac{\partial u}{\partial x'} + 8r'^3 y' \frac{\partial u}{\partial y'} . \]
This equation can be thought as an interior problem and considered by the DRM.
4 Numerical results

Table 1: Error values for the Example

<table>
<thead>
<tr>
<th>$N + L$</th>
<th>56</th>
<th>70</th>
<th>112</th>
<th>140</th>
<th>182</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>$7.34 \times 10^{-2}$</td>
<td>$4.08 \times 10^{-2}$</td>
<td>$2.13 \times 10^{-2}$</td>
<td>$1.94 \times 10^{-2}$</td>
<td>$1.01 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

We consider equation $\nabla^2 u = \frac{1}{x^2+y^2} u$, in an infinite domain outside a circle centred in the origin with radius one.

The function

$$f(r) = \frac{2c - r}{(r+c)^4},$$

is employed as a radial basis function. The exact solution is given by $u = \frac{1}{\sqrt{x^2+y^2}}$. The numerical errors are given in Table 1 for different number of points.

References


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Right-looking version of robust incomplete factorization preconditioner with pivoting

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Abstract
In this paper, we use a complete pivoting strategy for the right-looking version of Robust Incomplete Factorization preconditioner and study effect of this pivoting.

Keywords: ILU factorization, right-looking version of RIF preconditioner, Krylov subspace methods.

Mathematics Subject Classification[2010]: 65F10, 65F50, 65F08.

1 Introduction
Consider the linear system of equations of the form
\[ Ax = b, \]  
where the coefficient matrix \( A \in \mathbb{R}^{n \times n} \) is nonsingular, large, sparse and nonsymmetric and also \( x, b \in \mathbb{R}^{n} \). Krylov subspace methods can be used to solve this system [5]. An implicit preconditioner \( M \) for system (1) is an approximation of matrix \( A \), i.e., \( M \approx A \). If \( M \) is a good approximation of \( A \), then it can be used as the right preconditioner for system (1). In this case, instead of solving system (1) it is better to solve the right preconditioned system \( AM^{-1}u = b \) where \( M^{-1}u = x \) by the Krylov subspace methods.

Suppose that there is the factorization \( A = LDU \) for matrix \( A \); where \( L \) and \( U \) are unit lower triangular matrices and \( D \) is a diagonal matrix. Also suppose that dropping be applied on \( L, U \) and \( D \). Then, matrix \( M \) which is \( A \approx M = LDU \) is an implicit preconditioner for system (1). This preconditioner is also termed as an ILU preconditioner.

In this paper, we present a complete pivoting strategy for right-looking version of RIF preconditioner. To test effectiveness of such a pivoting, we have generated several linear systems. The coefficient matrices are taken from University of Florida sparse matrix collection [4]. Then we have computed the right-looking version of RIF with pivoting for such systems and have solved the right preconditioned linear systems by the GMRES(30) Krylov subspace method [5].
2 Right-looking version of RIF preconditioner with complete pivoting

An explicit preconditioner $M$ for system (1) is an approximation of matrix $A^{-1}$, i.e., $M \approx A^{-1}$. The most well-known explicit preconditioner is the AINV preconditioner [1]. This preconditioner has three factors in the form $A^{-1} \approx M = ZD^{-1}W^T$ where $Z$ and $W$ are unit upper triangular matrices and $D$ is a diagonal matrix. There are two left and right-looking versions for this preconditioner. In [3], Bollhoefer and Saad could present a complete pivoting strategy for the right-looking version of AINV preconditioner.

In 2003, Benzi and Tůma computed an incomplete factorization of a symmetric positive definite matrix $A$ in the form of $A \approx LDL^T$ as by-product of the AINV preconditioner. This preconditioner is termed Robust Incomplete Factorization or RIF [2]. There are also two left and right-looking versions for this preconditioner. In this paper, we focus on the nonsymmetric version of this preconditioner. The $L$ and $U$ factors of this preconditioner are computed independently. Algorithm 1, computes the right-looking version of this preconditioner and uses the complete pivoting strategy. At step $i$ of Algorithm 1, $i$-th column of matrix $L$ and $i$-th row of matrix $U$ are computed, independently. More precisely, matrix $L$ is computed column-wise and matrix $U$ is computed row-wise. At the end of this algorithm, factorization $IIAΣ \approx LDU$ is computed where $I$ and $Σ$ are the row and column permutation matrices, respectively.

3 Numerical Results

In this section, we report results of GMRES(30) method to solve the original and the right preconditioned linear systems. Preconditioners are right-looking version of RIF with and without pivoting. We have used the notation RLRIF to indicate the right-looking version of RIF preconditioner in Tables 2 and 3. We have also considered the notation RLRIFP(1.0) in these tables to indicate the right-looking version of RIF with pivoting that uses parameter $α = 1.0$ for column and row pivoting. We have generated 8 artificial linear systems $Ax = b$ with the nonsymmetric coefficient matrices from the University of Florida Sparse Matrix Collection [4]. Vector $b$ is $Ae$ in which $e = [1, ..., 1]^T$. We have written code of right-looking version of RIF with pivoting in Fortran 90. We have selected $τ_1, τ_9, τ_{10}$ and $τ_{20}$ equal to 0.1.

In Table 1, $n$ is the dimension and $anzt$ is the number of nonzero entries of the matrix. $it$ is the number of iterations of GMRES(30) method with no preconditioning and $It ime$ is its iteration time. This parameter is in seconds. $A +$ means that there is no convergence in 5000 number of iterations. In Tables 1 and 3, convergence criterion is satisfied when the relative residual is less than $10^{-10}$.

In Table 2, $Ptime$ is the preconditioning time and $density = \frac{anzt(L) + anzt(U)}{n^2}$. $Ptime$ is in seconds. For all matrices, the density of RLRIFP(1.0) and RLRIF preconditioners are so close to each other. This gives a proper atmosphere to compare number of iterations of GMRES(30), when these two preconditioners are used as the right preconditioner. For matrix orsarr_2, $Ptime$ of RLRIFP(1.0) preconditioner is smaller than $Ptime$ of RLRIF preconditioner. But for all other matrices, $Ptime$ of RLRIFP(1.0) preconditioner is greater than or equal to $Ptime$ of RLRIF preconditioner.

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Algorithm 1 (Right-looking version of RIF with complete pivoting)

Input: $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and $r_\alpha, r_\sigma \in (0, 1)$ be drop tolerances and prescribe a tolerance $\alpha \in (0, 1]$.
Output: $\text{PA}\Sigma = LDU$

1. $P = \Sigma = I_n$.
2. $s^{(0)}_j = a_{jj}$, $s^{(0)}_i = a_{ii}$, $1 \leq i \leq n$.
3. for $i = 1$ to $n$
4. $m_i = n_i = 0$
5. satisfied, $p = false$, satisfied, $q = false$.
6. while not satisfied, $p$ do
7. for $j = i$ to $n$
8. $p_j^{(i-1)} = (s_j^{(i-1)})^T (\Sigma A \Sigma^*) s_i^{(i-1)}$.
9. end for
10. if $|p_j^{(i-1)}| < \alpha \max_{m_n \geq j} |p_m^{(i-1)}|$ then
11. $m_i = m_i + 1$, $s_i^{(i-1)} = I_n$
12. satisfied, $q = false$, choose $k$ such that $|p_k^{(i-1)}| = \max_{m_n \geq k} |p_m^{(i-1)}|$.
13. Interchange columns $i$ and $k$ of $W - I$ and rows $i$ and $k$ of $Z - I$ and $\sigma^{(i-1)}$ and elements $p_i^{(i-1)}$ and $p_k^{(i-1)}$.
14. $\Sigma = \sigma^{(i-1)}$.
15. end if
16. satisfied, $p = true$.
17. for $j = i$ to $n$
18. $q_j^{(i-1)} = (s_j^{(i-1)})^T (\Sigma A) s_i^{(i-1)}$.
19. end for
20. if not satisfied, $q$ then
21. if $|q_j^{(i-1)}| < \alpha \max_{m_n \geq j} |q_m^{(i-1)}|$ then
22. $m_i = m_i + 1$, $\sigma^{(i-1)} = I_n$
23. satisfied, $q = false$, choose $l$ such that $|q_l^{(i-1)}| = \max_{m_n \geq l} |q_m^{(i-1)}|$.
24. Interchange columns $i$ and $l$ of $Z - I$, $U - I$ and $\sigma^{(i-1)}$ and elements $q_i^{(i-1)}$ and $q_l^{(i-1)}$.
25. $\Sigma = \Sigma_{n_i}^{(i-1)}$.
26. end if
27. end if
28. $s_{ii} = s_i^{(i-1)}$.
31. for $j = i + 1$ to $n$
32. $p_j^{(i-1)} = \frac{p_j^{(i-1)}}{s_i^{(i-1)}}$, $q_j^{(i-1)} = \frac{q_j^{(i-1)}}{s_i^{(i-1)}}$.
33. $L_{jj} = p_j^{(i-1)}$, $U_{jj} = q_j^{(i-1)}$.
34. apply dropping rule to $L_{jj}$ and to $U_{jj}$ if their absolute values are less than $r_\sigma$ and $r_\alpha$.
35. $u_{ji}^{(i)} = (s_i^{(i-1)} - p_j^{(i-1)} a_{ji})$, $q_j^{(i)} = q_j^{(i-1)} - q_j^{(i-1)} a_{ji}$.
36. for all $t \leq i$ apply a dropping rule to $w_{ti}^{(i)}$ and to $s_i^{(i)}$, if their absolute values are less than $r_\alpha$ and $r_\sigma$.
37. end for
38. end for
39. Return $L = (L_{ij})_{1 \leq i,j \leq n}$, $U = (U_{ij})_{1 \leq i,j \leq n}$, $D = \text{diag}(d_{ii})_{1 \leq i \leq n}$, $P$ and $\Sigma$.

In Table 3, results of $\text{GMRES}(30)$ method are presented. In this table, it is again the number of iterations of this Krylov subspace method and $Ttime$ is its total time which is defined as the iteration time plus the preconditioning time. Comparing columns 2 and 4 of this table, indicates that except for matrix $\text{raefsky6}$, the number of iterations of $\text{GMRES}(30)$, when one uses the $\text{RLRIFP}(1.0)$ preconditioner, is less than or equal to the number of iterations of this Krylov subspace method when one applies the $\text{RLRIF}$ preconditioner. Columns 3 and 5 of this table, indicate that for 6 matrices $Ttime$ of $\text{RLRIFP}(1.0)$ preconditioner is less than $Ttime$ of $\text{RLRIF}$ preconditioner and for 2 other matrices this is vice versa. Comparing column 2 of Table 3 with column 4 of Table 1, shows that $\text{RLRIFP}(1.0)$ preconditioner is useful to decrease the number of iterations of $\text{GMRES}(30)$ method.
4 Conclusion

In this paper, we presented right-looking version of RIF preconditioner with pivoting. We used parameter $\alpha = 1.0$ to compute this preconditioner. We also used this preconditioner as the right preconditioner to solve several linear systems. Numerical experiments indicate the fact that, although $Ptime$ of right-looking version of RIF preconditioner with pivoting for most of the tests is greater than $Ptime$ of right-looking version of RIF preconditioner, but the right-looking version of RIF with pivoting is more effective than right-looking version of RIF to decrease the number of iterations of GMRES(30) method.

<p>| Table 1: matrix properties and results of GMRES(30) with no preconditioning. |</p>
<table>
<thead>
<tr>
<th>Matrix</th>
<th>$n$</th>
<th>$mc$</th>
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<th>Time</th>
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<p>| Table 2: properties of RLRIFP(1,0) and RLRIF preconditioners. |</p>
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<p>| Table 3: results of GMRES(30) using RLRIFP(1,0) and RLRIF preconditioners. |</p>
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References


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Dynamical Systems
Dynamical behaviour of a Lotka-Volterra system

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Abstract
The dynamic behaviour of a Lotka-Volterra system, described by a planar map, is analytically and numerically investigated. We derive analytical conditions for stability and bifurcation of the fixed points of the system and compute analytically the normal form coefficients for the codimension 1 bifurcation points. Numerical simulations confirm our results and reveal further complex dynamical behaviours.

Keywords: Lotka-Volterra system, Planar map, Numerical simulation, Bifurcation point.

1 Introduction
The dynamic relationship between predators and their prey is one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. The prototype Lotka-Volterra predator-prey system has received a lot of attention from theoretical and mathematical biologists; see [2-3] This model is described by the following system of ordinary differential equations:

\[
\begin{align*}
x(t) &= x(r_1 - a_{11}x) - a_{12}xy, \\
y(t) &= y(-r_2 - a_{22}y) + a_{21}xy,
\end{align*}
\]

where \(x(t)\) and \(y(t)\) represent the densities of the prey and the predator, \(r_1, a_{12}, r_2\) and \(a_{21}\) are the intrinsic growth rate of the prey, the capture rate, the death rate of the predator, and the conversion rate, respectively, \(a_{11}\) and \(a_{22}\) denote the intraspecific competition coefficients of the prey and the predator, and \(r_1/a_{11}\) is the carrying capacity of the prey. We study the discrete version of (1), i.e:

\[
F : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} ax(1-x) - bxy \\ dxy \end{pmatrix}
\]

which is analogous to (1) for the case of predators and prey with nonoverlapping generations. We naturally start the bifurcation analysis of (2) with the calculation of the fixed points. These are the solutions \((x^*, y^*)\) to

\[
a x^*(1 - x^*) - b x^* y^* = x^*, \quad d x^* y^* = y^*.
\]

The origin \(E_1 = (0, 0)\) is a fixed point of (2) but is of little interest. Two further nontrivial fixed points are \(E_2 = ((a - 1)/a, 0)\) and \(E_3 = (1/d, 1/b(a(1 - (1/d)) - 1))\). We note that \(E_2\) is biologically possible only if its coordinates are nonnegative, that is, \(a \geq 1\). \(E_3\) is biologically possible only if the following conditions hold:
We start the local bifurcation analysis of the map \( (2) \) by linearization of \( F \) around each of its fixed points. The Jacobian matrix \( J(x, y) \) is given by
\[
J(x, y) = \begin{pmatrix} a - 2ax - by & -bx \\ dy & dx \end{pmatrix}.
\]

The characteristic equation of \( J(x, y) \) is given by
\[
\lambda^2 - tr(J) + det(J) = 0,
\]
where \( tr(J) = a - 2ax - by + dx \) and \( det(J) = dxa - 2dx^2a \).

Following, we study bifurcation of \( E_1, E_2 \) and \( E_3 \).

**Proposition 1.1.** The fixed point \( E_1 \) is asymptotically stable for \( 0 \leq a < 1 \). It loses stability via branching for \( a = 1 \) and there bifurcates to \( E_2 \).

**Proof.** see [6]

The Jacobian matrix of \( (2) \) at \( E_2 \) is given by
\[
J(E_2) = \begin{pmatrix} -a + 2 & -b(a - 1)/a \\ 0 & d(a - 1)/2 \end{pmatrix}.
\]

**Proposition 1.2.** The fixed point \( E_2 \) is linearly asymptotically stable if and only if \( a \in ]1, 3[ \) and \( d < a/(a - 1) \). Moreover, it loses stability:

(i) via branching for \( a = 1 \) and there bifurcates to \( E_1 \),

(ii) via branching for \( d = a/(a - 1) \) and there bifurcates to \( E_3 \) if \( 1 < a < 3 \),

(iii) via a supercritical flip for \( a = 3 \) if \( d < 3/2 \).

**Proof.** The proof becomes clear, by using The Jure’s stability condition.

**Proposition 1.3.**\( E_3 \) is linearly asymptotically stable if and only if one of the following mutually exclusive conditions holds:

(i) \( 3.2 < d < 9.4 \) and \( d/(d - 1) < a < 3d/(3 - d) \),

(ii) \( d = 9.4 \) and \( d/(d - 1) < a < d/(d - 2) = 3d/(3 - d) = 9 \),

(iii) \( d > 9.14 \) and \( d/(d - 1) < a < d/(d - 2) \).

**Proof.** see [6]

**Numerical Simulation**

To reveal the qualitative dynamical behaviours of \( (3) \) near the NS, we present a complete bifurcation sequence that is observed for different values of \( d \). We fix the parameters \( a = 3.5, b = 0.2 \) and assume that \( d \) is free.
Figure 1: The breakdown of the closed invariant curve of the system (2) for $a = 3.5, b = 0.2$, and $d = 3.13$.

As $d$ is increased further, however, the phase portrait starts to fold. We see that the circle, after being stretched, shrunk, and folded, creates new phenomena due to the breakdown of the closed curve; see Figure 1 for $d = 3.13$ For increasing $d$, we obtain the multiple invariant closed curves brought about the NS bifurcation point of iterates of (3). In these cases, higher bifurcations of the torus occurs as the system moves out of quasiperiodic region by increasing $d$. The dynamics move from one closed curve to another periodically, but the dynamics in each closed curve, may be periodic or quasiperiodic.

Figure 2: Chaotic attractor for the system (2) for $a = 3.5, b = 0.2$, and $d = 3.25$.

Figure 3: Chaotic attractor for the system 2 for $a = 3.5, b = 0.2$ and $d = 3.8$.

Moreover, the closed curves may break and lead to multiple fractal tori on which the dynamics of (3) are chaotic. Figure 2,3 present strange attractors for (3) with $d = 3.25$ and $d = 3.8$ respectively, which exhibit a fractal structure.

2 Main Result

In this paper, we studied a planar map that models a predator-prey interaction with nonoverlapping generations. We performed a numerical bifurcation analysis by using the MATLAB package Cl MatContM; see [4,5]. The bifurcation analysis is based on continuation methods, whereby we trace solution manifolds of fixed points while some of the parameters of themap vary; see [1].

We derived analytically a complete description of the stability regions of the fixed points of the
system, namely, $E_1$, $E_2$ and $E_3$. We showed that the system undergoes branching, period doubling, and Neimark-Sacker bifurcations. We further used numerical simulation methods to reveal chaotic behaviours and a strange attractor near the Neimark-Sacker bifurcation.

References


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A simple proof of the singularity-induced bifurcation theorem

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Abstract

The dynamics of a large class of physical systems can be represented by differential-algebraic equation (DAE) of the form $x' = f$ and $g = 0$. It has been shown that new type of bifurcation, called singularity induced bifurcation occurs when equilibrium point approaches the singular surface defined by $g = 0$ and $\det(D_{x}g) = 0$. It has been observed that some eigenvalues of the (DAE) jacobian at the equilibrium point tends to infinity. In this article a simple proof of the singularity induced bifurcation theorem and stronger conclusion is given.

Keywords: Differential-algebraic equation (DAE), Singularity-induced bifurcation (SIB)

1. INTRODUCTION

The dynamic of a large class of physical systems can be modeled by a parameter dependent differential-algebraic equation (DAE) of the form:

\[
\begin{align*}
\left\{ \\
x' &= f(x, y, \lambda) \\
0 &= g(x, y, \lambda)
\end{align*}
\]

\[
f : \mathbb{R}^{n+m+q} \to \mathbb{R}^{n} \\
g : \mathbb{R}^{n+m+q} \to \mathbb{R}^{m}
\]

(1)

For the (DAE), define the set of all equilibria to be $EQ$ and let $S$ denote the singular surface defined as

$EQ = \{(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q : f(x, y, \lambda) = 0, \quad g(x, y, \lambda) = 0\}$

$S = \{(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q : g(x, y, \lambda) = 0, \quad \Delta(x, y, \lambda) = 0\}$

Where $\Delta(x, y, \lambda) := \det[D_{x}g(x, y, \lambda)]$.

When the equilibrium is at the singularity, i.e, $EQ \cap S \neq \phi$ the jacobian of the (DAE)

$J := D_{x}f - D_{x}g(D_{x}g)^{-1}D_{x}g$

is not well defined because $D_{x}g$ is singular. It is obvious that the jacobian $J$ might have some eigenvalues tends to infinity near the bifurcation point, because $J$ involves $[D_{y}g]^{-1}$ which is
singular matrix.

The SIB theorem was first proved by Venkatasubramanian et al. in [2] and [3]. The proof is quite complicated and lengthy. After that an improved version of the singularity-induced bifurcation theorem was proved by Yang Lijun and Tang Yun which was based on a decomposition theorem of parameter dependent polynomials [1].

