Analysis
Generalized frames containing a $g$-Riesz basis

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Abstract
In this paper we introduce the concept of near $g$-Riesz basis and Besselian $g$-frame. We show that under some conditions the kernel of associated synthesis operator for a near $g$-Riesz basis is finite dimensional. Also we investigate the perturbation of near $g$-Riesz basis.

Keywords: $g$-Bessel sequence, $g$-frames, Besselian $g$-frames, near $g$-Riesz basis.

Mathematics Subject Classification: Primary 41A58, 42C15

1 Introduction
The concept of frame was introduced by Duffin and Schaeffer [4] in 1952. Afterward, several generalizations of frames in Hilbert spaces have been proposed [2, 7, 5, 3]. $G$-frames, the most recent generalization of frames, introduced by Wenchang Sun [9].

Definition 1.1. Let $H$ be a Hilbert space and $I$ be a subset of $\mathbb{N}$. A family $\{f_i\}_{i \in I} \subseteq H$ is a frame for $H$, if there exist two positive constants $A, B$ such that

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad (I)$$

for all $f \in H$.

Definition 1.2. A sequence $\{f_i\}_{i \in I} \subseteq H$ is called a Riesz basis for $H$, if $\{f_i\}_{i \in I}$ is complete in $H$, and there exist two positive constants $A$ and $B$ such that for any finite scalar sequence $\{c_i\}$,

$$A \sum_{i \in I} |c_i|^2 \leq \|\sum_{i \in I} c_if_i\|^2 \leq B \sum_{i \in I} |c_i|^2.$$

As usual, we denote by $l^2(I)$ the Hilbert space of all square-summable sequences of scalars $\{c_i\}_{i \in I}$. If $\{f_i\}_{i \in I}$ is a frame for $H$, then $\sum_{i \in I} c_if_i$ converges if $\{c_i\}_{i \in I} \in l^2(I)$. But the converse is not true in general, see [6].

Definition 1.3. We say that a frame $\{f_i\}_{i \in I}$ for $H$ is

- Besselian if, whenever $\sum_{i \in I} c_if_i$ converges, then $\{c_i\}_{i \in I} \in l^2(I)$;
- a near-Riesz basis, if there is a finite set $\sigma$ for which $\{f_i\}_{i \in I \setminus \sigma}$ is a Riesz basis for $H$.

In the rest of this paper, $H$ is a separable Hilbert space and $\{H_i\}_{i \in I}$ is a sequence of Hilbert spaces, where $I$ is a countable subset of $\mathbb{N}$. 
**Definition 1.4.** We call a sequence \( \{ \Lambda_i \in B(H, H_i) : i \in I \} \) a g-frame (or generalized frame) for \( H \) with respect to \( \{ H_i \}_{i \in I} \) if there exist two positive constants \( A \) and \( B \) such that

\[
A \| f \|^2 \leq \sum_{i \in I} \| \Lambda_i f \|^2 \leq B \| f \|^2,
\]

for all \( f \in H \). We call \( A \) and \( B \) the lower and upper g-frame bounds, respectively. We call \( \{ \Lambda_i \}_{i \in I} \) a tight g-frame if \( A = B \) and Parseval g-frame if \( A = B = 1 \). The sequence \( \{ \Lambda_i \in B(H, H_i) : i \in I \} \) is called the g-Bessel sequence if the right hand inequality in (2) holds for all \( f \in H \).

Let \( \Lambda_i \in B(H, H_i) \) be given for all \( i \in I \). Let us define the set

\[
\left( \sum_{i \in I} \oplus H_i \right)_{l_2} = \left\{ \{ f_i \} : f_i \in H_i, \sum_{i \in I} \| f_i \|^2 < \infty \right\}
\]

with this inner product given by \( \langle \{ f_i \}, \{ g_i \} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle \). It is clear that \( \left( \sum_{i \in I} \oplus H_i \right)_{l_2} \) is a Hilbert space with respect to the pointwise operations. It is proved in [8], if \( \Lambda_i \in B(H, H_i) : i \in I \) is a g-Bessel sequence for \( H \) then the operator

\[
T : \left( \sum_{i \in I} \oplus H_i \right)_{l_2} \to H
\]

defined by

\[
T(\{ f_i \}) = \sum_{i \in I} \Lambda_i^*(f_i)
\]

is well defined and bounded and its adjoint is \( T^*f = \{ \Lambda_i f \}_{i \in I} \) for all \( i \in I \). Also, a sequence \( \{ \Lambda_i \in B(H, H_i) : i \in I \} \) is a g-frame if and only if the operator \( T \) defined by (3), is a bounded and onto operator. We call the operators \( T \) and \( T^* \), synthesis and analysis operators, respectively.