The purpose of this paper provides a simple and short proof of the singularity induced bifurcation theorem and analyze the treat of the eigenvalue which is unbounded at the bifurcation point.

2. THE SIB THEOREM AND ITS SIMPLE PROOF

Now we present a simple proof of the singularity induced bifurcation theorem.

Theorem 2.1. (Singularity Induced Bifurcation Theorem): Consider (1) with one-dimensional parameter space. Suppose the following conditions are satisfied at $(x_0, y_0, \lambda_0)$:
1. $f(x_0, y_0, \lambda_0) = 0, g(x_0, y_0, \lambda_0) = 0$.
2. $D_xg$ is singular.
3. trace $(D_yf + D_yg D_xg D_x)$ is nonzero.
4. $A := \begin{pmatrix} D_xf & D_yf \\ D_xg & D_yg \end{pmatrix}$
is nonsingular.
5. $B := \begin{pmatrix} D_xf & D_yf & D_zf \\ D_xg & D_yg & D_zg \\ D_x\Delta & D_y\Delta & D_z\Delta \end{pmatrix}$
is nonsingular.
Then there exists a smooth surface of equilibria in $R^{n+m+1}$ which passes through $(x_0, y_0, \lambda_0)$ and unique eigenvalue $\mu_n(\lambda)$ of Jacobian $J$ such that $\lim_{\lambda \to \lambda_0} \frac{1}{\mu_n(\lambda)} = 0$ and if we define $\frac{1}{\mu_n(\lambda)} |_{\lambda=\lambda_0} = 0$, then $\frac{1}{\mu_n(\lambda)}$ is smooth at $(x_0, y_0, \lambda_0)$ and $\frac{\partial}{\partial \lambda} \left( \frac{1}{\mu_n(\lambda)} \right) |_{\lambda=\lambda_0} = -\frac{b}{c}$ where

\[ b := -\text{trace}[D_yf + D_yg D_xg D_xg] \]
\[ c := D_\lambda \Delta - (D_z\Delta D_y\Delta) \left( \begin{pmatrix} D_xf & D_yf \\ D_xg & D_yg \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} D_xf \\ D_yf \end{pmatrix} \right) \]
A simple proof of the singularity-induced bifurcation theorem

proof: The Jacobian in assumption 4), is nonsingular near \( \lambda = \lambda_0 \), hence by implicit function theorem, there exists a unique equilibrium locus \( EQ(\lambda) = [x(\lambda), y(\lambda), \lambda] \) which satisfies these equations

\[
 f(x(\lambda), y(\lambda), \lambda) = 0 \quad g(x(\lambda), y(\lambda), \lambda) = 0
\]

And

\[
 \left( \frac{dx}{d\lambda} \right) = - \left( \begin{array}{cc} D_x f & D_y f \\ D_x g & D_y g \end{array} \right)^{-1} \left( \begin{array}{c} D_x f \\ D_x g \end{array} \right)
\]

Now consider the derivative of \( \Delta[EQ(\lambda)] \)

\[
 \frac{d}{d\lambda} \Delta[x(\lambda), y(\lambda), \lambda] = D_x \Delta + \left( \begin{array}{cc} D_x \Delta & D_y \Delta \\ D_x \Delta & D_y \Delta \end{array} \right) \left( \frac{dx}{d\lambda}, \frac{dy}{d\lambda} \right)
\]

\[
 = D_x \Delta - \left( \begin{array}{cc} D_x \Delta & D_y \Delta \\ D_x \Delta & D_y \Delta \end{array} \right) \left( \begin{array}{cc} D_x f & D_y f \\ D_x g & D_y g \end{array} \right)^{-1} \left( \begin{array}{c} D_x f \\ D_x g \end{array} \right) = c
\]

Also consider these matrices:

\[
 C := \left( \begin{array}{ccc} I & 0 & 0 \\ 0 & I & 0 \\ D_x \Delta & D_y \Delta & I \end{array} \right) \quad D := \left( \begin{array}{ccc} D_x f & D_y f \\ D_x g & D_y g \end{array} \right) \left( \begin{array}{c} 0 \\ I \end{array} \right)
\]

Invertibility of matrices \( C \) and \( D \) at \((x_0, y_0, \lambda_0)\) and assumption 5), imply that \( CDB \) is also invertible at \((x_0, y_0, \lambda_0)\). Thus:

\[
 0 \neq \det(CDB) = \det \left( \begin{array}{ccc} I & 0 & 0 \\ 0 & D_x \Delta - \left( \begin{array}{cc} D_x \Delta & D_y \Delta \\ D_x \Delta & D_y \Delta \end{array} \right) \left( \begin{array}{cc} D_x f & D_y f \\ D_x g & D_y g \end{array} \right)^{-1} \left( \begin{array}{c} D_x f \\ D_x g \end{array} \right) \end{array} \right)
\]

\[
 = D_x \Delta - \left( \begin{array}{cc} D_x \Delta & D_y \Delta \\ D_x \Delta & D_y \Delta \end{array} \right) \left( \begin{array}{cc} D_x f & D_y f \\ D_x g & D_y g \end{array} \right)^{-1} \left( \begin{array}{c} D_x f \\ D_x g \end{array} \right) = c
\]

Therefore \( \frac{d}{d\lambda} \Delta[EQ(\lambda)]_{(x_0, y_0, \lambda_0)} = c \neq 0 \), so \( \Delta[EQ(\lambda)] = c(\lambda - \lambda_0) + O(\lambda - \lambda_0)^2 \). Now we show that the Jacobian \( J := D_x f - D_y f(D_y g)^{-1} D_x g \) must have an unbounded eigenvalue.

Consider:

\[
 \left( \begin{array}{cc} D_x f(\lambda) & D_y f(\lambda) \\ D_x g(\lambda) & D_y g(\lambda) \end{array} \right) \left( \begin{array}{cc} I_n & 0 \\ -D_y g(\lambda)^{-1}(\lambda)D_x g(\lambda) & I_m \end{array} \right) = \left( \begin{array}{cc} J(\lambda) & D_x f(\lambda) \\ 0 & D_y g(\lambda) \end{array} \right)
\]

In which:

\[
 \det \left( \begin{array}{cc} D_x f(\lambda) & D_y f(\lambda) \\ D_x g(\lambda) & D_y g(\lambda) \end{array} \right) = \det(D_y g(\lambda)) \det(J(\lambda)) = l \neq 0
\]
Again using assumption 4), and \( \Delta[EQ(\lambda)] = c(\lambda - \lambda_0) + O(\lambda - \lambda_0)^2 \) imply that the jacobian \( J \) must have some unbounded eigenvalues when \( \lambda \) tends to \( \lambda_0 \) or equilibrium locus passes through \((x_0, y_0, \lambda_0)\). Now it suffices to prove that the jacobian \( J \) has exactly one unbounded eigenvalue as \( \lambda \to \lambda_0 \). Consider the following polynomials

\[
P_1 := \det(kI - J^{-1}(\lambda)) = k^n + a_{n-1}(\lambda)k^{n-1} + ... + a_1(\lambda)k + a_0(\lambda)
\]

\[
P_2 := \det(kI - J(\lambda)) = k^n + \frac{a_1(\lambda)}{a_0(\lambda)}k^{n-1} + ... + \frac{a_{n-1}(\lambda)}{a_0(\lambda)}k + \frac{1}{a_0(\lambda)}
\]

when \( \lambda \to \lambda_0 \), \( J^{-1} \) has zero eigenvalue at \( \lambda_0 \) thus \( a_0(\lambda_0) = 0 \). We can conclude that \( J^{-1} \) has exactly one zero eigenvalue if \( a_1(\lambda_0) \neq 0 \).

\[
\begin{align*}
\text{trace}(J(\lambda)) &= (-1)^n \frac{a_1(\lambda)}{a_0(\lambda)} \quad \det(J(\lambda)) = (-1)^n \frac{1}{a_0(\lambda)} \\

a_1(\lambda) &= \frac{-\text{trace}(J(\lambda))}{\det(J(\lambda))} = \frac{-\text{trace}(J(\lambda))}{\det(D_u g(\lambda) J(\lambda))} = \frac{-\text{trace}(\det(D_u g(\lambda) J(\lambda)))}{l}.
\end{align*}
\]

Now for \( \lambda = \lambda_0 \) we have

\[a_1(\lambda_0) = -\frac{b}{l} \neq 0 \]

So the jacobian \( J \) has only one unbounded eigenvalue and \((n - 1)\) bounded eigenvalues at the bifurcation point.

Now can be deduced from what was shown that \( \lim_{\lambda \to \lambda_0} \frac{\text{trace}(J(\lambda))}{\mu_n(\lambda)} = 1 \) and it implies

\[
\lambda \to \lambda_0 \quad \text{trace}(J(\lambda)) \approx \mu_n(\lambda)
\]

Thus we have

\[
\begin{align*}
\frac{\partial}{\partial \lambda} \left( \frac{1}{\mu_n(\lambda)} \right) \bigg|_{\lambda=\lambda_0} &= \frac{\partial}{\partial \lambda} \left( \frac{1}{\text{trace}(J(\lambda))} \right) \bigg|_{\lambda=\lambda_0} = \frac{\partial}{\partial \lambda} \left( \frac{\Delta(\lambda)}{\text{trace}(J(\lambda))} \right) \bigg|_{\lambda=\lambda_0} \\
&= \frac{\text{trace}(\Delta(\lambda) J(\lambda)) (\frac{\partial}{\partial \lambda} \Delta(\lambda)) - \Delta(\lambda) \left( \frac{\partial}{\partial \lambda} \text{trace}(\Delta(\lambda) J(\lambda)) \right)}{\text{trace}(\Delta(\lambda) J(\lambda))^2} \bigg|_{\lambda=\lambda_0} \\
&= \frac{\text{trace}(\Delta(\lambda) J(\lambda)) (\frac{\partial}{\partial \lambda} \Delta(\lambda))}{\text{trace}(\Delta(\lambda) J(\lambda))^2} \bigg|_{\lambda=\lambda_0} = -\frac{c}{b}.
\end{align*}
\]
References


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Existence of positive solutions for a system of fractional boundary value problems

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Abstract
In this paper, we use the Leggett-Williams fixed point theorem, we prove the existence of at least three positive solutions of boundary value problems for system on an infinite interval.

Keywords: Boundary value problem, Fixed point theorem, Differential equations, infinite interval.

Mathematics Subject Classification: 34B40

1 Introduction
The purpose of this paper is to study the existence of at least three solutions for the following nonlinear fractional differential system boundary value problems

\[
\begin{align*}
D^\alpha_0 u(t) + a_1(t) f_1(t, u(t), v(t)) &= 0, \quad 0 < t < +\infty, \quad 2 < \alpha < 3, \\
D^\alpha_0 v(t) + a_2(t) f_2(t, u(t), v(t)) &= 0, \quad 0 < t < +\infty, \\
\end{align*}
\]

\[ u(0) = u'(0) = v(0) = v'(0) = 0, \quad D^{\alpha-1} u(+\infty) - \sum_{i=1}^{m-2} \beta_i u(\xi_i) = g_1(\int_0^{+\infty} u(s)d\phi_1(s), \int_0^{+\infty} v(s)d\phi_1(s)), \]

\[ D^{\alpha-1} v(+\infty) - \sum_{i=1}^{m-2} \gamma_i v(\eta_i) = g_2(\int_0^{+\infty} u(s)d\phi_2(s), \int_0^{+\infty} v(s)d\phi_2(s)), \]

where \( D^\alpha_0 \) is the standard Riemann-Liouville fractional derivative, \( 2 < \alpha < 3, \) \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty, \) \( 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < +\infty, \) \( \beta_1, \gamma_i \geq 0, i = 1, 2, \cdots, m-2 \) satisfies

\[ 0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} < \Gamma(\alpha), \quad 0 < \sum_{i=1}^{m-2} \gamma_i \eta_i^{\alpha-1} < \Gamma(\alpha). \]

Throughout this paper, we assume that the following conditions hold

(H1) \( f_1, f_2 \in C([0, +\infty) \times [0, +\infty) \times [0, +\infty), [0, +\infty]), f_1(t, 0, 0) \neq 0, f_2(t, 0, 0) \neq 0 \) on any subinterval of \( (0, +\infty) \) and \( f_1(t, (1 + t^{\alpha-1})u, (1 + t^{\alpha-1})v), f_2(t, (1 + t^{\alpha-1})u, (1 + t^{\alpha-1})v) \) is bounded when \( u, v \) is bounded on \( [0, +\infty); \)

(H2) \( a_1, a_2 \in C([0, +\infty), [0, +\infty)) \) are not identical zero on any closed subinterval of \( [0, +\infty), \\
0 < \int_0^{+\infty} a_i(t)ds < \infty, \quad \int_0^{+\infty} u(s)d\phi_i(t), \int_0^{+\infty} v(s)d\phi_i(t) \) denote the Riemann-Stieltjes integrals and \( \int_0^{+\infty} d\phi_i(t) < \infty, i = 1, 2. \)
Suppose that (H1) and (H2) hold. Let
\[ \frac{d}{dt} \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} = 1, \]
and
\[ \int_0^1 (t-s)^{\alpha-1} f(s) ds, \]
where \( \Gamma(\cdot) \) is the Euler gamma function.

Definition 1.1. ([1]) The Riemann-Liouville fractional integral operator of order \( \alpha > 0 \), of function \( f \in L^1(\mathbb{R}^+) \) is defined as
\[ \mathcal{I}_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \]
where \( \Gamma(\cdot) \) is the Euler gamma function.

Definition 1.2. ([1]). The Riemann-Liouville fractional derivative of order \( \alpha > 0 \), \( n-1 < \alpha < n \), \( n \in \mathbb{N} \) is defined as
\[ \mathcal{D}_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \]
where the function \( f(t) \) have absolutely continuous derivatives up to order \( n-1 \).

Lemma 1.3. Suppose that (H1) and (H2) hold. Let \( h \in C[0, +\infty), \Delta_1 = \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}, \Delta_2 = \sum_{i=1}^{m-2} \gamma_i \xi_i^{\alpha-1} \) and \( 0 < \int_0^1 h(s) ds < +\infty \), then fractional boundary value problem
\[
\begin{cases}
\mathcal{D}_0^\alpha u(t) + h(t) = 0, & 0 < t < +\infty, 2 < \alpha < 3, \\
\mathcal{D}_0^\alpha v(t) + h(t) = 0, & 0 < t < +\infty, 2 < \alpha < 3, \\
u(0) = u'(0) = 0, & D^{\alpha-1}u(+\infty) - \sum_{i=1}^{m-2} \beta_i u(\xi_i) = g_1 \left( \int_0^1 u(s) d\phi_1(s), \int_0^1 v(s) d\phi_1(s) \right), \\
v(0) = v'(0) = 0, & D^{\alpha-1}v(+\infty) - \sum_{i=1}^{m-2} \gamma_i v(\eta_i) = g_2 \left( \int_0^1 u(s) d\phi_2(s), \int_0^1 v(s) d\phi_2(s) \right)
\end{cases}
\]
has a unique solution \((u(t), v(t))\) that
\[
\begin{align*}
u(t) &= \int_0^1 G(t,s) h(s) ds + \frac{t^{\alpha-1}}{\Gamma(\alpha) - \Delta_1} g_1 \left( \int_0^1 u(s) d\phi_1(s), \int_0^1 v(s) d\phi_1(s) \right), \\
v(t) &= \int_0^1 H(t,s) h(s) ds + \frac{t^{\alpha-1}}{\Gamma(\alpha) - \Delta_2} g_2 \left( \int_0^1 u(s) d\phi_2(s), \int_0^1 v(s) d\phi_2(s) \right),
\end{align*}
\]
where
\[
G(t,s) = G_1(t,s) + G_{21}(t,s), \\
H(t,s) = G_1(t,s) + G_{22}(t,s),
\]
and
\[
G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t < +\infty, \\
\Gamma(\alpha) \frac{t^{\alpha-1}}{t^{\alpha-1}}, & 0 \leq t \leq s < +\infty,
\end{cases}
\]
and
\[
G_{21}(t,s) = \frac{\sum_{i=1}^{m-2} \beta_i t^{\alpha-1}}{\Gamma(\alpha) - \Delta_1} G_1(\xi_i, s), \\
G_{22}(t,s) = \frac{\sum_{i=1}^{m-2} \gamma_i t^{\alpha-1}}{\Gamma(\alpha) - \Delta_2} G_1(\eta_i, s).
\]
In this paper, we will use the following space $E$ to the study (13) which is denoted by

$$E = \left\{ (u, v) \in C([0, +\infty) \times [0, +\infty)) : \sup_{0 \leq t < +\infty} \frac{|u(t)| + |v(t)|}{1 + t^{\alpha-1}} < +\infty \right\}.$$ 

We know that $E$ is a Banach space, equipped with the norm

$$||(u, v)|| = ||u|| + ||v|| < \infty,$$

where,

$$||u|| = \sup_{0 \leq t < +\infty} \frac{|u(t)|}{1 + t^{\alpha-1}}, \quad ||v|| = \sup_{0 \leq t < +\infty} \frac{|v(t)|}{1 + t^{\alpha-1}}.$$

We define the cone $K \subset E$ by

$$K = \left\{ (u, v) \in E : u(t), v(t) \geq 0, \text{ on } [0, +\infty) \times [0, +\infty), \ \min_{\frac{1}{k} \leq t \leq k} \frac{u(t) + v(t)}{1 + t^{\alpha-1}} \geq \lambda(k)||u, v|| \right\},$$

where

$$\lambda(k) = \min_{\frac{1}{k} \leq t \leq k} \left\{ \frac{1}{4k^2(1 + k^{\alpha-1})}, \frac{1}{k^{2\alpha-2}(1 + k^{\alpha-1})} \right\}.$$

**Theorem 1.4.** (Leggett-Williams fixed point theorem). Let $A : \bar{P}_c \to \bar{P}_c$ be a completely continuous operator and let $\alpha$ be a nonnegative continuous concave functional on $K$ such that $\alpha(x) \leq ||x||$ for all $x \in \bar{P}_c$. Suppose there exist $0 < a < b < d < c$ such that

(A1) $x \in P(\alpha, b, d) \alpha(x) > b \neq \emptyset$, and $\alpha(Ax) > b$ for $x \in P(\alpha, b, d)$,

(A2) $\alpha(Ax) < a$ for $||x|| < a$, and

(A3) $\alpha(Ax) > b$ for $x \in P(\alpha, b, c)$ with $||x|| > d$.

Then $A$ has at least three fixed points $x_1$, $x_2$ and $x_3$ and such that $||x_1|| < a$, $b < \alpha(x_2)$ and $||x_3|| > a$, with $\alpha(x_3) < b$.

**2 Main Result**

We define the nonnegative continuous concave functional on $K$ by

$$\alpha(u, v) = \min_{\frac{1}{k} \leq t \leq k} \frac{(u(t) + v(t))}{1 + t^{\alpha-1}}.$$ 

It is obvious that, for each $(u, v) \in K, \alpha(u, v) \leq ||(u, v)||$.

Throughout this section, we assume that $p_i, q_i, i = 1, 2$, are four positive numbers satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{q_1} + \frac{1}{q_2} \leq 1,$$

and

$$L = \max\left( L_1 = \frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m-2} \beta_i \alpha^{\alpha-2} \Delta_1, L_2 = \frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m-2} \gamma_i \alpha^{\alpha-2} \Delta_2 \right).$$

Also, we give the following assumptions:

$$M_1 = L \int_0^{+\infty} a_1(s)ds, \quad M_2 = L \int_0^{+\infty} a_2(s)ds,$$

$$m_1 = \frac{\lambda(k)}{k^{\alpha-1}} \int_{\frac{1}{k}}^{k} a_1(s)ds, \quad m_2 = \frac{\lambda(k)}{k^{\alpha-1}} \int_{\frac{1}{k}}^{k} a_2(s)ds,$$

$$N_1 = \frac{1}{\int_0^{+\infty} d\phi_1(t)}(\Gamma(\alpha) - \Delta_1), \quad N_2 = \frac{1}{\int_0^{+\infty} d\phi_2(t)}(\Gamma(\alpha) - \Delta_2).$$

Clearly, we see that $0 < m_i < M_i < \infty; \text{ for } i = 1, 2$. 

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Theorem 2.1. Assume there exist nonnegative numbers $a, b, c$ such that $0 < a < b \leq \min\{\lambda, \frac{m_1}{p_1M_1}, \frac{m_2}{p_2M_2}\}c$, and $f_1(t, u, v), f_2(t, u, v)$ satisfy the following conditions:

(H3) $f_i(t, u, v) < \frac{1}{c_i} \cdot \frac{c_i}{M_i}$, $\forall t \in [0, +\infty), u + v \in [0, c], i = 1, 2,$

(H4) $f_i(t, u, v) < \frac{1}{c_i} \cdot \frac{c_i}{M_i}$, $\forall t \in [0, +\infty), u + v \in [0, a], i = 1, 2,$

(H5) $f_1(t, u, v) > \frac{b}{c_1}$ or $f_2(t, u, v) > \frac{b}{c_2}$, $\forall t \in \left[\frac{1}{k}, k\right], u + v \in [b, \frac{b}{\lambda(k)}].$

(H6) $g_i(x, y) \leq \frac{1}{q_i}N_i(x + y), \forall x + y \in [0, c], i = 1, 2.$

Then the system (13) has at least three positive solutions $(u_1, v_1), (u_2, v_2), (u_3, v_3)$ such that $\|u_1\| < a, b < \min\left[\frac{1}{k}, k\right] \frac{1}{1 + t\alpha - 1} \cdot \left(u_2(t) + v_2(t)\right)$, and $\|u_3\| > a, \text{ with } \min_{k \leq t \leq k + 1} \frac{(u_3(t) + v_3(t))}{1 + t\alpha - 1} < b.$

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The application of Lyapunov exponents in dynamical economic model

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Abstract

The paper describes the Hick Samuelson Keynes dynamical economic model with discrete time and consumer sentiment. We seek to demonstrate that consumer sentiment may create fluctuations in the economical activities.

Keywords: consumer sentiment, Hick Samuelson Keynes models, Lyapunov exponent.

Mathematics Subject Classification[2010]: 37N40

1 Introduction

The empirical evidence suggests that consumer sentiment influences the household expenditure and thus confirms Keynes’s suspicion that consumer “attitudes” and “animalic spirits” may cause actuations in the economic activity. Inspired by these observations, we develop a dynamic economic model in which the agents’ consumption expenditures depend on their sentiment.

2 The mathematical model with discrete time and consumer sentiment.

The mathematical model with discrete time and consumer sentiment is given by:

\[ y_t = a + d y_{t-2} + (b-d)y_{t-1} + v(y_{t-1}y_{t-2}) - w(y_{t-1}y_{t-2})^3 + \frac{g y_{t-1}}{1 + \exp(\alpha y_{t-2} - y_{t-1})}, \quad t \in \mathbb{N} \cap (2-1) \]

The parameter \( a \) represents the autonomous expenditures, \( d \in [0, 1] \), is the discount, \( c \in [0, 1] \), is the trend towards consumption, \( m \in [0, 1] \), is the trend towards the saving, \( b \in (0, 1) \), represents the consumer reaction against the increase or decrease of his income, \( v > 0 \) and \( w > 0 \), describe the investment function, \( \alpha \in [0, 1] \), describes a family of the sentiment functions.
3  The dynamical behavior of the model \((2 - 1)\).

considering the parameter \(a\), as bifurcation parameter. The associated map of \((2 - 1)\) is \(F : R_1 \rightarrow R_2\) given by:

\[
F \left( \frac{y}{z} \right) = \left( a + (b - d + v)y + (d - v)z - w(y - z)^3 + \frac{czy + m}{1 + c\exp(z - y)} \right) \tag{3 - 1}
\]

The map \((3 - 1)\) has the following properties:

i) If \((1 + \varepsilon)(1 - b) - c > 0\), then, for the map \((3 - 1)\), the fixed point with the positive components is \(E_0(y_0, z_0)\), where:

\[
y_0 = p_1 \alpha + p_2 \beta ; z_0 = y_0
\]

and

\[
p_1 = \frac{1 + \varepsilon}{(1 + \varepsilon)(1 + b) - c} , \quad p_2 = \frac{m}{(1 + \varepsilon)(1 + b) - c}
\]

(ii) The Jacobi matrix of the map \(F\) in \(E_0\) is given by:

\[
A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]

where

\[
a_{11} = p_3 \alpha - p_4 + p_5 ; \quad a_{12} = -p_3 \beta + p_4
\]

\[
p_3 = \frac{c}{1 + \varepsilon} p_1 \alpha , \quad p_4 = d - v - \frac{c m}{(1 + \varepsilon)(1 + \varepsilon)^2} p_2 \beta , \quad p_5 = b + \frac{c}{1 + \varepsilon}
\]

(iii) The characteristic equation of the matrix \(A\) is:

\[
\lambda^2 - a_{11} \lambda - a_{12} = 0 \tag{3-2}
\]

(iv) If the model parameters \(d, v, \varepsilon, b, c, m\) satisfy the following inequality:

\[
(1 + d - v)(1 + \varepsilon)(1 + \varepsilon)(1 - b) - c) - m \varepsilon (1 - b) > 0
\]

Then for equation \((3-2)\), the roots have the modulus less than 1, iff a \(c(a_1, a_2)\), where

\[
a_1 = \frac{2p_4p_5 - p_5 - 1}{2p_3} , \quad a_2 = \frac{1 + p_4}{p_3}
\]

4  The Lyapunov exponents.

If \(a_1 > 0\) then for a \(c(0, a_2)\) or \(ac(a_2, \infty)\) the system \((2-1)\) has a complex behavior and it can be established by computing the Lyapunov exponents.

We will use the algorithm QR and the determination of the Lyapunov exponents can be obtained by solving the following system:

\[
y_{t+1} = a + (b - d + v)y_t + (d - v)z_t - w(y_t - z_t)^3 + \frac{czy}{1 + c\exp(z - y)}
\]
\[ z_{t+1} = y_t \]
\[ x_{t+1} = \arctan\left(\frac{c_{t+1}}{f_{11}(y_t)\cos y_{t+1} + f_{12}(x_{t+1})\cos x_{t+1}}\right) \]
\[ \lambda_{t+1} = \lambda_t + \ln\left(|(f_{11} - \tan x_{t+1})\cos x_{t+1} - f_{12}\sin x_{t+1}|\right) \]
\[ \mu_{t+1} = \mu_t + \ln\left(|(f_{11} - \tan x_{t+1} + 1)\sin x_{t+1} - f_{12}\cos x_{t+1}|\right) \]

with
\[ f_{11} = \frac{\partial f}{\partial y}(y_t, z_t) = b - d + v - 3w(y_t - z_t)^2 + \frac{c + e(z_t - y_t)}{(1 + e\exp(z_t - y_t))^2} \]
\[ f_{12} = \frac{\partial f}{\partial z}(y_t, z_t) = d - v + 3w(y_t - z_t)^2 - \frac{e(z_t - y_t)}{(1 + e\exp(z_t - y_t))^2} \]

The Lyapunov exponents are:
\[ L_1 = \lim_{t \to \infty} \frac{\lambda_t}{t} \]
\[ L_2 = \lim_{t \to \infty} \frac{\mu_t}{t} \]

If one of the two exponents is positive then the system has a chaotic behavior.

5 Main Result

Using a Maple 11 program, the numerical simulation is done. We consider different values for the parameters which are used in the real economic processes. The bifurcation parameter is \( a \). For \( a = 250, v = 0.1, w = 0, c = 0.1, b = 0.45, \varepsilon = 1, m = 0.5 \) and \( d = 0 \) we obtain Figures, which represent the income \( y_t \) and the income variation, with respect to the income \( y_{t-1} \).

The system is not chaotic.

For \( a = 250, v = 0.1, w = 0, c = 0.1, b = 0.45, \varepsilon = 1, m = 0.5 \) and \( d = 0.6 \) we obtain Figures, which represent the income \( y_t \) and the income variation, with respect to the income \( y_{t-1} \). The system is not chaotic.

For \( a = 250, v = 0.1, w = 0, c = 0.1, b = 0.45, \varepsilon = 1, m = 0.5 \) and \( d = 0.8 \) we obtain Figures, which represent the income \( y_t \) and the income variation, with respect to the income \( y_{t-1} \) and the Lyapunov exponent. In this case the system has a chaotic behavior.
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Lyapunov direct method for distributed order fractional nonlinear dynamical systems

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Abstract
In this paper we study the stability of nonlinear single fractional-order and distributed-order dynamical systems. Further, we propose upper bounds for solutions of these classes of fractional equations and generalize the Lyapunov direct method for distributed-order fractional systems.

Keywords: Mittag-Leffler function, Lyapunov direct method, Distributed order.

Mathematics Subject Classification: 26A33; 37B55

1 Introduction
We consider the Caputo or the Riemann-Liouville fractional nonautonomous system [1,2,3]

\[ t_0 D^\alpha_t x(t) = f(t, x), \quad \alpha \in (0, 1) \quad (1) \]

with initial condition \( x(t_0) \), where \( D \) denotes either the Caputo or the Riemann-Liouville fractional operator. Also, \( f : [t_0, \infty) \times \Omega \rightarrow \mathbb{R}^n \) is piecewise continuous in \( t \) and locally Lipschitz in \( x \) on \([t_0, \infty) \times \Omega\), where \( \Omega \in \mathbb{R}^n \) is a domain that contains the origin \( x = 0 \).

For the system (1) Li et al. [1,2] studied the stability of nonlinear fractional-order dynamic systems. They used the Lipschitz condition to propose the Mittag-Leffler stability and the fractional Lyapunov direct method for finding an upper bound for solution of system (1). Following remarks are useful, to analyze the Mittag-Leffler stability and corresponding asymptotic stability around an equilibrium point \( x_0 \), i.e

\[ f(t, x_0) = t_0 D^\alpha_t x(t_0). \]

Lemma 1.1. The following two statements are equivalent:
(a) \( f(t, x) \) is Lipschitz with respect to \( x \).
(b) There exist a Lipschitz constant \( l \) satisfying

\[ \|f(t, x_1) - f(t, x_2)\| \leq l\|x_1 - x_2\|. \quad (2) \]

Lemma 1.2. For the real-valued continuous \( f(t, x) \) in (1), we have

\[ \|t_0 D^{-\alpha} f(t, x(t))\| \leq t_0 D^{-\alpha} \|f(t, x(t))\|, \quad (3) \]

where \( \alpha \geq 0 \) and \( \| \| \) denotes an arbitrary norm \([1,2]\).
Theorem 1.3. Let \( x = 0 \) be an equilibrium point of system (1) and \( f \) be Lipschitz on \( x \) with Lipschitz constant \( l \) and piecewise continuous with respect to \( t \), then the solution of (1) satisfies
\[
\| x(t) \| \leq \| x(t_0) \| E_\alpha (l(t-t_0)\alpha), \quad \alpha \in (0, 1),
\]
where the Mittag-Leffler function is defined as
\[
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad \alpha > 0, \quad z \in C.
\]

Proof. For proof see [1,2].

Definition 1.4. (Mittag-Leffler Stability). The solution of (1) is said to be Mittag-Leffler stable if
\[
\| x(t) \| \leq \{m[x(t_0)]E_\alpha(-\lambda(t-t_0)^\alpha)\}^b, \quad \alpha \in (0, 1), \quad \lambda \geq 0, \quad b > 0, \quad m(0) = 0,
\]
where \( t_0 \) is the initial time, \( m(x) \geq 0 \) and \( m(x) \) is locally Lipschitz on \( x \in B \subset \mathbb{R}^n \) with Lipschitz constant \( m_0 \) [1,2].


By using the Lyapunov direct method, we can get the asymptotic stability of the corresponding systems. Li et al. [1,2] extend the Lyapunov direct method fractional-order systems which leads to the Mittag-Leffler stability.

Theorem 1.6. Let \( x = 0 \) be an equilibrium point for the system (1), where \( t_0 \mathcal{D}_d^\alpha = m \mathcal{D}_t^\alpha \), and \( \mathbb{D} \subset \mathbb{R}^n \) be a domain containing the origin. Let \( V(t, x(t)) : [0, \infty] \times \mathbb{D} \to \mathbb{R} \) be a continuously differentiable function and locally Lipschitz with respect to \( x \) such that
\[
\alpha_1 \| x \|^\alpha \leq V(t, x(t)) \leq \alpha_2 \| x \|^\beta, \quad t \geq 0, \quad x \in \mathbb{D}, \quad \beta \in (0, 1),
\]
where \( \alpha_1, \alpha_2, \alpha_3, a \) and \( b \) are arbitrary positive constants. Then \( x = 0 \) is stable in the sense of Mittag-Leffler.

Proof. For proof see [1,2].

2 Main Results

The fractional differential operator of distributed order \( \mathcal{D}_d^\alpha = \int_0^1 b(\alpha) \mathcal{D}_d^{\alpha \beta} \), \( \alpha > l \geq 0, \ b(\alpha) > 0 \), is a generalization of the single order \( \mathcal{D}^\alpha = \frac{d^\alpha}{dt^\alpha} \), which is obtained by considering a continuous or discrete distribution of fractional derivative. In this paper we consider the distributed order fractional nonlinear differential equations systems (DOFNDEs) with respect to the density function \( b(\alpha) \geq 0 \) as follows:
\[
\mathcal{D}_d^\alpha x(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad 0 < \alpha \leq 1,
\]
where \( x(t) \in \mathbb{R}^n \), \( \mathcal{D}_d^\alpha = \int_0^1 b(\alpha)^l \mathcal{D}_d^{\alpha \beta} x(t) d\alpha \) is the Caputo fractional derivative operator of distributed order with respect to the order-density function \( b(\alpha) \). Further \( f : [t_0, \infty] \times \Omega \to \mathbb{R}^n \) is piecewise continuous at \( t \) and locally Lipschitz in \( x \) on \([t_0, \infty] \times \Omega \), and \( \Omega \in \mathbb{R}^n \) is a domain which contains the origin \( x = 0 \). For the above distributed order fractional system an equilibrium point is defined as [4], \( f(t, x_0) = \mathcal{D}_d^\alpha x_0 \). We now obtain an upper bound for the solution of the system (8) and next we extend the Lyapunov direct method to distributed order fractional systems.
Theorem 2.1. If $x = 0$ be an equilibrium point of system (8) and $f$ be Lipschitz on $x$ with Lipschitz constant $l$ and piecewise continuous with respect to $t$, then the solution of (8) satisfies

$$\|x(t)\| \leq \|x(t_0)\| \star \mathcal{L}^{-1}\left\{\frac{B(s)}{B(s)} \right\},$$

(9)

where $0 < \alpha \leq 1$, $B(s) = \int_0^1 b(\alpha)s^\alpha da$ and $\star$ denotes the convolution product of the Laplace transform.

Theorem 2.2. Let $x = 0$ be an equilibrium point for the system (8) and $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Let $V(t,x(t)) : [0, \infty) \times \mathbb{D} \rightarrow \mathbb{R}$ be an continuously differentiable function and locally Lipschitz with respect to $x$ such that

$$\alpha_1 \|x(t)\|^a \leq V(t,x(t)) \leq \alpha_2 \|x(t)\|^b,$$

(10)

$$\frac{\partial}{\partial_0 D_t^\beta} V(t,x(t)) \leq -\alpha_3 \|x(t)\|^b, \quad t \geq 0, \quad x \in \mathbb{D}, \quad \beta \in (0,1),$$

(11)

where $\alpha_1$, $\alpha_2$, $\alpha_3$, $a$ and $b$ are arbitrary positive constants. Then $x = 0$ is asymptotic stability.

Proof. It follows from equations (11) and (12) that

$$\frac{\partial}{\partial_0 D_t^\beta} V(t,x(t)) \leq -\frac{\alpha_3}{\alpha_2} V(t,x(t)).$$

(12)

There exists a nonnegative function $M(t)$ satisfying

$$\frac{\partial}{\partial_0 D_t^\beta} V(t,x(t)) + M(t) = -\frac{\alpha_3}{\alpha_2} V(t,x(t)).$$

(13)

Taking the Laplace transform of (14) gives

$$V(s) = \frac{1}{s} \frac{B(s)V(0)}{B(s) + \alpha_2} - \frac{M(s)}{B(s) + \alpha_2},$$

(14)

where nonnegative constant $V(0) = V(0,x(0))$ and $V(s) = \mathcal{L}\{V(t,x(t))\}$. If $x(0) = 0$ namely $V(0) = 0$ then solution of (8) is $x = 0$. If $x(0) \neq 0$ then $V(0) > 0$. Applying the inverse Laplace transform to (15) gives

$$V(t) \leq V(0) \star \mathcal{L}^{-1}\left\{\frac{B(s)}{B(s) + \alpha_2} \right\},$$

(15)

which by substituting (16) into (11) yields

$$\|x(t)\| \leq \left(\frac{V(0)}{\alpha_1} \star \mathcal{L}^{-1}\left\{\frac{B(s)}{B(s) + \alpha_2} \right\}\right)^{\frac{1}{\alpha}},$$

(16)

where $V(0) > 0$ for $x(0) \neq 0$. If we consider $x(0) = 0$, since $V(t,x)$ is locally Lipschitz with respect to $x$ and $\frac{V(0,x(0))}{\alpha_1}$ is also locally Lipschitz with respect to $x(0)$, these implies the stability of system (8).

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Operations Research
On the existence of multiple positive solutions for a class of multi-singular nonlinear elliptic equations

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Abstract
In this paper, we investigate a class of semilinear elliptic equations involving multi-singular and concave-convex terms. We obtain some asymptotic behavior of the positive solutions and prove the existence and multiplicity of positive solutions for such problems.

Keywords: Multiple positive solutions; Multi-singular; Concave-convex terms; Asymptotic behavior

Mathematics Subject Classification: 35J25, 35J20, 35J61

1 Introduction
In this paper, we deal with the existence and multiplicity of positive solutions to the following problem

\[ \begin{aligned}
-\Delta u - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}u &= u^{2^*-1} + \lambda u^{q-1} & & x \in \Omega \setminus \{a_1, \ldots, a_k\} \\
 u(x) &= 0 & & x \in \Omega \setminus \{a_1, \ldots, a_k\} \\
 u(x) &= 0 & & x \in \partial \Omega 
\end{aligned} \]

(1)

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a smooth bounded domain such that \( a_i \in \Omega, i = 1, 2, \ldots, k, k \geq 2 \), are different points, \( 0 \leq \mu_i < \mu := \left( \frac{N-2}{2} \right)^2 \) that \( \mu \) is the best constant in the Hardy inequality, \( \lambda > 0, 1 \leq q < 2 \), and \( 2^* := \frac{2N}{N-2} \) is the critical Sobolev exponent.

Definition 1.1. For \( \sum_{i=1}^{k} \mu_i \in [0, \mu) \), we use \( H^1_0(\Omega) \) to denote the completion of \( C^\infty_0(\Omega) \) with respect to the norm

\[ \| u \|_\mu = \left( \int_{\Omega} \left( |\nabla u|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}u^2 \right) dx \right)^{1/2}. \]

The corresponding energy functional of the problem (1) is defined by

\[ J_\lambda(u) = \frac{1}{2} \int_{\Omega} \left( |\nabla u|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2}u^2 \right) dx - \frac{1}{2^*} \int_{\Omega} \left( u^+ \right)^{2^*} dx - \frac{\lambda}{q} \int_{\Omega} \left( u^+ \right)^q dx \]

where \( u^+ = \max\{u, 0\} \).

Then \( J_\lambda(u) \) is well defined and of class \( C^1 \) on \( H^1_0(\Omega) \).

The function \( u \in H^1_0(\Omega) \) is said to be a weak solution of the problem (1), if for any \( v \in H^1_0(\Omega) \), \( u \)
satisfies
\[ \int_{Ω} \left( \nabla u \nabla ν - \sum_{i=1}^{k} \frac{μ_i}{|x - a_i|^2} u ν - (u^+)^{2s-1} ν - λ(u^+)^{q-1} ν \right) dx = 0 \]
and by the standard elliptic regularity argument, we have that \( u \in C^2(Ω \setminus \{a_1, \ldots, a_k\}) \cap C^1(Ω \setminus \{a_1, \ldots, a_k\}) \).