**Definition 1.5.** A sequence \( \{ \Lambda_i \in B(H, H_i) : i \in I \} \) is called a g-Riesz basis for \( H \) with respect to \( \{ H_i \}_{i \in I} \), if there exist two positive constants \( A \) and \( B \) such that for any finite subset \( F \subseteq I \) and \( g_i \in H_i \)

\[
A \sum_{i \in F} \| g_i \|^2 \leq \sum_{i \in F} \| \Lambda_i^* g_i \|^2 \leq B \sum_{i \in F} \| g_i \|^2
\]

and \( \{ \Lambda_i \in B(H, H_i) : i \in I \} \) is g-complete i.e.

\[
\{ f \mid \Lambda_i f = 0, i \in I \} = \{ 0 \}.
\]

**2 Main Result**

**Definition 2.1.** We say that a g-frame \( \{ \Lambda_i \}_{i \in I} \) for \( H \) w.r. to \( \{ H_i \}_{i \in I} \) is

1. a Besselian g-frame, if \( \sum_{i \in I} \Lambda_i^* g_i \) converges then \( g_i \in \left( \sum_{i \in I} \oplus H_i \right)_{l_2} \) for each \( i \in I \) and \( g_i \in H_i \);

2. a near g-Riesz basis, if there exists a finite set \( \sigma \) for which \( \{ \Lambda_i \}_{i \in I \setminus \sigma} \) is a g-Riesz basis for \( H \) w. r. to \( \{ H_i \}_{i \in I \setminus \sigma} \).

**Example 2.2.** Let \( A = [0, +\infty) \) with the Lebesgue measure \( \mu \) and \( A_1 = [0, 5) \), \( A_2 = [5, 10) \) and \( A_n = [n - 3, n - 2) \) for all integers \( n \geq 3 \). Let \( H = L^2(A) \), \( H_i = L^2(A_i) \) and \( \Lambda_i \) be the orthogonal projection from \( H \) onto \( H_i \). Then \( \{ \Lambda_i \}_{i \in \mathbb{N}} \) is near g-Riesz basis for \( H = L^2(A) \), because \( \{ \Lambda_i \}_{i \geq 3} \) is g-Riesz basis for \( H = L^2(A) \).
Theorem 2.3. Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a $g$-frame for $H$ with respect to $\{H_i\}_{i \in I}$ and let $T$ be the associated synthesis operator for $\Lambda$ such that $\dim(Ker\, T) < \infty$. Then there is a $g$-Riesz basis $\{\Theta_i\}_{i \in I}$ for $H$ with respect to $\{W_i\}_{i \in I}$, where $W_i$ is a closed subspace of $H_i$, such that $\Theta_i = \Lambda_i$ and $W_i = H_i$ for all $i \in I$ except finitely many $i$.

Proposition 2.4. Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a $g$-frame for $H$ with respect to $\{H_i\}_{i \in I}$. If $\Lambda = \{\Lambda_i\}_{i \in I}$ is a near $g$-Riesz basis, then $\Lambda$ is a Besselian $g$-frame.

Theorem 2.5. Suppose that $\dim H_i < \infty$ for each $i \in I$. Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a $g$-frame for $H$ w. r. to $\{H_i\}_{i \in I}$ if $\Lambda = \{\Lambda_i\}_{i \in I}$ is a near $g$-Riesz basis then $\dim(Ker\, T) < \infty$.

Proposition 2.6. Let $\{\Lambda_i\}_{i \in I}$ be a Besselian $g$-frame for $H$ with bounds $A,B$ and $\{\Theta_i\}_{i \in I} \in B(H, H_i)$ be a sequence of bounded operators such that for any finite subset $J \subseteq I$,

$$
\left\| \sum_{i \in J} (\Lambda_i^* f_i - \Theta_i^* f_i) \right\| \leq \lambda \left\| \sum_{i \in J} \Lambda_i^* f_i \right\| + \mu \left\| \sum_{i \in J} \Theta_i^* f_i \right\|,
$$

where $0 \leq \lambda, \mu < 1$ and $f_i \in H_i$ for all $i \in J$. Then $\{\Theta_i\}_{i \in I}$ is a Besselian $g$-frame for $H$ with the bounds

$$
\left( \frac{(1 - \lambda)\sqrt{A}}{1 + \mu} \right)^2 \text{ and } \left( \frac{(1 + \lambda)\sqrt{B}}{1 - \mu} \right)^2.
$$