**Lemma 1.2.** Let \( u(x) = |x - a_i|^s ν(x) \) where \( s = -\left(\sqrt{μ} - \sqrt{μ_i}\right), \quad 1 \leq i \leq k \).
If \( u \in H^1_0(Ω) \) is a solution of the problem (1), then, as we mentioned in the definition, \( u \in C^2(Ω \setminus \{a_1, \ldots, a_k\}) \cap C^1(Ω \setminus \{a_1, \ldots, a_k\}) \) then we have that \( ν(x) \in C^2(Ω \{a_1, \ldots, a_k\}) \cap C^1(Ω \setminus \{a_1, \ldots, a_k\}) \) and satisfies
\[
\begin{cases}
-\text{div}(|x - a_i|^{2s} \nabla ν(x)) = |x - a_i|^{2s} ν^{2s-1} + λ|x - a_i|^q ν^{q-1} \\
+ \sum_{j=1, j \neq i}^{k} \frac{μ_j}{|x - a_j|^2} |x - a_i|^{2s} ν \\
ν(x) > 0 \\
ν(x) = 0
\end{cases}
\] in \( Ω \setminus \{a_1, \ldots, a_k\} \) and satisfis \( \text{on} \partial Ω \) (2)

**Proposition 1.3.** If \( ν \in C^2(Ω \setminus \{a_1, \ldots, a_k\}) \cap C^1(Ω \setminus \{a_1, \ldots, a_k\}) \) is positive and satisfies (2), then there exists a small number \( r_0 > 0 \), such that \( ν(x) ≥ \min_{|x - a_i| = r_0} ν(x) = C_0 > 0 \) for any \( x \in B_r(a_i) \setminus \{a_i\} \).

**Proposition 1.4.** If \( ν \in C^2(Ω \setminus \{a_1, \ldots, a_k\}) \cap C^1(Ω \setminus \{a_1, \ldots, a_k\}) \) is positive and satisfies (2), then \( ν \in L^∞(B_r(a_i)) \) for \( r > 0 \) small enough.

**Theorem 1.5.** Suppose that \( 0 ≤ μ_1 ≤ μ_2 ≤ \ldots ≤ μ_k < \bar{μ} \) and \( \sum_{i=1}^{k} μ_i < \bar{μ} \) then for any solution \( u \in C^2(Ω \setminus \{a_1, \ldots, a_k\}) \cap C^1(Ω \setminus \{a_1, \ldots, a_k\}) \) of the problem (1), there exist positive constants \( N_1, N_2 \) such that \( N_1|x - a_i|^{s} ≤ u(x) ≤ N_2|x - a_i|^{s} \) that \( s = -\left(\sqrt{μ} - \sqrt{μ_i}\right), \quad 1 \leq i \leq k \) holds for any \( x \in B_r(a_i) \setminus \{a_i\} \) and \( r \) sufficiently small.

Proof. By the proposition (1.3) we have that \( u(x) = |x - a_i|^s ν(x) ≥ |x - a_i|^s \min_{|x - a_i| = r_0} ν(x) = N_1|x - a_i|^s \) for any \( x \in B_{r_0}(a_i) \setminus \{a_i\} \). From proposition (1.4), we have that \( u(x) = |x - a_i|^s ν(x) ≤ N_2|x - a_i|^s \) for \( x \in B_r(a_i) \setminus \{a_i\} \), where \( r ≤ r_0 \) is sufficiently small.

1.6. Theorem (1.5) present a asymptotic behavior of positive solutions of the problem (1)
Now using theorem (1.5) we can prove our main result:

2 Main result

**Theorem 2.1.** Suppose that \( 0 ≤ μ_1 ≤ μ_2 ≤ \ldots ≤ μ_k < \bar{μ} \) and \( \sum_{i=1}^{k} μ_i < \bar{μ} \), then there exists \( Λ > 0 \), such that for any \( λ \in (0, Λ) \), the problem (1) has at least two positive solutions in \( H^1_0(Ω) \).
References


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A collocation method for approximate boundary optimal control of the heat equations via Haar wavelets

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Abstract
This paper presents a computational collocation approach to obtain the approximate optimal control of heat equations based on properties of Haar wavelets. This technique transforms the state and control variables into nonlinear programming (NLP) parameters at collocation points. An appropriate technique of optimization has been applied to tackle the created nonlinear programming problem. The method ability has illustrated presenting numerical results.

Keywords: Wavelets theory, Optimal control, Approximation, Haar wavelet, Parabolic equations

Mathematics Subject Classification: 49J20, 65T60

1 Introduction
This paper offers a wavelet collocation method utilizing the advantages of applying wavelet theory concepts in solving Heat equations. The method is based on the method of solving Fisher’s equations which was introduced in by G. Hariharan, et. al., [2], and wavelet collocation approach of solving optimal control problems, [1], to elicit approximate optimal control $u(t)$ and its corresponding trajectory $y(x,t)$, where this pair of control and state satisfy in Heat equation

$$y_t = \lambda \Delta y, \ x \in \Omega, \ t \in (0,T),$$

with boundary conditions

$$y(x,0) = f(x), \ x \in \Omega,$$

$$y(0,t) = g(t), \ t \in (0,T)$$

$$y(1,t) = u(t), \ t \in (0,T)$$

and control minimizes the functional

$$J = \int_{x \in \Omega} (y(x,T) - \zeta(x))^2 dx.$$ 

Here $\Omega$ denotes a bounded open region in $\mathbb{R}^d$ and $T > 0$ is the terminal time. The functional (5) means that the control functions should lead the trajectory $y(x,t)$ in terminal time $T$ to desired state $\zeta(x)$ in $\Omega$. 

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2 Approximation, analysis and discretization of problem

Any function $g(x) \in L^2[0,1]$ can be decomposed as

$$g(x) = \sum_{n=0}^{\infty} \alpha_n^0 h_n(x).$$  \hspace{1cm} (6)

If $g(x)$ is piecewise constant by itself or it may be approximated as piecewise constant during each subinterval, then $g(x)$ may be approximated at finite terms, that is

$$g(x) = \sum_{n=0}^{m-1} \alpha_n^0 h_n(x) = \alpha_{(M)}^T h_{(M)}(x),$$  \hspace{1cm} (7)

where the row vector of coefficients $\alpha_{(M)}^0$ and the Haar function vector $h_{(M)}(x)$ are denominated as

$$\alpha_{(M)}^0 = [\alpha_0^0, \alpha_1^0, ..., \alpha_{M-1}^0],$$  \hspace{1cm} (8)

and

$$h_{(M)}(x) = [h_0(x), h_1(x), ..., h_{M-1}(x)]^T.$$  \hspace{1cm} (9)

Therefore, the function $g(x)$ may be estimated as

$$g(x_t) = C_{t \times M}^T H_{M \times M}.$$  \hspace{1cm} (10)

Let us divide the interval $(0,1]$ into $M$ equal parts of length $x_l = (l-0.5)/M$, $l = 1, 2, ..., M$ and divide the interval $(0,T]$ into $N$ equal parts of length $t_s = (s-1) \Delta t$, $s = 1, 2, ..., N$ and $\Delta t = T/N$. We assume that $\dot{y}$ and $\ddot{u}$ can be approximated in terms of Haar wavelets as formula

$$\dot{u} \approx \sum_{n=0}^{M-1} \alpha_n^u h_{(M)}(x) = C_{u}^T h_{(M)}(x)$$  \hspace{1cm} (11)

where , and $t$ mean differentiation with respect to $t$ and $x$, respectively and the row vectors $C_{u}^T$ and $C_{y}^T$ are constants in the subinterval $t \in (t_s, t_{s+1}]$. Integrating formula (11) with respect to $t$ from $t_s$ to $t$ results

$$u(t) = (t-t_s)C_{u}^T h_{(M)}(x) + u(t_s).$$  \hspace{1cm} (13)

Integrating formula (12) with respect to $t$ and twice with respect to $x$ results:

$$\dddot{y}(x_t, t_{s+1}) = (t_{s+1} - t_s) C_{y}^T Q_{(M)}(x_t) + y''(x_t, t_s).$$  \hspace{1cm} (14)

$$g(x_t, t_{s+1}) = (t_{s+1} - t_s) C_{y}^T Q_{(M)} h_{(M)}(x_t) + g(x_t, t_s) - g(t_s) + g(t_{s+1})$$  \hspace{1cm} (15)

$$\dot{y}(x_t, t_{s+1}) = C_{y}^T Q_{(M)} h_{(M)}(x_t) + x_t \left( C_{y}^T h_{(M)}(x_t) - C_{y}^T P_{(M)} f - \ddot{y}(t_{s+1}) \right) + \ddot{y}(t_{s+1}).$$  \hspace{1cm} (16)

where the $M \times 1$ vector $f$ is defined as $f = [1,0,0,...,0]^T$. In this case, (1) can be written as follows:

$$C_{y}^T Q_{(M)} h_{(M)}(x_t) + x_t \left( C_{y}^T h_{(M)}(x_t) - C_{y}^T P_{(M)} f - \ddot{y}(t_{s+1}) \right) + \ddot{y}(t_{s+1}) = \lambda \dot{y''}(x_t, t_{s+1}).$$  \hspace{1cm} (17)
Looking at a discretized form of the cost functional (5) by a scheme of numerical integration in collocation points gives rise to

$$ J = \sum_{i=1}^{M} \omega_i \left( y(x_i, T) - \zeta(x_i) \right)^2 dx, \quad (18) $$

where $y(x_i, T)$ can be achieved from (13). Now, implementing a nonlinear programming solver to nonlinear optimization problem of minimizing the objective function (17) with the constraints (16) and vectors of unknowns variables $C_u^T$ and $C_y^T$ may be led to approximate optimal control and trajectory of original problem.

### 3 Numerical results

Because the aim of the objective functional in problem (1)-(5) is achievement to the state $\zeta(x)$ in final time $T$, an error function to evaluate the approximation is defined as

$$ e(x) = |y(x, T) - \zeta(x)|. \quad (19) $$

We consider a numerical example as the optimal control problem governed by the following Heat equation

$$ y_t = 4y_{xx}, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 0.5, $$

with boundary conditions

$$ y(x, 0) = \sin(2\pi x) + \cos(2\pi x), \quad 0 \leq x \leq 1, $$

$$ y(0, t) = e^{-\pi t}, \quad 0 \leq t \leq 0.5, $$

$$ y(1, t) = u(t). $$

The aim of problem is to determine the best boundary control $u(t)$ for enforcing the trajectory $y(x, t)$ to achieve the state

$$ \zeta(x) = e^{-\frac{x^2}{2}} (\sin(2\pi x) + \cos(2\pi x)), $$

in final time $T = 0.5$. Taking into $M = 16$ collocation points the solution of nonlinear programming problem with objective function corresponding to (17) and constraints (16) will be acquired as

$$ C_u^T = [-769.549, -645.935, -918.533, -80.751, -1303.843, -105.059, -49.252, -29.981, ... $$

$$ -1724.806, -163.798, -67.496, -40.253, -27.907, -20.925, -15.983, -12.518], $$

$$ C_y^T = [-631.655, -275.415, -117.502, -198.326, -78.957, -45.478, -78.957, -112.436, ... $$

$$ -46.138, -32.82, -23.402, -23.401, -32.82, -46.137, -55.556, -55.556]. $$

The optimal value of objective is obtained $1.565E - 27$. Also in Figures 1-2, one can see the error function and comparison of approximate and exact states and approximate optimal control function, respectively.
Figure 1: Left diagram shows error function and right diagram shows approximate and exact optimal states $y(x, T)$.

Figure 2: The approximate optimal control $u(t)$ for $M = 16$ and $T = 0.5$.

References


The system optimization perspective for multiproduct supply chain network

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Abstract

In this paper, multiproduct supply chain network model is developed with system optimization perspective. Each kind of products, has an individual cost function and, at the same time, contributes to its own and other product’s cost function in an individual way. The well known equilibrium algorithm is extended to find system optimization pattern for such multiproduct supply chain network.

Keywords: system optimization, multiproduct supply chain, total cost minimization

Mathematics Subject Classification: 90C25

Introduction

A supply chain is a network of retailers, distributors, transporters, storage facilities, and suppliers that participate in the production, delivery, and sale of products to the consumer.

Note that Min and Zhou [2] provided a synopsis of supply chain modeling and the importance of planning, designing, and controlling the supply chain as a whole.

In this paper, multiproduct supply chain network model is developed with system optimization perspective, so that the total cost is minimized in network.

The total cost shipment $f_a^j$ unit for $j$ typical product on link $a$ that are denoted by $\bar{c}_a(f_a^1, \ldots, f_a^J)$, define as following:

$$\bar{c}_a(f_a^1, \ldots, f_a^J) = c_a^j(f_a^1, \ldots, f_a^J) \times f_a^j, \quad j = 1, \ldots, J, \quad \forall a \in L.$$ 

and given by quadratic model as following

$$\bar{c}_a^j(f_a^1, \ldots, f_a^J) = \sum_{l=1}^J g_{l_a}^j f_a^l f_a^j + h_{a}^j f_a^j, \quad \forall a \in L, \quad j = 1, \ldots, J$$

where $g_{l_a}^j$, $h_{a}^j$ are given constants.

The multiproduct supply chain cost minimization problem can be formulated jointly as follows:

$$\text{Minimize} \sum_{a \in L} \sum_{j=1}^J \bar{c}_a^j(f_a^1, \ldots, f_a^J)$$
subject to constraints is developed in paper. The well known equilibrium algorithm [1] is extended to find system optimization flow pattern for such multiproduct supply chain network so that total cost in network is minimized for sending products and is developed as following.

Let $Z[T]$ stand for the set of all feasible flow patterns of the supply chain network. An operator $E : Z[T] \rightarrow Z[T]$ is defined as the composition

$$E = E_{w(m)} \circ \ldots \circ E_{w(1)}$$

of operators

$$E_{w(l)} : Z[T] \rightarrow Z[T], \quad l = 1, \ldots, m,$$

where $w(1), \ldots, w(m)$ is an arbitrary ordering of the set origin/destination pairs. In turn, $E_{w(l)}$ will be defined as the composition

$$E_{w(l)} = E_{w(l)}^J \circ \ldots \circ E_{w(l)}^1$$

of operators

$$E_{w(l)}^j : Z[T] \rightarrow Z[T], \quad j = 1, \ldots, J,$$

end equation where $E_{w(l)}^j$ sends a feasible flow pattern $x$ to another feasible flow pattern $\hat{x} = E_{w(l)}^j x$, which is constructed by the following procedure.

Among the elements of $P_w$ which $P_w$ denote the set of paths connecting the O/D pair of nodes, determine the paths $q$ and $r$ requiring

$$\hat{C}_q^j(f) = \min_{p \in P_{q_k}} \{ \hat{C}_p^j(f) \}$$

$$\hat{C}_r^j(f) = \max_{p \in P_{r_k}, x_{p}^j > 0} \{ \hat{C}_p^j(f) \}$$

where $\hat{C}_p^j(f)$ denotes the total marginal cost on path $p$ for product $j$, given by

$$\hat{C}_p^j(f) \equiv \sum_{l=1}^{J} \sum_{a \in L} \frac{\partial q_p^j(f)}{\partial f_a^j} \delta_{ap},$$

and then set

$$\hat{x}_p^j = x_p^j \quad l \neq j, \quad p \in P$$

$$\hat{x}_p^j = x_p^j \quad p \neq q, \quad p \neq r$$

$$\hat{x}_q^j = x_q^j + \delta$$

$$\hat{x}_r^j = x_r^j - \delta,$$

where $\delta$ is selected so that the total cost is minimized over the class of admissible load patterns $\hat{f}$ that are induced by the class of flow patterns $\hat{x}$. In the case of the quadratic model, $\delta$ can be determined explicitly through the formula

$$\delta = \min \{ x_r^j, \frac{\hat{C}_q^j(f) - \hat{C}_q^j(f)}{2 \sum_{a \in L} g_a^j (\delta_{aq} - \delta_{ar})^2} \}.$$
Main Result

In this paper, multiproduct supply chain network model is constructed utilizing a system-optimization perspective and the total cost is minimized in network. Therefore, system-optimizing pattern is needed to optimized the network. Since, multiproduct is dealt, modified equilibrium algorithm is defined as composition of operators. In fact, system-optimizing flow pattern can be determined with the extended equilibrium algorithm, so that total cost for sending products in supply chain network is minimized.

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Biological computation of the solution to the assignment problem

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Abstract

The Assignment problem is an NP-hard problem. In this paper, we describe a DNA algorithm for the DNA solution of the Assignment problem.

keywords: DNA Algorithm, Assignment problem, Biological computing.

1 Introduction

DNA computing methods were employed in complex computational problems such as the Hamilton path problem (HPP) (Adleman 1994), maximal clique problem (Ouyang et al 1997), satisfiability problem (SAT) (Liu et al 2000). The advantage of these approaches is the huge parallelism inherent in DNA-based computing, which has the potential to yield vast speedups over conventional electronic-based computers for such problems. We desirable an algorithm using stick model for Assignment problem.

2 The Assignment problem:

The Assignment problem can be described as follows. There are \( n \) tasks which must be performed by \( n \) individuals. The cost of individual \( i \) performing task \( j \) is \( c_{ij} \). We wish to assign people to the tasks so as to minimize the cost of completing the tasks. We can formulate the problem as follows: let

Then we wish to minimize the total cost subject to constraints which state that each person must be assigned one task and each task must be assigned to one person. That is,

\[
\text{minimize} \quad f = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij} \\
\text{s.t.:} \\
\sum_{j=1}^{n} x_{ij} = 1 \quad i = 1, 2, ..., n \\
\sum_{i=1}^{n} x_{ij} = 1 \quad j = 1, 2, ..., n \\
x_{ij} = 0 \text{ or } 1 \quad i, j = 1, 2, ..., n.
\]
If we study this formulation, we can see that it is a transportation problem with $m = n$ and all supplies and demands equal to 1.

3 DNA computing:

DNA is the major example of a biological molecule that stores information and can be manipulated, via enzymes and nucleic acid interactions, to retrieve information. similarly, as a string of binary data is encoded with zeros and ones, a strand of DNA is encoded with four bases (known as nucleotides), represented by the letters A,T,C and G. Each strand, according to chemical convention, has a 5' and a 3' end; hence, any single strand has a natural orientation. Bonding occurs by the pairwise attraction of bases; A bonds with T and G bonds with C. the pairs (A,T) and (G,C) are therefore known as complementary base pairs.

3.1 operations:

The operation of general solutions are summarized as follows, in the form of Adleman's "Sticker-based" DNA models.
- Extract $(T, x_i, T_1, T_2)$: Given a test tube $T$ and bit $X_i$, get two test tubes $T_1$ and $T_2$, where $T_1$ and $T_2$ contain all those strands in $T$ with value 1 and 0 in the bit $x_i$, respectively.
- Merge $(T, T_1, T_2)$: Given two test tubes $T_1$, $T_2$, get a test tube $T$ containing all strands in $T_1$ and all strands in $T_2$.
- Amplify $(T, T_1, T_2)$: Given a test tube $T$, this operation is used to yield two new identical tubes $T_1,T_2$ of $T$, and then to empty the tube $T$.
- Append $(T, s)$: Given a test tube $T$ and strand $s$, the operation logically adds "$s$" onto the end of very strand in tube $T$.
- Set $(T, x_i)$: Given a test tube $T$ and a bit $x_i$, get a test tube $T$ by setting bit $x_i$ of all strands in $T$ as 1.
- Clear $(T, x_i)$: Given a test tube $T$ and a bit $x_i$, get a test tube $T$ by setting bit $x_i$ of all strands in $T$ as 0.
- Detect $(T)$: Given a test tube $T$, which assumes value yes or no according as there exists a DNA strand in $T$ or not.
It returns (Y) if it contains DNA strands: otherwise it returns (N).
These operations are used to "molecular programs", whose input is a tube with DNA or molecules is "yes","no" or (setof) tube (s).

3.2 DNA Algorithm:

This algorithm consists of four stages:
- Prepare the set $T$ of all possible assignment candidates.
- Determine the set $T$ of all feasible assignment by excluding all those unfeasible assignment from $T$. 

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- calculate the value for objective function for all feasible assignments in $T$.
- Compare the objective function value in $T$ so that it’s minimized.

Initially, $2m$ strands are poured into test tube $T_0$ to represent the $2m$ constraints. $2m$ strands with lengths of $(m^2k)$ bits are produced. For each of the $m^2$ variables $x_i$, ($i = 1, 2, ..., m^2$), two distinct "value sequences" of 20 nucleotides are assigned to represent value "1" and "0". (table 1)

Each strand is subdivided with eight non-overlapping regions in which each region is explained in Table 1.

1. DNA initialization

```
procedure Init($T_0, m$)
    $T_1 = T_2 = \phi$
    Amplify($T_0, T_0, r * 2^m$)
    for $i := 1$ to $m$ do
        for $j := 1$ to $m$ do
            separate($T_0, T_1, T_2$)
            set($T_1, x_{(i-1)m+j}$)
            Merge($T_0, T_1, T_2$)
            next $j$
        next $i$
```

2. DNA checking

```
procedure fit – constraint($T_0, Q, k$)
    $T_{yes} = T_{drop} = \phi$
    for $i := 1$ to $m$ do
        for $j := 1$ to $m$ do
            Extract($T_0, x_{(i-1)m+j}, T_{yes}, T_0$)
            if (Detect($T_{yes}$) = $y$) then
                for $k := 1$ to $m$ do
                    if ($k \neq i$) then
                        Extract($T_{yes}, x_{(k-1)m+j}, T_{drop}, T_{yes}$)
                    end if
                next $k$
            end if
            merge($T_0, T_{yes}$)
            wash($T_{yes}$)
            next $j$
        next $i$
```

strands $c_i$ are appended to tube $T_0$, as governed by $x_i$ in the execution of procedure Append-$c_i(T_0, m, k)$. to change $CX$ to a binary number is, primary valuating the places ($(k + 1)m^2 + 1$) to ($(k + 2)m^2 + k$), which is done by the following process.
procedure Append \(- c_i(T_0, m, k)\)
\[
T_1 = T_2 = \emptyset
\]
\[
\text{for } i := 1 \text{ to } m \text{ do }
\]
\[
\text{for } j := 1 \text{ to } m \text{ do }
\]
\[
\text{Extract}(T_0, x_{i(j-1)+j}, T_1, T_2)
\]
\[
\text{Append}(T_1, c_{i(j-1)+j})
\]
\[
\text{Append}(T_2, c_0)
\]
\[
\text{Merge}(T_0, T_1, T_2)
\]
\[
\text{next } j
\]
\[
\text{next } i
\]
\[
c_0 = 0
\]

now, to change $CX$ to a binary number is parallel-add process, which passes parameters $(\alpha, \beta, f)$ with $((k+2)m^2 + k, (k+3)m^2 + 2k, (k+2)m^2 + k)$. The procedure parallel-comparator$(T_0, m^2, k)$ is applied to determine which strands have maximum value by checking all strands in tube $T_0$.

procedure parallel \(- add(T_0, k, \alpha, \beta, f)\)
\[
\text{for } i := 0 \text{ to } Q \text{ do }
\]
\[
\text{for } j := 0 \text{ to } (k-1) \text{ do }
\]
\[
\text{Extract}(T_0, Y_{(\alpha+k)i-j}, T_p, T_q)
\]
\[
\text{Extract}(T_p, Y_{(\beta-j)j}, T_a, T_b)
\]
\[
\text{set}(T_b, Y_{\beta-j})
\]
\[
\text{clear}(T_a, Y_{\beta-j})
\]
\[
\text{set}(T_a, Y_{-j+1})
\]
\[
\text{Merge}(T_0, T_a, T_b, T_q)
\]
\[
\text{next } j
\]
\[
\text{for } j := 1 \text{ to } (Q+k) \text{ do }
\]
\[
\text{Extract}(T_0, Y_{1j-i}, T_f, T_b)
\]
\[
\text{Extract}(T_f, Y_{1-j}, T_{A_1}, T_{A_0})
\]
\[
\text{set}(T_{A_0}, Y_{\beta-j})
\]
\[
\text{clear}(T_{A_1}, Y_{\beta-j})
\]
\[
\text{clear}(T_{A_2}, Y_{-j})
\]
\[
\text{set}(T_{A_1}, Y_{-j+1})
\]
\[
\text{clear}(T_{A_1}, Y_{-j})
\]
\[
\text{merge}(T_0, T_{A_1}, T_{A_0}, T_f)
\]
\[
\text{next } j
\]
\[
\text{next } i
\]

procedure parallel \(- comparator(T_0, m^2, k)\)
\[
s = 0
\]
\[
D0
\]
\[
s + +
\]
\[
\text{Extract}(T_0, x_{\alpha+i}, T_{ans}, T_a)
\]
\[
\text{if } \text{Number}(T_{ans}) = 1 \text{ then exit}
\]
\[
\text{if } \text{Number}(T_{ans}) > 1 \text{ then }
\]
\[
\text{for } i := s + 1 \text{ to } m^2 \text{ do }
\]
\[
\text{Extract}(T_{ans}, x_{\alpha+i}, T_f, T_{ans})
\]
\[
\text{if } \text{Number}(T_f) = 1 \text{ then exit}
\]
\[
\text{Amplify}(T_f, T_{ans})
\]
\[
\text{if } \text{Number}(T_f) > 1 \text{ then }
\]
\[
\text{Amplify}(T_f, T_{ans})
\]
\[
\text{wash}(T_{ans})
\]
\[
\text{next } i
\]
\[
\text{until}(\text{number}(T_{ans}) > 0)
\]
\[
\alpha = (k+3)m^2 + 2k
\]

4 Conclusions:

In this study, We present a different DNA computing approach for solving the transportation problem, which is an improvement on the conventionally adopted exhaustive
search. The authors hope that, in future studies, more highly effective DNA operations will be exploited to drive a DNA computing model with time efficiency and is complete for NP-hard problems.

References


Combinatorics
The independence polynomial of a graph

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Abstract
Let $i_k(G)$ denote the number of independent sets of size $k$ of graph $G$. Then the independence polynomial of $G$ is

$$I(G, x) = \sum_{k=0}^{n} (-1)^k i_k(G)x^k.$$

We show that, for every $G$, the polynomial $I(G, x)$ has only one root of smallest modulus, and it is real and unique.

Keywords: Independence polynomial, clique polynomial

Mathematics Subject Classification: 05C31

1 Introduction
Throughout the paper $G$ is a graph without multiple edges and loops. The independence polynomial of the graph $G$ is defined as follows. Let $i_k(G)$ denote the number of independent sets of size $k$ of $G$, where $i_0(G) = 1$. Then the independence polynomial of $G$ is

$$I(G, x) = \sum_{k=0}^{n} (-1)^k i_k(G)x^k.$$

Let $\beta(G)$ be the smallest real root of the independence polynomial. It is known that it exists and it is in the interval $(0, 1]$ (see [1,2]). It is also well-known that if $\rho$ is another root of the independence polynomial, then $|\rho| \geq \beta(G)$ [2,3]. It was only proved much later by Goldwurm and Santini [5] that in fact, the much stronger statement $|\rho| > \beta(G)$ holds. On the other hand, their proof is very complicated.

Notation. Throughout the paper we will consider only simple graphs. As usual $G = (V, E)$ will denote a graph with vertex set $V(G)$ and edge set $E(G)$. Let $e(G)$ be the number of edges, i.e., $|E(G)| = e(G)$. The set of neighbors of the vertex $v$ will be denoted by $N_G(v)$; if it makes no confusion we will write simply $N(v)$ instead of $N_G(v)$. The closed neighborhood of the vertex $v$ is $N_G[v] = N_G(v) \cup \{v\}$. The degree of the vertex $v$ will be denoted by $d(v) = |N_G(v)|$. For $S \subseteq V(G)$ the graph $G - S$ denotes the subgraph of $G$ induced by the vertices $V(G) \cap S$ while $G - e$ denotes the subgraph of $G$ induced by the vertex set $S$. If $e \in E(G)$ then $G - e$ denotes the graph with vertex set $V(G)$ and edge set $E(G) - \{e\}$.
2 Lemmas from complex analysis

In this part we collect the required background from complex analysis. We use only two theorems from complex analysis, both theorems are well-known and they can be found in [4].

Lemma 2.1 (Pringsheim’s theorem). [?] If \( f(z) \) is representable at the origin by a series expansion that has non-negative coefficients and radius of convergence \( R \), then the point \( z = R \) is a singularity of \( f(z) \).

Definition 2.2. Let

\[
f(z) = \sum_{n=0}^{\infty} f_n z^n.
\]

Then \( \text{Supp}(f) = \{ k | f_k \neq 0 \} \). The sequence \( (f_n) \) as well as \( f(z) \) admits a span \( d \) if for some \( r \), there holds

\[
\text{Supp}(f) \subseteq r + d\mathbb{Z}_{\geq 0} = \{ r, r + d, r + 2d, \cdots \}.
\]

The largest span, \( p \), is the period, all other spans being divisors of \( p \). If the period is equal to 1, then the sequence \( (f_n) \) and \( f(z) \) are said to be aperiodic.

Lemma 2.3 (Daffodil lemma). [?] Let \( f(z) \) be analytic in \( |z| < R \) and have non-negative coefficients at 0. Assume that \( f \) does not reduce to a monomial and that for some non-zero non-positive \( s \), one has \( |f(s)| = f(|s|) \). Then the following holds,

(i) The argument of \( s \) must be commensurate to \( 2\pi \), i.e., \( s = |s|e^{i\theta} \) with \( \theta/2\pi = \frac{r}{p} \in \mathbb{Q} \) (an irreducible fraction) and \( 0 < r < p \);

(ii) \( f \) admits \( p \) as a span.

3 Generator functions of the independence polynomial

Throughout this section we will us the following simple recursive formulas for the independence polynomial.

Lemma 3.1. (a) If \( G \) is non-connected graph with connected components \( H_1, \cdots, H_k \) then

\[
I(G, z) = \prod_{j=1}^{k} I(H_j, z).
\]

(b) Let \( v \) be an arbitrary vertex of the graph \( G \). Then

\[
I(G, z) = I(G - v, z) - zI(G - N[v], z).
\]

(c) If \( e = \{u, v\} \in E(G) \), then we have

\[
I(G, z) = I(G - e, z) - z^2I(G - N[u] - N[v], z).
\]

The next lemma is the main tool to prove main result. It is a generalization of Fisher’s theorem [2,3].

Lemma 3.2. Let \( G \) and \( H \) be graphs and set

\[
\frac{I(H, z)}{I(G, z)} = \sum_{k=0}^{n} r_k(H, G)z^k.
\]

Then

(a) If \( H \) is a proper induced subgraph of \( G \) then \( r_k(H, G) > 0 \) for \( k \geq 0 \).

(b) If \( H \) is a proper subgraph of \( G \) then \( r_k(H, G) > 0 \) for \( k \geq 2 \).
Corollary 3.3. Let $\beta(G)$ be convergence radius of $\frac{1}{I(G,z)}$. Then $\beta(G)$ is the root of independence polynomial $I(G,z)$ and it has the smallest modulus among the roots of $I(G,z)$. Let $H$ be a subgraph of $G$. Then $\beta(G) \leq \beta(H)$.

Our next aim is to prove that if $H$ is a proper subgraph of the connected graph $G$ then $\beta(G) < \beta(H)$ holds. This is the last step to prove the uniqueness of the root $\beta(G)$.

Lemma 3.4. Let $G$ be a connected graph and let $H$ be a proper subgraph of $G$. Then $\beta(G) < \beta(H)$ and the multiplicity of the root $\beta(G)$ is 1.

In this paper we give a much simpler proof for this theorem.

Theorem 3.5. Let $\rho$ be a root of the independence polynomial $I(G,z)$ different from $\beta(G)$ then $|\rho| > \beta(G)$.

Proof. Clearly, it is enough to prove the claim for a connected graph $G$ since the roots of $I(G,z)$ are the union of the roots $I(H_j,z)$ where $H_j$ are the connected components of the graph $G$. We can also assume that $|V(G)| \geq 2$, since the claim is trivial for $K_1$.

Let $u$ be an arbitrary vertex of the graph $G$. Note that $G - N[u]$ is a proper induced subgraph of $G - u$ since $G$ is connected graph on at least two vertices.

Once again we use the identity

$$g(z) = \frac{I(G - u, z)}{I(G, z)} = \frac{1}{1 - z\frac{I(G - N[u], z)}{I(G, z)}}.$$  

Let us examine

$$f(z) = z\frac{I(G - N[u], z)}{I(G - u, z)}.$$  

Since $G - N[u]$ is a proper induced subgraph of $G - u$, the convergence radius of $f(z)$ is $\beta(G - u)$ and all the coefficients of $f(z)$ are positive except the coefficient of $z^0$. If $\rho$ is a root of $I(G,z)$ of modulus $\beta(G)$ then it must be a singularity of $g(z)$ since $I(G - u, z)$ has no root of smaller modulus than $\beta(G - u)$ and $\beta(G - u) > \beta(G)$. Hence $f(\rho) = 1$. Hence $|f(\rho)| = f(|\rho|)$ so we can use the Daffodil Lemma (Lemma 2.3) to obtain that $\rho = \beta(G)e^{2\pi i r/p}$ where $p$ is some period of $f(z)$. But $f(z)$ is aperiodic: all the coefficients are positive. Thus $p = 1$ and $\rho = \beta(G)$. 

\[\Box\]

References


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Total domination in cartesian product $P_m \square C_n$

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Abstract
Let $G = (V, E)$ be a graph with vertex set $V$ of order $n(G)$ and edge set $E$ of size $m(G)$. A subset $S$ of $V$ is a total dominating set of $G$ if every vertex $v \in V$ is adjacent to at least one vertex in $S$. The total domination number $\gamma_t(G)$ of $G$ is the minimum cardinality of a total dominating set of $G$. The Cartesian product $G \square H$ of two graphs $G$ and $H$ is the graph with the vertex set $V(G) \times V(H)$ where two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. In this presentation, we will find $\gamma_t(P_m \square C_n)$. In the complete graph $K_n$, the closed neighborhood of a vertex $v$ is also $N_G(v)$.

Keywords: Total domination number, Cartesian product of graphs.

Mathematics Subject Classification: 05C69

1 Introduction
All graphs considered here are finite, undirected and simple. Let $G = (V, E)$ be a graph with vertex set $V$ of order $n(G)$ and edge set $E$ of size $m(G)$. The open neighborhood and the closed neighborhood of a vertex $v \in V$ are $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The degree of a vertex $v$ is also $deg_G(v) = |N_G(v)|$. The minimum and maximum degree of $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write $K_n$ and $P_n$ for the complete graph and the path of order $n$, respectively, while $G[S]$ denotes the subgraph induced by $G$ by a vertex set $S$ of $G$.

As stated in many references, for example in [1], the Cartesian product $G \square H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. In the product $G \square H$, we define $H_x$ to be the subgraph induced by $x \times V(H)$, for any $x \in V(G)$, $G_y$ can be defined similarly for any $y \in V(H)$.

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi and Slater [1, 2]. Among the variations of domination, the $k$-tuple total domination number were introduced by Henning and Kazemi [3], which is the extension of total domination number. Let $k$ be a positive integer. A subset $S$ of $V$ is a $k$-tuple total dominating set of $G$ if for every vertex $v \in V$ has at least $k$ neighbors in $S$. Also the $k$-tuple total domination number $\gamma_{k,t}(G)$ of $G$ is the minimum cardinality of a $k$-tuple total dominating set of $G$. When $k = 1$, the $k$-tuple total domination number is known as total domination number.

In this presentation, we will find $\gamma_t(P_m \square C_n)$. Some of our work are given in the following.

2 Results

Theorem 2.1. Let $m$ be an odd positive integer and let $n$ be a positive integer that is a multiple of 4. Then $\gamma_t(P_m \square C_n) = \frac{n(m+1)}{4}$.
Theorem 2.2. If $m$ is an even positive integer, then $\gamma_t(P_m \square C_4) = m + 2$.

Theorem 2.3. If $m \not\equiv 1 \pmod{4}$ is a positive integer and $n$ is a multiple of $m + 1$, then

$$\gamma_t(P_m \square C_n) = \begin{cases} (n - \frac{n}{m+1}) \left\lceil \frac{m}{4} \right\rceil & \text{if } m \equiv 2 \pmod{4} \\ n \left\lceil \frac{m}{4} \right\rceil & \text{if } m \equiv 0, 3 \pmod{4} \end{cases}.$$

Theorem 2.4. For any integer $n \geq 3$, we have

$$\gamma_t(P_2 \square C_n) = \begin{cases} \left\lceil \frac{2n}{3} \right\rceil + 1 & \text{if } n \equiv 1 \pmod{3} \text{ and } n \not= 7, \\ \left\lceil \frac{2n}{3} \right\rceil & \text{otherwise.} \end{cases}$$

Theorem 2.5. For any $n \geq 3$, we have $\gamma_t(P_3 \square C_n) = n$.

Theorem 2.6. For any integer $n \geq 3$, we have

$$\gamma_t(P_5 \square C_n) = \begin{cases} 6 \left\lceil \frac{n}{4} \right\rceil & \text{if } n \equiv 0 \pmod{4}, \\ 10 & \text{if } n = 6, \\ 5 & \text{if } n = 3, \end{cases}$$

and

$$3 \left\lceil \frac{n-1}{2} \right\rceil \leq \gamma_t(P_5 \square C_n) \leq 3 \left\lceil \frac{n+1}{2} \right\rceil - 3$$
if $n \equiv 3 \pmod{4}$,

$$3 \left\lceil \frac{n+1}{2} \right\rceil + 1 \leq \gamma_t(P_5 \square C_n) \leq 3 \left\lceil \frac{n-2}{2} \right\rceil + 4$$
if $n \equiv 2 \pmod{4}$,

$$3 \left\lceil \frac{n-1}{2} \right\rceil \leq \gamma_t(P_5 \square C_n) \leq 3 \left\lceil \frac{n-1}{2} \right\rceil + 3$$
if $n \equiv 1 \pmod{4}$.

References


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Probability and Statistics
Randomly weighted average with beta random proportion

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Abstract

A weighted average of two independent continuous random variables $X_1$ and $X_2$ with random proportions obtained by Beta distribution is introduced. A formula between the Stieltjes transforms of the distribution functions of the weighted average and $X_1$ and $X_2$ is established. We review Van Assche’s results and related issues for this weighted average and we also find characterizations for Beta, Cauchy, arcsin, semicircle and power semicircle distributions.

Keywords: Arcsin, Cauchy, Random proportion, Stieltjes transform, Schwartz theory, Weighted average.

Mathematics Subject Classification: 34K33

1 Introduction

In a fundamental paper, Van Assche (1987) considered a random variable $Z$ uniformly distributed between two independent random variables $X_1$ and $X_2$, in the sense that the conditional distribution of $Z$ given $X_1 = x_1$ and $X_2 = x_2$ is uniform over $[\min\{x_1, x_2\}, \max\{x_1, x_2\}]$. He employed the Stieltjes transform and derived that: (i) $-S'(H, z) = S(F, z)S(G, z)$; (ii) for $X_1$ and $X_2$ on $[-1, 1]$, $Z$ is uniform on $[-1, 1]$ if and only if $X_1$ and $X_2$ have an arcsin distribution; (iii) $Z$ possesses the same distribution as $X_1$ and $X_2$ if and only if $X_1$ and $X_2$ are degenerated or have a Cauchy distribution. Johnson and Kotz (1990) noticed that the random variable $Z$ is indeed a weighted average of $X_1$ and $X_2$ with random proportions $U$ and $1 - U$; $Z = UX_1 + (1 - U)X_2$, $U$ uniform on $[0, 1]$ independent of $X_1$ and $X_2$. Also they raised the question of the possibility of extending the (Van Assche, 1987) results for a randomly weighted average. Soltani and Homei (2009) extended the work of Van Assche (1987). They let $X_1, ..., X_n$ to be independent, and considered

$$S_n = R_1X_1 + R_2X_2 + \cdots + R_{n-1}X_{n-1} + R_nX_n, \quad n \geq 2,$$

where random proportions are $R_i = U(i) - U(i-1)$, $i = 1, ..., n - 1$ and $R_n = 1 - \sum_{i=1}^{n-1} R_i$, $U(1), ..., U(n-1)$ order statistics from a uniform distribution on $[0, 1]$, $U(0) = 0$. These random proportions are said to be uniformly distributed over the unit simplex. They employed Stieltjes transform and derived that: (i) \( \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}S(F_{S_n}, z) = \prod_{i=1}^{n} S(F_{X_i}, z) \); (ii) $S_n$ possesses the same distribution as $X_1, ..., X_n$ if and only if $X_1, ..., X_n$ are degenerated or have a Cauchy distribution; (iii) the arcsin’s result of Van Assche (1987) is only true for $S_2$. In the present work, we extend the work of Van Assche to other form. We suppose that $U$ is the distribution of $Beta(n_1, n_2)$, where $n_1, n_2$ are integers.
In section 2, we derive the distribution of $Z$, for given distinct values $X_1 = x_1$ and $X_2 = x_2$. Theorem 2.1. In section 3, we establish the main result of this article, where the $(n_1 + n_2 - 1)$-th derivative of the Stieltjes transform of the distribution of $Z$ is expressed in terms of the product of the $(n_1 - 1)$-th and $(n_2 - 1)$-th derivative of the Stieltjes transform of the distributions of $X_1$ and $X_2$. In section 4, we observe that the (Van Assche, 1987) arcsin result and the characterization for the Cauchy distribution is true for some Beta distributions, so the results given by Van Assche is not only for uniform distribution. In section 5, we also mention some applications of this result.