Theorem 2.7. Let $\{\Lambda_i\}_{i \in I}$ be a $g$-Riesz basis for $H$ with bounds $A, B$ and $\{\Theta_i\}_{i \in I} \in B(H, H_i)$ be a sequence of bounded operators such that for any finite subset $J \subseteq I$,

$$
\left\| \sum_{i \in J} (\Lambda_i^* f_i - \Theta_i^* f_i) \right\| \leq \lambda \left\| \sum_{i \in J} \Lambda_i^* f_i \right\| + \mu \left\| \sum_{i \in J} \Theta_i^* f_i \right\| + \gamma \left( \sum_{i \in J} \| f_i \|^2 \right)^{\frac{1}{2}},
$$

where $0 \leq \max\{\lambda + \frac{2}{\sqrt{A}}, \mu\} < 1$ and $f_i \in H_i$ for all $i \in J$. Then $\{\Theta_i\}_{i \in I}$ is a $g$-Riesz basis for $H$ with the bounds

$$
\left( \frac{(1 - \lambda)\sqrt{A}}{1 + \mu} - \gamma \right)^2 \text{ and } \left( \frac{(1 + \lambda)\sqrt{B} + \gamma}{1 - \mu} \right)^2.
$$

Especially, if $\{\Lambda_i\}_{i \in I}$ is a near $g$-Riesz basis for $H$, then $\{\Theta_i\}_{i \in I}$ is a near $g$-Riesz basis for $H$.

References


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Some coincidence point results for $E$-contractions in uniform spaces

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Abstract

In this talk, some well-known fixed and common fixed point results for weak contractions and weak expansions are generalized from metric to uniform spaces equipped with an $A$- or $E$-distance. Also, we present some coincidence point results in partially ordered uniform spaces.

Keywords: Separated uniform space; $A$-distance; Fixed point; Coincidence point.

Mathematics Subject Classification: 47H10, 54H25, 05C40.

1 Introduction

In [4], the concept of a $w$-distance was introduced on a uniform space and a few fixed point theorems were generalized from metric to uniform spaces. Later in [1], $A$- and $E$-distances were defined in view of extending some well-known fixed point results of weak contractions and weak expansions as well as $w$-distance results.

We first recall a few number of basic notions in uniform spaces. All of them can be found, e.g., in [5, Chapter 9Aris Aghanians and Kourosh Nourouzi].

A uniform space is a nonempty set $X$ together with a nonempty collection of subsets of $X \times X$, called entourages, satisfying certain properties. A uniform space $X$ is separated if the intersection of all entourages coincides with the diagonal \{(x, x) : x \in X\}.

A sequence \{x_n\} in a uniform space $X$ is said to be convergent to $x \in X$ if for each entourage $U$, there exists an $n_0 \geq 1$ such that $(x_n, x) \in U$ for all $n \geq n_0$, and Cauchy if for each entourage $U$, there exists an $n_0 \geq 1$ such that $(x_m, x_n) \in U$ for all $m, n \geq n_0$.

Recall that two self-maps $T$ and $S$ on $X$ are said to be commuting if $TS = ST$ and weakly compatible if they commute at their coincidence points, i.e., $Tx = Sx$ implies $TSx = STx$.

2 Main Results

Throughout this section, we suppose that $X$ is a separated uniform space and $T$ and $S$ are two self-maps on $X$.

Definition 2.1 ([1]). A function $p : X \times X \to \mathbb{R}^+$ is called an $A$-distance on $X$ if given any entourage $U$, there exists a $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $(x, y) \in U$. If, further, $p$ satisfies the triangular inequality, then $p$ is called an $E$-distance on $X$.

If $p$ is an $A$-distance on $X$, then a sequence \{x_n\} is said to be $p$-convergent to $x \in X$, denoted by $x_n \xrightarrow{p} x$, whenever $p(x_n, x) \to 0$ as $n \to \infty$, and $p$-Cauchy whenever $p(x_m, x_n) \to 0$ as $m, n \to \infty$.

The uniform space $X$ is called $p$-complete if every $p$-Cauchy sequence in $X$ is $p$-convergent to a point of $X$. 

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The following lemma contains the most important and useful properties of $A$-distances defined on separated uniform spaces.

**Lemma 2.2** ([1]). Let $p$ be an $A$-distance on $X$ and let $\{x_n\}$ and $\{y_n\}$ be two arbitrary sequences in $X$. Then the following assertions hold:

1) If $x_n \xrightarrow{p} x$ and $x_n \xrightarrow{p} y$, then $x = y$. In particular, if $p(z,x) = p(z,y) = 0$ for some $z$, then $x = y$.

2) If $x_n \xrightarrow{p} x$ and $p(x_n,y_n) \rightarrow 0$, then $\{y_n\}$ converges to $x$.

3) If $\{x_n\}$ is $p$-Cauchy, then it is Cauchy.

We next have two new concepts.

**Definition 2.3** ([1]). Let $p$ be an $A$-distance on $X$.

ii) The uniform space $X$ is called $p$-bounded if $\sup\{p(x,y) : x,y \in X\} < \infty$.

ii) The mapping $T$ is called $p$-continuous on $X$ if $x_n \xrightarrow{p} x$ implies $Tx_n \xrightarrow{p} Tx$.

**Theorem 2.4** ([2]). Suppose that $p$ is an $A$-distance on $X$ such that $X$ is $p$-bounded and $T$ and $S$ satisfy the following conditions:

i) $T$ and $S$ are weakly compatible;

ii) The range of $T$ is contained in the range of $S$ and $T(X)$ or $S(X)$ is $p$-complete;

iii) The contractive condition

$$p(Tx,Ty) \leq \varphi(p(Sx,Sy))$$

holds for all $x,y \in X$, where $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing and satisfies $\varphi^n(t) \rightarrow 0$ for each $t > 0$.

Then $T$ and $S$ have a unique common fixed point.

As two immediate consequences of Theorem 2.4, we have:

**Corollary 2.5** ([2]). Let $p$ be an $A$-distance on $X$ such that $X$ is $p$-complete and $p$-bounded and $T$ satisfy the contractive condition

$$p(Tx,Ty) \leq \varphi(p(x,y))$$

for all $x,y \in X$, where $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing and satisfies $\varphi^n(t) \rightarrow 0$ for each $t > 0$. Then $T$ has a unique fixed point.

**Corollary 2.6** ([2]). Let $p$ be an $A$-distance on $X$ such that $X$ is $p$-complete and $p$-bounded and $T$ be surjective and satisfy the contractive condition

$$p(x,y) \leq \varphi(p(Tx,Ty))$$

for all $x,y \in X$, where $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing and satisfies $\varphi^n(t) \rightarrow 0$ for each $t > 0$. Then $T$ has a unique fixed point.

**Theorem 2.7** ([2]). Suppose that $p$ is an $E$-distance on $X$ and $T$ and $S$ satisfy the following conditions:

i) $T$ and $S$ are weakly compatible;

ii) The range of $T$ is contained in the range of $S$ and $T(X)$ or $S(X)$ is $p$-complete;
iii) The contractive condition

$$\psi(p(Tx, Ty)) \leq p(Sx, Sy)$$

holds for all \(x, y \in X\), where \(\psi : \mathbb{R}^+ \to \mathbb{R}^+\) satisfies \(t < \psi(t)\) for each positive number \(t\) and the following property:

For each nonincreasing sequence \(\{t_n\}\) of nonnegative numbers, if \(\{t_n\}\) and \(\{\psi(t_n)\}\) have the same limit, then \(t_n \to 0\) as \(n \to \infty\).

Then \(T\) and \(S\) have a unique common fixed point.

Now, we mention our results in partially ordered uniform spaces. In the next results, the uniform space \(X\) is endowed with a partial order \(\preceq\) and comparability and monotonicity are with respect to \(\preceq\).

**Theorem 2.8** ([3]). Let \(p\) be an \(A\)-distance on \(X\) such that \(X\) is \(p\)-bounded and \(T\) and \(S\) satisfy the following conditions:

i) \(T\) and \(S\) are \(p\)-continuous and commute on \(X\);

ii) The range of \(T\) is contained in the range of \(S\) and \(T(X)\) or \(S(X)\) is \(p\)-complete;

iii) \(T\) is \(S\)-nondecreasing and there exists an \(x_0 \in X\) such that \(gx_0 \preceq fx_0\);

iv) The contractive condition

$$p(Tx, Ty) \leq \varphi(p(Sx, Sy))$$

holds for all \(x, y \in X\) such that \(Sx\) and \(Sy\) are comparable, where \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) is nondecreasing and satisfies \(\varphi^n(t) \to 0\) for each \(t > 0\).