2 Conditional distribution

We note that for $x_2 < z \leq x_1$, the conditional distribution of $Z$ given $X_1 = x_1$ and $X_2 = x_2$ at $z$, denoted by $K(z|x_1, x_2)$, is equal to

$$\frac{1}{B(n_1, n_2)} \sum_{k=0}^{n_2-1} \left( \frac{(n_2-1)}{k} (-1)^k \right) \frac{z-x_2}{n_1+k} \frac{1}{x_1-x_2} \left( z-x_2 \right)^{n_1+k},$$

and the conditional distribution for $x_1 < z \leq x_2$ is equal to

$$\frac{1}{B(n_1, n_2)} \sum_{k=0}^{n_1-1} \left( \frac{(n_1-1)}{k} (-1)^k \right) \frac{z-x_1}{n_2+k} \frac{1}{x_2-x_1} \left( z-x_1 \right)^{n_2+k}.$$

Then

$$K(z|x_1, x_2) = \frac{1}{B(n_1, n_2)} \sum_{k=0}^{n_2-1} \left( \frac{(n_2-1)}{k} (-1)^k \right) \frac{z-x_2}{n_1+k} \frac{1}{C(x_2-i ; x_1, x_2)} \left( z-x_2 \right)^{n_1+k}, \quad x_2-i < z \leq x_{i+1}, \quad i = 0, 1$$

where

$$C(x_2-j ; x_1, x_2) = \prod_{k=1}^{2-j-1} (x_k - x_2-j) \prod_{k=2-j+1}^2 (x_k - x_2-j).$$

By using the Heaviside function: $U(x) = 0, x < 0, = 1, x \geq 0$, we obtain that for any given distinct values $x_1, x_2$, the conditional distribution is given by

$$K(z|x_1, x_2) = \frac{1}{B(n_1, n_2)} \sum_{i=0}^{n_2-1} \sum_{k=0}^{n_2-1} \left( \frac{(n_2-1)}{k} (-1)^k \frac{z-x_2-i}{n_1+k} \right) \frac{1}{C(x_2-i ; x_1, x_2)} \left( z-x_2-i \right)^{n_1+k}.$$

For $z \in [\min \{x_1, x_2\}, \max \{x_1, x_2\}]$, together with $K(z | x_1, x_2) = 0$ for $z < \min \{x_1, x_2\}$ and $z = 1$, for $z > \max \{x_1, x_2\}$, we arrive at the following result.

**Theorem 2.1.** Assume $Z$ is a randomly weighted average with Beta($n_1, n_2$) random proportion. Then the conditional distribution of $Z$, for given distinct values $X_1 = x_1$ and $X_2 = x_2$ at $z$, $-\infty < z < +\infty$ will be given by (1).

3 Stieltjes transform of weighted average

In this section we present the main results of this article. Let us first develop some basic tools. We first prove the following lemma.
Lemma 3.1. For given distinct real \( x_1, x_2, z \) and integers \( m_j \geq 1, j = 1, 2 \), we have the following formula:
\[
\sum_{j=1}^{2} \frac{(-1)^{m_j}}{(m_j - 1)!} \left[ \frac{d^{m_j-1}}{dx_j^{m_j-1}} \frac{1}{(z - x_j)} \right] a_j = \frac{1}{(x_1 - z)^{m_1}(x_2 - z)^{m_2}},
\]
where
\[
a_j = \prod_{i \neq j} \frac{1}{(x_i - x_j)^{m_i}}.
\]

Proof. This formula is perhaps known, however can easily be verified by Leibnitz formula. \( \square \)

The second item is the following formula taken from the Schwartz distribution theory, namely,
\[
\int_{-\infty}^{\infty} \varphi(x) \Lambda^{[n]}(dx) = \frac{(-1)^n}{n!} \int_{-\infty}^{\infty} \frac{d^n}{dx^n} \varphi(x) \Lambda(dx),
\]
(2)

\( \Lambda \) is a distribution function and \( \Lambda^{[n]} \) is the \( n \)-th distributional derivative of \( \Lambda \).

The conditional distribution \( K(z|x_1, x_2) \) obtained by (1) leads us to the following linear functional
on complex-valued functions \( f \), defined on the set of real numbers \( \mathbb{R} \);
\[
K(f|x_1, x_2) = \frac{1}{B(n_1, n_2)} \sum_{i=0}^{n_2-1} \sum_{k=0}^{n_2-i-1} \binom{n_2-i-1}{k} (-1)^k f(x_{2-i}) (n_{i+1} + k)(C(x_{2-i}; x_1, x_2))^{n_{i+1}+k},
\]
(3)

It easily follows that
\[
K(af + bg | x_1, x_2) = aK(f | x_1, x_2) + bK(g | x_1, x_2),
\]
(4)

for any choice of complex-valued functions \( f, g \) and of complex constants \( a, b \). We note that
\[
K(z|x_1, x_2) = K(f_z|x_1, x_2), \\
\]
where \( f_z(x) = (z - x)^{n_1+n_2-1}U(z - x) \) and
\[
K(f_z|x_1, x_2) = \frac{1}{B(n_1, n_2)} \sum_{i=0}^{n_2-1} \sum_{k=0}^{n_2-i-1} \binom{n_2-i-1}{k} (-1)^k \frac{d^{n_2-i-1-k}}{dz^{n_2-i-1-k}} f_z(x_{2-i}),
\]
(4)

where
\[
C_i(n_1, n_2) = (n_{i+1} + k)(n_{i+1} + k + 1)...(n_1 + n_2 - 1).
\]

Also we note that \( U(z - x) = \frac{(-1)^{n_1+n_2-1}}{(n_1+n_2-1)!} \int \frac{d^{n_1+n_2-1}}{dx^{n_1+n_2-1}} f_z(x) \). Thus
\[
P(Z \leq z) = \int_{\mathbb{R}} U(z - x) dF_Z(x) = \int_{\mathbb{R}^2} K(z|x_1, x_2) \prod_{i=1}^{2} F_{X_i}(dx_i),
\]
(5)

can be viewed as:
\[
\int_{\mathbb{R}} \frac{(-1)^{n_1+n_2-1}}{(n_1+n_2-1)!} \frac{d^{n_1+n_2-1}}{dx^{n_1+n_2-1}} f_z(x) dF_Z(x) = \int_{\mathbb{R}^2} K(f_z|x_1, x_2) \prod_{i=1}^{2} F_{X_i}(dx_i),
\]
(5)

Therefore by using property (3) along with (5) and a standard argument in the integration theory,
we obtain that
\[
\int_{\mathbb{R}} \frac{(-1)^{n_1+n_2-1}}{(n_1+n_2-1)!} \frac{d^{n_1+n_2-1}}{dx^{n_1+n_2-1}} f(x) dF_Z(x) = \int_{\mathbb{R}^2} K(f|x_1, x_2) \prod_{i=1}^{2} F_{X_i}(dx_i),
\]
(6)
Proof. Theorem 3.2. Under the assumption that $X_1, X_2$ are independent and continuous,

$$-\text{Beta}(n_1, n_2) \ S^{(n_1 + n_2 - 1)}(F_Z, z) = S^{(n_1 - 1)}(F_{X_1}, z) S^{(n_2 - 1)}(F_{X_2}, z), \quad z \in \mathbb{C} \cap (\text{supp} F_{X_i})^c.$$  

for a suitable $f$. Now (6) together with (2) lead us to

$$\int_R f(x) dF_Z^{(n_1 + n_2 - 1)}(x) = \int_{R^2} K(f|_{X_1}, x_2) \prod_{i=1}^2 F_{X_i}(dx_i),$$

for a suitable $f$, where $F_Z^{(n_1 + n_2 - 1)}$ is the $(n_1 + n_2 - 1)$-th distributional derivative of the distribution of $Z$.

Let us denote the Stieltjes transform of a distribution $H$ by

$$S(H, z) = \int_R \frac{1}{z - x} H(dx),$$

for every $z$ in the set of complex numbers $\mathbb{C}$ which does not belong to the support of $H$, $z \in \mathbb{C} \cap (\text{supp} H)^c$.

The following theorem indicates how the Stieltjes transforms of $Z$ and $X_1, X_2$ are related.

**Theorem 3.2.** Under the assumption that $X_1, X_2$ are independent and continuous,

$$-\text{Beta}(n_1, n_2) \ S^{(n_1 + n_2 - 1)}(F_Z, z) = S^{(n_1 - 1)}(F_{X_1}, z) S^{(n_2 - 1)}(F_{X_2}, z), \quad z \in \mathbb{C} \cap (\text{supp} F_{X_i})^c.$$  

**Proof.** It follows from (7) that

$$S(F_Z^{(n_1 + n_2 - 1)}, z) = \int_{R^2} K(g_z|x_1, x_2) \prod_{i=1}^2 F_{X_i}(dx_i),$$

for $g_z(x) = \frac{1}{z - x}$. Since

$$(n_1 + n_2 - 1)! K(g_z|x_1, x_2) = -\frac{1}{B(n_1, n_2)} \sum_{i=0}^{n_2-1} \sum_{k=0}^{n_1} \frac{(n_i + 1 + k - 1)! (n_2 - i - k - 1)!}{(z - x_{2-i})^{n_2-1}} \frac{(-1)^{n_2-k}}{(C(x_{2-i}; x_1, x_2))^{n_1+k}}$$

$$= -\frac{1}{B(n_1, n_2)} \prod_{i=1}^{n_2} \frac{(-1)^{n_j} (n_j - 1)!}{(z - x_j)^{n_j}}.$$  

Therefore

$$-B(n_1, n_2) \frac{d^{n_1 + n_2 - 1}}{dz^{n_1 + n_2 - 1}} S(F_Z, z) = \int_{R^2} \prod_{j=1}^{n_2} \frac{(-1)^{n_j} (n_j - 1)!}{(z - x_j)^{n_j}} \prod_{i=1}^2 F_{X_i}(dx_i),$$

and

$$-B(n_1, n_2) S^{(n_1 + n_2 - 1)}(F_Z, z) = S^{(n_1 - 1)}(F_{X_1}, z) S^{(n_2 - 1)}(F_{X_2}, z), \quad z \in \mathbb{C} \cap (\text{supp} F_{X_i})^c.$$  

$\square$
Remark 3.3. Van Assche’s result follows from theorem 3.2, when \( n_1 = 1 \) and \( n_2 = 1 \):

\[-S'(F_Z, z) = S(F_{X_1}, z)S(F_{X_2}, z).\]

4 Van Assche’s results

Now we are to review Van Assche’s results for some beta distributions, so the results given by Van Assche is not only for uniform distribution.

**Theorem 4.1.** Assume \( Z \) is randomly weighted average with Beta(2,1) random proportion and \( X_1 \) and \( X_2 \) are random variables with a common distribution function \( F \), then \( Z \) has distribution \( F \) if and only if \( X_1 \) and \( X_2 \) are almost surely constant or have a Cauchy distribution.

**Proof.** The random variable \( Z \) has the same distribution as \( X_1 \) and \( X_2 \). Then it follows from Theorem 3.2 that

\[-\frac{1}{2} S''(F_z, z) = S(F, z)S'(F, z).\]

By using a reduction order in ODE method, the solution of this differential equation in \( \{ z \in \mathbb{C} | \text{Im} z > 0 \} \) is

\[ S(F, z) = \frac{1}{z + c}, \]

where \( c \) is a constant. In order that this is the Stieltjes transform of a probability distribution, we need to have that \( \text{Im} S(F, z) < 0 \) whenever \( \text{Im}(z) > 0 \), therefore

\[ S(F, z) = \frac{1}{z - a + ib}, \quad \text{Im}(z) > 0 \]

where \( a \) is real and \( b \geq 0 \). The case \( b = 0 \) corresponds to \( F(x) = U(x - a) \) whence \( X_1 \) and \( X_2 \) are almost surely constant. When \( b > 0 \) we have

\[ F(x) = \frac{1}{\pi} \int_{-\infty}^{x} \frac{b}{b^2 + (t - a)^2} dt, \]

which is the Cauchy distribution with center \( a \) and spread \( b \).

**Theorem 4.2.** Let \( X_1 \) and \( X_2 \) be i.i.d random variables on \([-1,1]\) and \( Z \) be randomly weighted average with Beta(2,1) random proportion, then \( Z \) is uniformly distributed on \([-1,1]\) if and only if \( X_1 \) and \( X_2 \) have an arcsin distribution on \([-1,1]\).

**Proof.** The random variable \( Z \) has a uniform distribution on \([-1,1]\). Then it follows from theorem 3.2 that

\[ S(F_{X_1}, z)S'(F_{X_1}, z) = -\frac{1}{2} S''(F_Z, z) \]

\[ = -\frac{z}{(z^2 - 1)^2}. \]

The solution \( S(F_{X_1}, z) \) is

\[ S(F_{X_1}, z) = \frac{1}{\sqrt{z^2 - 1}}, \]

which is the Stieltjes transform of the arcsin distribution.
Remark 4.3. It will also easily follow from theorem 3.2 that the arcsin’s result of Van Assche (1987) is not true for $Z$ with some Beta random proportion $(\text{Beta}(3,1))$.

5 Some characterization

In this section, we also mention some applications of theorem 3.2.

Theorem 5.1. Let $X_1$ and $X_2$ be independent random variables and $Z$ be randomly weighted average with $\text{Beta}(2,1)$ random proportion. Then if $X_1$ has uniform distribution on $[-1,1]$, then $X_2$ has arcsin distribution on $[-1,1]$ if and only if $Z$ has semicircle distribution on $[-1,1]$.

Proof. Since the random variable $X_1$ has uniform distribution and $X_2$ has arcsin distribution on $[-1,1]$, then
\[
S(F_{X_1}, z) = \frac{1}{2} (\ln |z + 1| - \ln |z - 1|),
\]
and
\[
S(F_{X_2}, z) = \frac{1}{\sqrt{z^2 - 1}}.
\]
Then it follows from theorem 3.2 that
\[
-\frac{1}{2} S''(F_Z, z) = S(F_{X_2}, z) S'(F_{X_1}, z)
\]
\[
= \frac{1}{\sqrt{z^2 - 1}} \frac{1}{1 - z^2}.
\]
The solution $S(F_Z, z)$ is
\[
S(F_Z, z) = 2(z - \sqrt{z^2 - 1}),
\]
which is the Stieltjes transform of semicircle distribution on $[-1,1]$.

Theorem 5.2. Let $X_1$ and $X_2$ be i.i.d. random variables on $[-1,1]$ and $Z$ be randomly weighted average with $\text{Beta}(2,2)$ random proportion. Then $X_1$ and $X_2$ have uniform distribution on $[-1,1]$, if and only if $Z$ has power semicircle distribution, i.e.,
\[
f(z) = \frac{3(1 - z^2)}{4}, \quad -1 \leq z \leq 1.
\]

Proof. The random variable $Z$ has a power semicircle distribution and
\[
S(F_Z, z) = \frac{3}{2} z + \frac{3}{4} (1 - z^2)(\ln |z + 1| - \ln |z - 1|).
\]
Then it follows from theorem 3.2 that
\[
(S'(F_{X_1}, z))^2 = \frac{1}{6} S'''(F_Z, z)
\]
\[
= \frac{1}{(1 - z^2)^2}.
\]
The solution $S(F_{X_1}, z)$ is
\[
S(F_{X_1}, z) = \frac{1}{2} (\ln |z + 1| - \ln |z - 1|),
\]
which is the Stieltjes transform of uniform distribution on $[-1,1]$.

\[
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\]
Theorem 5.3. Let $X_1$ and $X_2$ be i.i.d random variables on $[0,1]$ and $Z$ be randomly weighted average with Beta(2,2) random proportion. Then $Z$ has Beta(2,2) distribution if and only if $X_1$ and $X_2$ have uniform distribution on $[0,1]$.

Proof. The random variable $Z$ has Beta(2,2) distribution and

$$S(F_Z, z) = 6(z - z^2)(\ln |z| - \ln |z - 1|) + 6z - 3.$$  

Then it follows from theorem 3.2 that

$$(S'(F_{X_1}, z))^2 = -\frac{1}{6} S'''(F_Z, z)$$

$$= \frac{1}{(1-z)^2z^2}.$$  

The solution $S(F_{X_1}, z)$ is

$$S(F_{X_1}, z) = \ln |z| - \ln |z - 1|,$$

which is the Stieltjes transform of uniform distribution on $[0,1]$.

Theorem 5.4. Let $X_1$ and $X_2$ be independent random variables and $Z$ be randomly weighted average with Beta(2,1) random proportion. Then if $X_1$ has uniform distribution on $[0,1]$, then $X_2$ has Beta(2,2) distribution if and only if $Z$ has Beta(2,2) distribution.

Proof. Since the random variable $X_1$ has uniform distribution and $Z$ has Beta(2,2) distribution, then

$$S(F_{X_1}, z) = \ln |z| - \ln |z - 1|,$$

and

$$S(F_Z, z) = 6(z - z^2)(\ln |z| - \ln |z - 1|) + 6z - 3.$$  

Then it follows from theorem 3.2 that

$$\frac{S'(F_{X_1}, z)S(F_{X_2}, z)}{z(1-z)} = \frac{1}{2} S''(F_Z, z)$$

$$= \frac{1}{z(z-1)}(6(z^2 - z)(\ln |z| - \ln |z - 1|) - 6z + 3).$$  

The solution $S(F_{X_2}, z)$ is

$$S(F_{X_2}, z) = 6(z - z^2)(\ln |z| - \ln |z - 1|) + 6z - 3,$$

which is the Stieltjes transform of Beta(2,2) distribution.

Theorem 5.5. Let $X_1$ and $X_2$ be independent random variables and $Z$ be randomly weighted average with Beta(2,2) random proportion. Then if $X_1$ has uniform distribution on $[0,1]$, then $X_2$ has Beta(3,3) distribution if and only if $Z$ has Beta(3,3) distribution.

Proof. Since the random variable $X_1$ has uniform distribution on $[0,1]$ and $Z$ has Beta(3,3) distribution, then

$$S(F_{X_1}, z) = \ln |z| - \ln |z - 1|,$$

and

$$S(F_Z, z) = 30(z^2 - 2z^3 + z^4)(\ln |z| - \ln |z - 1|) - 30z^3 + 45z^2 - 10z - \frac{5}{2}.$$  

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Then it follows from theorem 3.2 that
\[
S'(F_{X_1}, z)S'(F_{X_2}, z) = -\frac{1}{6} S''(F_Z, z)
\]
\[
\frac{1}{z(1-z)} S'(F_{X_2}, z) = \frac{60(z - 3z^2 + 2z^3)(\ln|z| - \ln|z - 1|) - 120z^2 + 120z - 10}{z(1-z)}.
\]
The solution \(S(F_{X_2}, z)\) is
\[
S(F_{X_2}, z) = 30(z^2 - 2z^3 + z^4)(\ln|z| - \ln|z - 1|) - 30z^3 + 45z^2 - 10z - \frac{5}{2},
\]
which is the Stieltjes transform of Beta(3,3) distribution.