Then \(T\) and \(S\) have a coincidence point. Moreover, if \(Tu\) and \(T^2u\) are comparable for some coincidence point \(u\), then \(T\) and \(S\) have a common fixed point.

**Corollary 2.9** ([3]). Suppose that \(p\) is an \(A\)-distance on \(X\) such that \(X\) is \(p\)-complete and \(p\)-bounded. Let \(T\) be nondecreasing and \(p\)-continuous on \(X\) and satisfy the contractive condition

$$p(Tx, Ty) \leq \varphi(p(x, y))$$

for all comparable \(x, y \in X\), where \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) is nondecreasing and satisfies \(\varphi^n(t) \to 0\) for each \(t > 0\). If there exists an \(x_0 \in X\) with \(x_0 \preceq Tx_0\), then \(T\) has a fixed point.

**References**


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Compact homomorphisms between extended Lipschitz algebras

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Abstract

Let \((X, d)\) be a compact metric space and let \(K\) be a compact subset of \(X\). For \(\alpha \in (0, 1]\), we denote by \(\text{Lip}(X, K, d^\alpha)\) the complex algebra of complex-valued continuous functions \(f\) on \(X\) for which

\[
\rho_{\alpha, K}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in K, x \neq y \right\} < \infty.
\]

In particular, if \(K = X\), we write \(\text{Lip}(X, d^\alpha)\) instead of \(\text{Lip}(X, K, d^\alpha)\). The algebras \(\text{Lip}(X, d^\alpha)\) and \(\text{Lip}(X, K, d^\alpha)\) are called Lipschitz algebras and extended Lipschitz algebras, respectively. In this note, we study unital compact homomorphisms between extended Lipschitz algebras and determine the spectrum of unital compact endomorphisms of these algebras.

Keywords: Compact homomorphism, Extended Lipschitz algebra, Natural Banach function algebra, Unital homomorphism.

Mathematics Subject Classification: Primary 46J10; Secondary 46J15.

1 Introduction

Let \(A\) and \(B\) be commutative semi-simple unital Banach algebras with maximal ideal spaces \(M_A\) and \(M_B\). If \(T\) is a unital homomorphism from \(A\) into \(B\), then \(T\) is continuous and there exists a continuous map \(\psi : M_B \to M_A\) such that \(\hat{T} f = \hat{f} \circ \psi\) for all \(f \in A\), where \(\hat{g}\) is the Gelfand transform of \(g\).

Let \(X\) and \(Y\) be compact Hausdorff spaces and \(A\) and \(B\) be Banach function algebras on \(X\) and \(Y\), respectively. If \(\varphi : Y \to X\) is a continuous map such that \(f \circ \varphi \in B\) for all \(f \in A\), then \(T : A \to B\), defined by \(Tf = f \circ \varphi\), is a unital homomorphism and called the homomorphism induced by \(\varphi\). If \(A\) is natural, then every unital homomorphism \(T : A \to B\) is induced by a unique continuous map \(\varphi : Y \to X\).

**Definition 1.1.** Let \((X_1, d_1)\) and \((X_2, d_2)\) be metric spaces. The map \(\varphi : X_2 \to X_1\) is called a Lipschitz mapping from \((X_2, d_2)\) into \((X_1, d_1)\), if there exists a positive constant \(M\) such that

\[
d_1(\varphi(s), \varphi(t)) \leq M d_2(s, t),
\]

for all \(s, t \in X_2\).

Let \((X, d)\) be a compact metric space. For \(\alpha \in (0, 1]\), we denote by \(\text{Lip}(X, d^\alpha)\) the standard Lipschitz algebra of order \(\alpha\), which first introduced by D. R. Sherbert in 1963 [4]. D. R. Sherbert studied the unital homomorphisms between Lipschitz algebras in [4].