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Heart pacemakers synchronization by using Hopf bifurcation

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Abstract

In this work we investigate dynamic of heart pacemakers modeled by the van der Pol oscillators. Since time delays in signal transmission are unavoidable, we consider a system of two van der Pol oscillators with delayed coupling. Then, the system is analyzed from the viewpoint of Hopf bifurcation by using center manifold and bifurcation theory. This bifurcation is necessary to determine synchronization region. Studying of synchronization is important to prevent the blocking arrhythmias which are caused by lack of synchronization between the two major heart pacemakers.

Keywords: Delay differential equations, Bifurcation, Periodic solutions, Synchronization.

Mathematics Subject Classification: 34K99, 34K18, 34K13, 34D06.

1 Introduction

The dynamics of coupled oscillators have been of interest because of their applications in physics, laser dynamic, biological science and so on. For example authors in [1, 2, 3, 4, 5, 6, 7, 8] focused on the nonlinear dynamics of oscillators. One of the most important oscillators is van der Pol which was originally employed to describe relaxation oscillators in electronic circuits, and has been frequently used in theoretical models of the cardiac rhythm [2, 3, 6, 7].

This equation is very useful in the phenomenological modeling of natural systems, especially the heartbeat, since it displays many of those features supposed to occur in the biological setting as limit cycle, synchronization and chaos. The normal cardiac rhythm is primarily generated by the SA (Sino-Atrial) node, which is considered as the normal pacemaker. Besides, the AV (Atrio-Ventricular) node is another pacemaker. Each one of these presents an actuation potential that is fundamental to the heart dynamics. Usually, two oscillators are considered representing the SA and AV nodes, however, it is observed that these two oscillators are not enough to reproduce the ECG (Electrocardiography) signal. The authors in [3] proposed a mathematical model to describe heart rhythms considering three modified van der Pol oscillators connected with time delay couplings. They showed numerical simulations are carried out presenting qualitative agreement with the general heart rhythm behavior but they did not study analytically this model. Moreover, Gholizade-Narm [2] studied the coupled van der Pol oscillators with different parameters - as model of SA-AV nodes in heart -

\[
\begin{align*}
\dot{x} + \varepsilon_1 (x^2 - 1) \dot{x} + \omega_1^2 x &= \alpha_1 (y(t) - x(t)) \\
\dot{y} + \varepsilon_2 (y^2 - 1) \dot{y} + \omega_2^2 y &= \alpha_2 (x(t) - y(t))
\end{align*}
\]  

and

\[
\begin{align*}
\dot{x} + \varepsilon_1 (x^2 - 1) \dot{x} + \omega_1^2 x &= \alpha_1 (y(t - \tau_1) - x(t)) \\
\dot{y} + \varepsilon_2 (y^2 - 1) \dot{y} + \omega_2^2 y &= \alpha_2 (x(t - \tau_2) - y(t)).
\end{align*}
\]
Then the characteristic equation of the linearization of system (3) at the trivial solution is

\[ T(i) = \pm \frac{\sqrt{3}}{2} \{ (2(1 - \gamma_i) - \epsilon^2) \pm [\epsilon^4 + 4 \alpha_i^2 - 4(1 - \gamma_i)\epsilon^2]^{\frac{1}{2}} \} \]^\frac{1}{2} \]

where \( T = T_1 + T_2 \) and \( \alpha_1 \alpha_2 = \alpha^2 \). This equation has a pair of simple purely imaginary roots when

\[ w_{\pm} = \frac{\sqrt{2}}{2} \{ (2(1 - \gamma_1) - \epsilon^2) \pm [\epsilon^4 + 4 \alpha_1^2 - 4(1 - \gamma_1)\epsilon^2]^{\frac{1}{2}} \} \]

at \( T = T_j^\pm \), such that

\[ T_j^+ = \left\{ \begin{array}{ll}
\frac{1}{w_+} \{ 2\pi - \arcsin \left( \frac{\epsilon}{|\alpha|} w_+ \right) + j\pi \}; & w_+^2 - 1 + \gamma_1 \geq 0,
\frac{1}{w_+} \{ \arcsin \left( \frac{\epsilon}{|\alpha|} w_+ \right) + j\pi \}; & w_+^2 - 1 + \gamma_1 < 0.
\end{array} \right. \]

and

\[ T_j^- = \frac{1}{w_-} \{ \arcsin \left( \frac{\epsilon}{|\alpha|} w_- \right) + j\pi \} \]

where \( j \in \{0, 1, 2, \ldots \} \) and \( T_j^+ > 0 \).

**Lemma 2.1.** Suppose that \( w_{\pm} \) and \( T_j^\pm \) are defined by (5), (6) and (7).

(i) If either \((1 - \gamma_1)^2 - \alpha^2 < 0\), or \(2(1 - \gamma_1) - \epsilon^2 > 0\) and \( \epsilon^4 + 4 \alpha_1^2 - 4(1 - \gamma_1)\epsilon^2 = 0\), then system (3) undergoes a Hopf bifurcation at the origin when \( T = T_j^+ \), \( j = 0, 1, \ldots \).

(ii) If \((1 - \gamma_1)^2 - \alpha^2 > 0\) , \(2(1 - \gamma_1) - \epsilon^2 > 0\) and \( \epsilon^4 + 4 \alpha_1^2 - 4(1 - \gamma_1)\epsilon^2 > 0\), then system (3) undergoes a Hopf bifurcation at the origin when \( T = T_j^- \), \( j = 0, 1, \ldots \).

**Remark 2.2.** The curve which are stated in Lemma 2.1 shows the synchronization boundary in (3). We reduce system (3) to a system of ordinary differential equations on a finite-dimensional center manifold the corresponding to an infinite-dimensional phase space.

If conditions (ii) or (iii) in lemma 2.1 hold, then system (3) has a pair of purely imaginary roots \( \pm i w_{\pm} \) at the critical value of time delay \( T = T_* \). The corresponding equation to system (3) on the center manifold at \( \mu = 0 \) is

\[ \dot{z}(t) = i w T_* z(t) + g_{20} \frac{z^2}{2} + g_{11} zz + g_{02} \frac{\dot{z}^2}{2} + g_{21} \frac{z^2 \dot{z}}{2} + \ldots \]
such that $g_{20} = g_{11} = g_{02} = 0$ and

$$g_{21} = T_*(q_2^* - q_3^* - 2q_3q_4 + q_2q_4^*)$$

$(9)$

where $q = (q_1, q_2, q_3, q_4)$ and $q^* = (q_1^*, q_2^*, q_3^*, q_4^*)$ are corresponding eigenvectors to $\pm iw$. For analyzing (3), we need to compute the following quantities:

$$C_1 = \frac{g_{21}}{2}$$

$(10)$

$$K_2 = -Re\{C_1(0)\}$$

$(11)$

$$\beta_2 = 2Re\{C_1(0)\}$$

$(12)$

$$T_2 = -Im\{C_1(0)\} + K_2 Im\{\lambda'(T_*)\}$$

$(13)$

$$\frac{d\lambda}{dT}\bigg|_{T=T_*} = \frac{-iw(-w^2 - 1 + \gamma_1 + i\epsilon w)}{(-\epsilon - T_* w^2 + T_* - T_* \gamma_1) + i\epsilon (2 - T_* \epsilon)}$$

$(14)$

**Theorem 2.3.** Suppose that system (3) satisfied lemma 2.1 (ii) or (iii). Then the Hopf bifurcation occurs for system (3) and according to the sign of $K_2, T_2, \beta_2$, we have

- if $K_2 > 0$ ($K_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist at $T = T_*$,

- if $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable (unstable),

- if $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating solutions increases (decreases).

It is easy to see that the periodic solutions in (8) are equivalent to quasiperiodic solutions in system (3). This quasiperiodic solutions reproduce the ECG signals. Increasing (Decreasing) of their periods can show beating too slow or bradycardia (beating too fast or tachycardia).

**References**


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Synchronization of commensurate fractional-order chaotic systems via sliding mode controllers

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Abstract
This paper proposes a sliding mode controller to synchronize chaotic fractional-order systems in master–slave structure that may be identical or different. Based on the Lyapunov direct method, the stability analysis is performed for the proposed control method. By using this stability theorem, we show that error dynamics between the two chaotic fractional-order systems is globally asymptotically stable under certain conditions. Finally, three numerical simulations are presented to show the effectiveness of the proposed controller.

Keywords: Chaos synchronization, chaotic fractional-order systems, sliding mode controller.

1 Introduction

Over recent decades, several thousand publications have appeared due to the fact that chaotic behavior was discovered in numerous systems in mechanics, laser and radio physics, hydrodynamics, chemistry, biology and medicine, electronic circuits, economical systems [1]. Over the last two decades, synchronization of chaotic systems has become more interesting to researchers in different fields. The problem of designing a system, whose behavior mimics another chaotic system, is called synchronization. Synchronization has been extensively studied in various disciplines such as physics, chemistry, biology and neuroscience since the Dutch physicist Chistiaan Huygens first observed the anti-phase synchronization phenomenon of two pendulum clocks in the 17th century [2]. The two chaotic systems are generally called master and slave systems. Recently, synchronization of chaotic fractional-order systems starts to attract increasing attention due to its potential applications in secure communication and control processing [2]. It should be noted that most of previous works focus on synchronization between identical fractional-order systems. However, in practice most systems are non-identical and parameter mismatches are inevitable because of noise or other uncertain factors. In fact, it has been reported that parameter mismatches are one of the most frequent types of errors made by end-user web application developers [3]. In recent years, some papers proposed active sliding mode controllers to synchronize chaotic fractional-order systems in master slave structure. Master and slave systems may be identical or different. In [4] based on stability theorems in the fractional calculus, analysis of stability for such configurations have been performed via an active sliding mode control method. In [4] according to the active control design procedure, the nonlinear part of the error dynamics has been eliminated. The active sliding mode control technique is a discontinuous control strategy that relies on two stages. The first stage is to select an appropriate active controller to facilitate the design of the sequent sliding
mode controller. The second stage is to design a sliding mode controller to achieve the synchronization. The elimination of nonlinear terms in synchronization systems and its error dynamics via active control design procedures, applied in [5] [6]. But in this paper we will not use active control to eliminate the nonlinear terms. Our designed mechanism will be useful in implementation of controllers, that we will not need active control designing procedures in implementation. We propose a proper sliding surface to stabilize the error dynamics asymptotically. In some papers like [7] that the sliding mode control is used for a financial chaotic fractional-order system, the form of the system is not general, and the sliding surface is introduced exclusively for this system. In this paper, we propose a general form for chaotic fractional-order systems (both master and slave that may be identical or different) and introduce the sliding surface with flexibility in design of terms for stabilizing the error dynamic asymptotically. We find a Lyapunov function for this designed controller and show that under certain conditions the error dynamics is asymptotically stable and the system is synchronized. The rest of this paper is organized as follows: Section 2 provides the preliminary backgrounds. In section 3, two existing techniques of approximated solution of fractional differential equations are briefly explained. The general form of the chaotic fractional-order systems that work as master and slave is described in section 4. The design of fractional-order sliding mode controller and its stability analysis is given in section 5. In section 6, the numerical simulations are presented. Finally, some concluding remarks are given in section 7.

2 Preliminary background

As an extension of ordinary integration and differentiation, fractional calculus has been known since the development of the classic calculus. Although there are several different definitions for fractional derivatives of order $\alpha (\alpha > 0)$, the Grünwald–Letnikov, the Riemann–Liouville and the Caputo definitions are three most commonly used ones [8]. The Grünwald-Letnikov (GL) fractional derivative with fractional-order is defined by

$$GLD^q f(t) = \lim_{h \to 0, mh=t} h^{-q} \sum_{k=0}^{m} \frac{(-1)^k}{k!} f(t-kh)$$

The Riemann-Liouville (RL) derivative can be written as

$$RLD^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau$$

where $n$ is the smallest integer larger than $q$, i.e., and denotes the gamma function defined by:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

The Caputo derivative is defined by

$$CD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q-n+1}} d\tau$$

where $n-1 < q < n$.

If the initial condition assumed homogenous then the equations with Riemann–Liouville operators are equivalent to those with Caputo operators [8]. The Laplace transforms of the Riemann–Liouville fractional integral and derivative are given as follows [9]:

$$L \{0D^q f(t) \} = \begin{cases} s^q f(s), & q \leq 0 \\ \sum_{k=0}^{n-1} s^k D_0^{q-k} f(0), & n-1 < q \leq n \in N \end{cases}$$
Contrary to the Laplace transform of the Riemann-Liouville fractional $f(t)$ derivative, only integer order derivatives of function are appeared in the Laplace transform of the Caputo fractional derivative because its Laplace transform is:

$$L \{\alpha D^\alpha_t f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{q-1-k} f^{(k)}(0)$$

(6)

where $n - 1 < q \leq n \in N$.

For a wide class of functions, the Grünwald-Letnikov definition and the Riemann-Liouville definition are equivalent. In this paper we note $\alpha D^\alpha_t$ to indicate the Caputo fractional derivative.

### 3 Solution approximation methods

The numerical calculation of a fractional differential equation (FDE) is more complicated than that of an ordinary differential equation. The approximation method developed in the related literature is numerically stable and can be applied to both linear and nonlinear FDEs. This time-domain method is a generalization of the Adams-Bashford-Moulton algorithm [10], [11]. This method is based on a predictor-corrector scheme using the Caputo definition [12], [13]. Consider the following differential equation:

$$\alpha D^\alpha_t y(t) = r(t, y(t)), \quad 0 \leq t \leq T$$

$$y^{(k)}(0) = y^k_0, \quad k = 0, 1, \ldots, m - 1 \quad (m = [\alpha])$$

(7)

(8)

This differential equation is equivalent to Volterra integral equation [14]

$$y(t) = \sum_{k=0}^{[q]-1} y^k_0 \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t \frac{f(s, y(s))}{(\tau - s)^{1-q}} \, ds.$$  

(9)

Set $h = T/N, t_n = nh, \ n = 0, 1, \ldots, N$ Then Eq. (9) can be discretized as

$$y_n(t_{n+1}) = \sum_{k=0}^{[q]-1} y^k_0 \frac{h^k}{k!} + \frac{h^q}{\Gamma(q+1)} \left( r(t_{n+1}) + y_n^p(t_{n+1}) \right)$$

$$\quad + \frac{h^q}{\Gamma(q+1)} \sum_{j=0}^{n} a_{j,n+1} r(t_j, y_h(t_j)),$$

(10)

Where predicted value $y_n^p(t_{n+1})$ is determined by

$$y_n^p(t_{n+1}) = \sum_{k=0}^{[q]-1} y^k_0 \frac{h^k}{k!} + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} r(t_j, y_h(t_j)),$$

(11)

in which

$$a_{j,n+1} = \begin{cases} 
  n^{q+1} - (n-q)(n+1)^q, & j = 0 \\
  (n-j+2)^q + (n-j)^{q+1} - 2(n-j+1)^q - 2(n-j+1)^{q+1}, & 1 \leq j \leq n \\
  1, & j = n + 1 
\end{cases}$$

(12)

$$b_{j,n+1} = \frac{h^q}{q} ((n+1-j)^q - (n-j)^q)$$

(13)

This approximation has an estimation error that is described as follows:

$$\max_{j=0,1,\ldots,N} |y(t_j) - y_h(t_j)| = O(h^p)$$

(14)
To achieve the goal we define the synchronization errors dynamics as follows
\[
\begin{align*}
D_i^q x &= f_i(x, y, z) \\
D_i^{q_i} y &= f_i(x, y, z) ; \quad 0 < q_i \leq 1, \quad i = 1, 2, 3 \\
D_i^{q_3} z &= f_i(x, y, z)
\end{align*}
\]

with initial condition \((x_0, y_0, z_0)\). The above system can be discretized as follows:
\[
\begin{align*}
x_{n+1} &= x_0 + \frac{h^n}{\Gamma(n+1)} \left[ f_1(x_{n+1}, y_{n+1}, z_{n+1}) + \sum_{j=0}^{n} \alpha_{1,j,n+1} f_1(x_j, y_j, z_j) \right] \\
y_{n+1} &= y_0 + \frac{h^n}{\Gamma(n+1)} \left[ f_2(x_{n+1}, y_{n+1}, z_{n+1}) + \sum_{j=0}^{n} \alpha_{2,j,n+1} f_2(x_j, y_j, z_j) \right] \\
z_{n+1} &= z_0 + \frac{h^n}{\Gamma(n+1)} \left[ f_3(x_{n+1}, y_{n+1}, z_{n+1}) + \sum_{j=0}^{n} \alpha_{3,j,n+1} f_3(x_j, y_j, z_j) \right]
\end{align*}
\]

where,
\[
\begin{align*}
x_{n+1} &= x_0 + \frac{1}{\Gamma(n+1)} \sum_{j=0}^{n} \beta_{1,j,n+1} f_1(x_j, y_j, z_j) \\
y_{n+1} &= y_0 + \frac{1}{\Gamma(n+1)} \sum_{j=0}^{n} \beta_{2,j,n+1} f_2(x_j, y_j, z_j) \\
z_{n+1} &= z_0 + \frac{1}{\Gamma(n+1)} \sum_{j=0}^{n} \beta_{3,j,n+1} f_3(x_j, y_j, z_j)
\end{align*}
\]

\[
\alpha_{i,j,n+1} = \begin{cases} 
\frac{n^{q_i+1} - (n - q_i) (n + 1)^{q_i}}{(n - j + 2)^{q_i+1} + (n - j)^{q_i+1}} - 2(n - j + 1)^{q_i+1}, & j = 0 \\
\frac{1}{n - j} \frac{n^{q_i+1} - (n - j)^{q_i}}{(n - j)^{q_i+1}}, & 1 \leq j \leq n \\
\end{cases}
\]

and
\[
\beta_{i,j,n+1} = \frac{h^n}{\Gamma(q_i)} ((n + 1 - j)^{q_i} - (n - j)^{q_i}).
\]

In the simulations of this paper, we use the predictor-correctors scheme based method to solve the fractional-order differential equations.

4 System description

Consider a general form for chaotic commensurate fractional-order system of order \(q\) (0 < \(q\) < 1) described by the following nonlinear fractional-order differential equation:
\[
t_0 D_t^q x_1 = f_1(x_1)
\]

where \(x_1 \in \mathbb{R}^3\) denotes the system’s 3-dimensional state vector and \(f_1 : \mathbb{R}^3 \to \mathbb{R}^3\) show both linear and nonlinear part of the system. Eq. 20 represents the master system. The controller is added into the slave system, so it is given by
\[
t_0 D_t^q x_2 = f_2(x_2) + u(t)
\]

where, \(x_2 \in \mathbb{R}^3\) is the slave system’s 3-dimensional state vector, and \(f_2 : \mathbb{R}^3 \to \mathbb{R}^3\) implies the same role as the \(f_1\) for the master system. Synchronization of the systems means finding a control signal \(u(t) \in \mathbb{R}^3\) that makes states of the slave system to evolve as the states of the master system. To achieve the goal we define the synchronization errors dynamics as follows
\[
t_0 D_t^q e = t_0 D_t^q x_2 - t_0 D_t^q x_1 = f_2(x_2) - f_1(x_1) + u(t) = \varphi(x_1, x_2) + u(t)
\]
where \( e = x_2 - x_1 \) and \( \varphi(x_1, x_2) = f_2(x_2) - f_1(x_1) \). The aim is to design the signal control \( u(t) \in \mathbb{R}^3 \) such that

\[
\lim_{t \to \infty} \| e(t) \| = 0
\]  

Hence, synchronization between two commensurate fractional order chaotic systems will be done.

## 5 Fractional-order sliding mode controller design and stability analysis

To design a sliding mode controller, we have two steps: first constructing a sliding surface that represents a desired system dynamics, and next develop a switching control law such that a sliding mode exists on every point of the sliding surface, and any states outside the surface are driven to reach the surface in a finite time.

### 5.1 Fractional-order sliding mode controller design

The sliding surface is chosen as:

\[
S(x_1, x_2, t) = t_0 D_t^{\alpha-1} e - D^{-1}(g(x_1, x_2))
\]

If the condition \( S(x_1, x_2, t_0) = 0 \) in any point of trajectories of errors dynamics verified then the reaching state trajectories to the sliding surface will be guaranteed. If the condition \( \dot{S}(x_1, x_2, t) = 0 \) is satisfied for all \( t \geq t_0 \) of trajectories of errors dynamics then settling state trajectories on the sliding surface will be guaranteed.

So, one easily conclude that

\[
\dot{S}(t) = \frac{d}{dt} S(t) = \frac{d}{dt} \left[ t_0 D_t^{\alpha-1} e - D^{-1}g(x_1, x_2) \right] = t_0 D_t^{\alpha} e - g(x_1, x_2) = 0
\]

Then, one conclude that

\[
t_0 D_t^{\alpha} e = g(x_1, x_2)
\]

We define the general structure of the controller as

\[
u(t) = u_{eq}(t) + u_r(t)
\]

According to sliding mode control theory and using Equations (22) and (24), the equivalent control law is concluded as

\[
u_{eq}(t) = t_0 D_t^{\alpha} e - \varphi(x_1, x_2) = g(x_1, x_2) - \varphi(x_1, x_2)
\]

To satisfy the sliding condition, the discontinuous reaching law is chosen as follows

\[
u_r(t) = K_r sign(S)
\]

Where \( K \) is the gain of the sliding mode controller and

\[
\text{sign}(S) = \begin{cases} 
+1 & S > 0 \\
0 & S = 0 \\
-1 & S < 0 
\end{cases}
\]

According to Equation (27), the total control law can be concluded as

\[
u(t) = g(x_1, x_2) - \varphi(x_1, x_2) + K_r sign(S)
\]
5.2 Stability analysis of the proposed control system

**Theorem 5.1.** Consider the master-slave chaotic commensurate fractional-order system described in (20), (21) and the control law (31). If the controller gain $K_r < 0$, then the closed loop system is synchronized.

**Proof.** The lyapunov function candidate assumed as follow

$$V = \frac{1}{2} S^2$$  \hspace{1cm} (32)

Then, the lyapunov function $V(x_1, x_2, t)$ is positive definite. Its derivative $V$ along any system trajectory is

$$\dot{V} = S \dot{S} = S [\dot{x}(t) e - g(x_1, x_2)] = S [\dot{x}(x_1, x_2) + u(t) - g(x_1, x_2)]$$

$$= S [\dot{x}(x_1, x_2) + g(x_1, x_2) - \frac{\partial g(x_1, x_2)}{\partial x_1} \frac{\partial x_1}{\partial x_1} - K_r \text{sign}(S) - g(x_1, x_2)]$$

$$= S [K_r \text{sign}(S)] = K_r |S| < 0$$  \hspace{1cm} (33)

So, a Lyapunov function has been found that satisfies the conditions of the Lyapunov theorem. Thus, the errors dynamics system in the presence of the controller (31) is globally asymptotically stable and the closed loop system is synchronized. \hfill \Box

6 Numerical simulations

In this section we present three illustrative examples that simulated in MALAB software to verify the proficiency of the designed control procedure. In the first example we consider an identical fractional-order Lu’s system as master and slave that found by Lu and Chen [15]. The so-called Lu’s system is known as a bridge between the Lorenz system and Chen’s system. In the second simulation we applied our proposed method to an identical fractional-order Liu’s system as master and slave that found by Liu [16]. In the third simulation the synchronization scheme is applied to non-identical fractional-order Lu (as master) and fractional-order Liu systems.

**Example 6.1.** synchronization between two fractional-order Lu’s systems. In first simulation we consider two fractional-order Lu’s systems as follows [1]:

$$\begin{align*}
0D^q_t x(t) &= a(y(t) - x(t)) \\
0D^q_t y(t) &= -x(t)z(t) + cy(t) \\
0D^q_t z(t) &= x(t)y(t) - bz(t)
\end{align*}$$

(34)

Now, we apply our approach to synchronize two fractional-order Lu’s systems with the following initial conditions regarding equations (20), (21) and (34): $(x_{10}, y_{10}, z_{10}) = (18, -20, 25)$, $(x_{20}, y_{20}, z_{20}) = (-22, -19, -17)$ and $(a, b, c) = (28, 5, 16)$. Assume that order of master and slave systems is $0.3$ $(q = 0.1)$. The simulation results are given in Fig. 1. The results show that the designed controller is effectively able to synchronize two fractional-order Lu systems.
Example 6.2. Synchronization between two fractional-order Liu’s systems. In second simulation we consider two fractional-order Liu’s systems that described as follows [1]:

\[
\begin{align*}
0D_t^q x(t) &= -ax(t) - ey^2(t), \\
0D_t^q y(t) &= by(t) - kx(t)z(t), \\
0D_t^q z(t) &= -cz(t) + mx(t)y(t), \\
\end{align*}
\]

We apply our approach to synchronize two fractional-order Liu’s systems with the following initial conditions regarding equations (20), (21) and (35): \((x_{10}, y_{10}, z_{10}) = (3, 1, -5), (x_{20}, y_{20}, z_{20}) = (0, 5, 9)\) and \((a, b, c, e, k, m) = (1, 3, 5, 1, 4, 4)\). Assume that order of master and slave systems is 0.3 \((q = 0.1)\). The simulation results are given in Fig. 2 The results show that the designed controller is effectively able to synchronize two fractional-order Liu systems.
Example 6.3. Synchronization between two fractional-order Lu and Liu systems. In third simulation we apply our approach to synchronize fractional-order Lu’s and fractional order Liu systems that have been referred to in equations (34) and (35). In this case, we assume that the Lu system drives the Liu system. The initial conditions that applied in this simulation are the same that present in the simulations 6.1 and 6.2 (the initial condition in master system is considered). The numerical simulation has carried out using MATLAB subroutines written based on predictor-correctors method. The time step size employed in this simulation is 0.001 (h = 0.001). The simulation results are given in Fig.3.
References


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Study two-layered blood flow In straight artery

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Abstract
A two-fluid model for blood flow through a straight tube has been developed; the model consists of a core (suspension of RBCs) and peripheral plasma layer. Blood flow is assumed to be represented by a couple stress fluid. The main idea of this work is a mathematical study of this phenomenon in order to obtain analytical expressions for flow rate.

Keywords: Blood Flow, Body Acceleration, Two-Fluid Model

Mathematics Subject Classification: 92B05, 92C35

1 Introduction
In normal life the human body is quite often subjected to acceleration. In many situations like travel in vehicles, aircraft or spacecraft, while jogging using Lathe machine or jackhammer or driving a tractor, athletes and sportsmen for their sudden movements, the human body is subjected to vibrations. N.Mustapha [1] has investigated two layered poiseuille flow model for blood flow through arteries of small diameter and arterioles.

2 Analysis
Stokes[2], has given the equation of motion and constitutive equations for a couple stress fluid on the basis of couple stress in elastic materials. The equations of motion for the flow of an incompressible fluid with couple stress are[2].

\[ T_{ji,j} + \rho f_i = \rho \frac{DV_i}{Dt}, \quad E_{i,j,k} T_{jk} + M_{ji,j} + \rho C_i = 0 \] (1)

where \( f_i \) is body force per unit mass, \( C_i \) is body moment per unit mass, \( V_i \) is velocity vector, \( T^S_{jk} \) and \( T^A_{jk} \) are the symmetric and anti-symmetric parts of the stress tensor \( T_{jk} \) respectively, \( \rho \) is the density of the fluid, \( M_{ij} \) is the couple stress tensor and the terms have their usual meaning of tensor analysis. The constitutive for an isotropic incompressible fluid with couple stress are

\[ T^S_{ji} = -p S_{ij} + 2 \mu d_{ij}, \quad \mu_{ij} = 4 \eta W_{j,i} + 4 \eta' W_{j,i} \] (2)

Where \( \mu \) is the shear viscosity which is different from the solvent, \( \eta, \eta' \) are constants associated with couple stress, \( p \) is pressure, \( \mu_{ij} \) is deviatoric part of \( M_{ij}, d_{ij} \) is symmetric part of velocity gradient and \( W_{i} \) vorticity vector. The governing equations for steady viscous, incompressible couple stress fluid through narrow circular tube is given by[2]

\[ \eta \nabla^4 u - \mu \nabla^2 u = -\frac{\partial p}{\partial z} \] (3)
The pressure gradient and body acceleration are given by [3]

\[- \frac{\partial p}{\partial z} = A_0 + A_1 \quad , \quad G = a_0 \cos \phi \quad (4)\]

Where \( A_0 \) is the steady-state part of the pressure gradient \( A_1 \) is the amplitude of the oscillatory part of the pressure gradient, \( a_0 \) is the amplitude of body acceleration \( \phi \) is its phase difference \( z \) is the axial distance.

The steady couple stress equation, stokes [2]in cylindrical polar coordinates under the periodic body acceleration can be written in the form

\[\nabla^4 u_1 - \tilde{\alpha}^2 \nabla^2 u_1 - \bar{\alpha}^2 (A_0 + A_1 + a_0 \cos \phi) = 0 \quad (5)\]

\[\nabla^2 u_2 + (A_0 + A_1 + a_0 \cos \phi) = 0 \quad (6)\]

Where \( \tilde{\alpha}^2 = R^2(\bar{\alpha}) \) couple stress parameter. The initial boundary condition for this problem are:

\( u_1 \) and \( \nabla^2 u_2 \) are all finite at \( r = a \)

\( u_1 = u_2 \) and \( \frac{\partial u_2}{\partial r} = 0 \) at \( r = h \)

\( u_2 = 0 \) and \( \nabla^2 u_2 = 0 \) at \( r = 1 \).

Employing the hankel transform to equations (5),(6) and using the boundary condition we get [4]

\[u_1(r) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\lambda_n \bar{\alpha}^2}{(\lambda_n^2 + \lambda_n^2)} (A_0 + A_1 + a_0 \cos \phi)J_1(h \lambda_n) \cdot \frac{J_0(r \lambda_n)}{[\lambda_n^4 + \lambda_n^2 \bar{\alpha}^2 - \frac{2 \lambda_n^2}{\lambda_n^2} h \lambda_n^2] \cdot \left[ \frac{2}{h} - \frac{\lambda_n J_1(h \lambda_n)}{h J_0(h \lambda_n)} + h \lambda_n (\bar{\alpha}^2 + \bar{\alpha}^2) \frac{J_1(h \lambda_n)}{J_0(h \lambda_n)} \right]} \quad (7)\]

and

\[u_2(r) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\lambda_n \bar{\alpha}^2}{(\lambda_n^2 + \lambda_n^2)} (A_0 + A_1 + a_0 \cos \phi)J_1(h \lambda_n) \cdot \frac{1}{[\lambda_n^4 + \lambda_n^2 \bar{\alpha}^2 - \frac{2 \lambda_n^2}{\lambda_n^2} h \lambda_n^2] \cdot \left[ \frac{2}{h} - \frac{\lambda_n J_1(h \lambda_n)}{h J_0(h \lambda_n)} + h \lambda_n (\bar{\alpha}^2 + \bar{\alpha}^2) \frac{J_1(h \lambda_n)}{J_0(h \lambda_n)} \right]} \quad (8)\]

Where \( \lambda_n \) are roots of Bessel function \( J_0(\lambda_n) = 0 \).

The expression for flow rate \( Q \), which is the volume of the suspension flowing per unit time across a cross-section of the tube, for two phase fluid is given by the equation

\[Q = Q_1 + Q_2 \quad (9)\]

where

\[Q_1 = 2\pi \int_{h}^{1} ru_1(r)dr \quad \text{and} \quad Q_2 = 2\pi \int_{h}^{1} ru_1(r)dr \quad (10)\]

then

\[Q_1 = \frac{4\pi}{\pi} \sum_{n=1}^{\infty} \frac{\lambda_n \bar{\alpha}^2}{(\lambda_n^2 + \lambda_n^2)} (A_0 + A_1 + a_0 \cos \phi)J_1(h \lambda_n) \cdot \frac{1}{[\lambda_n^4 + \lambda_n^2 \bar{\alpha}^2 - \frac{2 \lambda_n^2}{\lambda_n^2} h \lambda_n^2] \cdot \left[ \frac{2}{h} - \frac{\lambda_n J_1(h \lambda_n)}{h J_0(h \lambda_n)} + h \lambda_n (\bar{\alpha}^2 + \bar{\alpha}^2) \frac{J_1(h \lambda_n)}{J_0(h \lambda_n)} \right]} \quad (11)\]

\[Q_2 = \frac{2\pi}{\pi} \sum_{n=1}^{\infty} \frac{\lambda_n \bar{\alpha}^2}{(\lambda_n^2 + \lambda_n^2)} (A_0 + A_1 + a_0 \cos \phi)J_1(h \lambda_n) \cdot \frac{1}{[\lambda_n^4 + \lambda_n^2 \bar{\alpha}^2 - \frac{2 \lambda_n^2}{\lambda_n^2} h \lambda_n^2] \cdot \left[ \frac{2}{h} - \frac{\lambda_n J_1(h \lambda_n)}{h J_0(h \lambda_n)} + h \lambda_n (\bar{\alpha}^2 + \bar{\alpha}^2) \frac{J_1(h \lambda_n)}{J_0(h \lambda_n)} \right]} \quad (12)\]
3 Main Result

The present analytical study of the steady two-layered blood flow with periodic body acceleration through rigid, straight tube has been computed for derived expression of flow rate if both the layers (core and plasma region). Hence, the present mathematical model gives a simple form of velocity expression for the blood flow to that it will help not people working in the field of physiological fluid dynamics but also to the medical practitioners having elementary knowledge of mathematical.

References


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On fuzzy subgroups in fuzzy algebra and group theory

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Abstract
In this paper, we introduce the concept of fuzzy sets, and fuzzy subgroups. Then by using the definition of fuzzy subgroups and some of their basic properties we define some new subgroups of a group with respect to a fuzzy subgroup. First we introduce some new concepts of fuzzy group theory and then by using the definitions we prove some theorems and results on fuzzy group theory.

Keywords: fuzzy sets, fuzzy algebra, fuzzy subgroups, fuzzy cosets.

Mathematics Subject Classification: 53A15

1 Introduction
The concept of fuzzy sets was first introduced by Zadeh in [1]. The concept of fuzzy relation on a set was defined by Kuroki [3] and other authors like Rosenfeld [2]. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld [2]. Fuzzy normal subgroups were studied by Wu [5,6] and Dib [7] and Kumar [4]. In this study we define some new special subgroups with respect to a fuzzy subgroup.

2 Preliminaries

Definition 2.1. Let $X$ be a nonempty set. A fuzzy subset of $X$ is a function from $X$ into $[0,1]$. The set of all fuzzy subsets of $X$ is called the fuzzy power set of $X$ and is denoted by $FP(X)$.

Definition 2.2. Let $\mu \in FP(X)$. Then the set $\{\mu(x)|x \in X\}$ is called the image of $\mu$ and is denoted by $\mu(X)$ or $Im(\mu)$. The set $\{x|x \in X, \mu(x) > 0\}$, is called the support of $\mu$ and denoted by $\mu^*$.

Definition 2.3. Let $f$ be a function from $X$ into $Y$, and $\mu \in FP(X)$ and $\nu \in FP(Y)$. define the fuzzy subset $f(\mu) \in FP(Y)$ and $f^{-1}(\nu) \in FP(X)$ by $\forall y \in Y$:

$$f(\mu)(y) = \begin{cases} \lor\{\mu(x)|x \in X, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and $\forall x \in X$,

$$f^{-1}(\nu)(x) = \nu(f(x)).$$

Definition 2.4. Let $G$ be a group and $\mu \in FP(G)$. Then $\mu$ is called a fuzzy subgroup of $G$ if:

1. $\mu(xy) \geq \mu(x) \land \mu(y) \forall x, y \in G$
2. $\mu(x^{-1}) = \mu(x) \forall x \in G$
Definition 2.5. Let \( G \) be a group we denote the set of all fuzzy subgroups of \( G \) by \( F(G) \). We let \( \mu_* := \{ x \in G | \mu(x) = \mu(e) \} \) and \( \mu^* := \{ x \in G | \mu(x) > 0 \} \).

Theorem 2.6. [9] Let \( G \) be a group and \( \mu \in F(G) \) then \( \mu_* \) and \( \mu^* \) are subgroups of \( G \).

Theorem 2.7. [8] Let \( H \) be a group and \( \nu \in F(H) \). Let \( f \) be a homomorphism of \( G \) into \( H \) then \( f^{-1}(\nu) \in F(G) \).

Definition 2.8. Let \( G \) be a group and \( \mu \in F(G) \). For any \( a \in G \), \( \mu \) defined by \( (a_\mu)(x) = \mu(a^{-1}x) \), \( \forall x \in G \) is called the right fuzzy coset of \( G \) determined by \( a \) and \( \mu \). The left fuzzy coset is defined similarly.

Theorem 2.9. [9] Let \( \mu \in F(G) \). Then for all \( x, y \in G \) we have:

1. \( x\mu = y\mu \Leftrightarrow x\mu_* = y\mu_* \)
2. \( \mu x = \mu y \Leftrightarrow \mu_* x = \mu_* y \)

Definition 2.10. [9] Let \( \mu \in F(G) \). Then for all \( x, y \in G \) we have:

1. \( x\mu = y\mu \Leftrightarrow x\mu^* = y\mu^* \)
2. \( \mu x = \mu y \Leftrightarrow \mu^* x = \mu^* y \)

3 Main theorems

In this section we introduce some definitions and some subsets of a group and then prove some theorems and results in group theory.

Theorem 3.1. Let \( G \) be an abelian group and \( \mu \in F(G) \). Define the set \( \mu^n_* := \{ x \in G | \mu(x^n) > 0 \} \). Then \( \mu^n_* \) is a subgroup of \( G \).

Theorem 3.2. Let \( G \) be an abelian group and \( \mu \in F(G) \). Define the set \( \mu^n_* := \{ x \in G | \mu(x^n) = \mu(e) \} \). Then \( \mu^n_* \) is a subgroup of \( G \).

Lemma 3.3. Let \( G \) be a group and \( \mu \in F(G) \) Then \( \mu_* \) is a normal subgroup of \( \mu^n_* \).

Lemma 3.4. Let \( G \) be a group and \( \mu \in F(G) \) Then \( \mu^* \) is a normal subgroup of \( \mu^n_* \).

Theorem 3.5. [9] Let \( G \) be an abelian group and \( \mu \in F(G) \) and \( f \in \text{Aut}(G) \). Then we have:

1. \( (f^{-1}(\mu))^*_n = f^{-1}(\mu^*_n) \)
2. \( (f^{-1}(\mu))^* = f^{-1}(\mu^*) \)

Theorem 3.6. Let \( G \) be an abelian group and \( \mu \in F(G) \) and \( f \in \text{Aut}(G) \). Then we have:

\( (f^{-1}(\mu))_n^* = f^{-1}(\mu^n_*). \)

Theorem 3.7. Let \( G \) be an abelian group and \( \mu \in F(G) \) and \( f \in \text{Aut}(G) \). Then we have:

\( (f^{-1}(\mu))^n_* = f^{-1}(\mu^n_*). \)

Definition 3.8. A fuzzy subgroup \( \mu \) of a group \( G \) is said to be normalized if there exist \( x \in G \) such that \( \mu(x) = 1 \).

Theorem 3.9. Let \( G \) be a group and \( \mu \) is a normalized fuzzy subgroup of \( F(G) \). We let \( \mu := \{ x \in G | \mu(x) = 1 \} \) then \( \mu \) is a subgroup of \( G \).
Theorem 3.10. Let \( G \) be a group and \( \mu \) is a normalized fuzzy subgroup of \( F(G) \) and \( f \in \text{Aut}(G) \) then we have:
\[
(f^{-1}(\mu)) = (f^{-1}(\mu)).
\]

Theorem 3.11. Let \( G \) be an abelian group and \( \mu \) is a normalized fuzzy subgroup of \( F(G) \). We let \( \mu_n := \{x \in G | \mu(x^n) = 1\} \) then \( \mu_n \) is a subgroup of \( G \).

Theorem 3.12. Let \( G \) be an abelian group and \( \mu \) is a normalized fuzzy subgroup of \( F(G) \) and \( f \in \text{Aut}(G) \) then we have:
\[
(f^{-1}(\mu_n)) = (f^{-1}(\mu))_n.
\]

References


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