Let \((X, d)\) be a compact metric space and let \(K\) be a compact subset of \(X\). For \(\alpha \in (0, 1]\), we define

\[
\text{Lip}(X, K, d^\alpha) = \{ f \in C(X) : f|_K \in \text{Lip}(K, d^\alpha) \}.
\]
Then $\text{Lip}(X, K, d^\alpha)$ is a Banach function algebra on $X$ under the norm

$$||f||_{\alpha, K} = ||f||_X + p_{\alpha, K}(f),$$

where

$$||f||_X = \sup\{|f(x)| : x \in X\},$$

and

$$p_{\alpha, K}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in K, \ x \neq y \right\}.$$

Throughout this section, we assume that $(X_j, d_j)$ is a compact metric space, $K_j$ is a compact subset of $X_j$, $\alpha_j \in [0, 1]$ and $A_j = \text{Lip}(X_j, K_j, d_j^\alpha)$, where $j \in \{1, 2\}$.

We first give a sufficient condition for which a continuous mapping $\varphi$ from $(X_2, d_2)$ into $(X_1, d_1)$ induces a unital compact homomorphism $T$ from $A_1$ into $A_2$.

**Theorem 2.1.** Let $\varphi$ be a continuous mapping from $(X_2, d_2)$ to $(X_1, d_1)$. If $\varphi(K_2) \subseteq K_1$ and $\varphi|_{K_2}$ is a supercontraction from $(K_2, d_2^\alpha)$ to $(K_1, d_1^\alpha)$, then $\varphi$ induces a unital compact homomorphism $T$ from $A_1$ into $A_2$.

The following result is a modification of a result due to H. Kamowitz [2, Theorem 1.7] for natural Banach function algebras on connected compact Hausdorff spaces.

**Theorem 2.2.** Let $X$ be a connected, compact Hausdorff space and let $B$ be a natural Banach function algebra on $X$. If $T : B \to B$ is a unital compact endomorphism of $B$, then there exists a continuous self-map $\varphi$ of $X$ such that $\varphi$ induces $T$ and $\bigcap_{n=1}^{\infty} \varphi_n(X)$ is a singleton, where $\varphi_n$ denotes the $n$th iterate of $\varphi$ for all $n \in \mathbb{N}$.

**Theorem 2.3.** Let $(X, d)$ be a compact metric space and let $K$ be a compact subset of $X$. Suppose $\alpha \in (0, 1]$. If $\varphi$ is continuous self-map of $X$ with $\varphi(X) \subseteq K$ and $\varphi|K$ is a supercontraction from $(K, d^\alpha)$ to $(K, d^\alpha)$, then $\varphi$ induces a compact endomorphism $T$ of $\text{Lip}(X, K, d^\alpha)$. Moreover, if $X$ is connected, then $\sigma(T) = \{0, 1\}$.

Kamowitz, Scheinberg and Wortman in [3] and Kamowitz and Scheinberg in [3] constructed an example of nontrivial unital compact endomorphism of two regular commutative semi-simple Banach algebras with connected maximal ideal spaces. Since, as we now show, there exist a compact connected metric space $(X, d)$ and a nonempty proper compact subset $K$ of $X$ and a nonconstant continuous self-map $\varphi$ of $X$ with $\varphi(X) \subseteq K$ and $\varphi|K$ is a supercontraction from $(K, d)$ on $(X, d)$ the resulting Banach algebra $\text{Lip}(X, K, d)$ is then another example of a regular commutative semi-simple Banach algebra with connected maximal space that has a nontrivial unital compact endomorphism $T$ with $\sigma(T) = \{0, 1\}$.

**Example 2.4.** Let $X = [\frac{1}{4}, 1]$ and define $d : X \times X \to \mathbb{R}$ by

\[
\begin{align*}
    d(x, y) &= \sqrt{|x - y|} \quad (\frac{1}{4} \leq x \leq \frac{1}{2}, \quad \frac{1}{4} \leq y \leq \frac{1}{2}), \\
    d(x, y) &= \sqrt{\frac{1}{2} - x} + \sqrt{\frac{1}{2} - y} \quad (\frac{1}{4} \leq x \leq \frac{1}{2}, \quad \frac{1}{2} \leq y \leq 1), \\
    d(x, y) &= (x - \frac{1}{2}) + \sqrt{\frac{1}{2} - y} \quad (\frac{1}{4} \leq x \leq 1, \quad \frac{1}{4} \leq y \leq \frac{1}{2}), \\
    d(x, y) &= |x - y| \quad (\frac{1}{2} \leq x \leq 1, \quad \frac{1}{2} \leq y \leq 1).
\end{align*}
\]
Suppose $K = \left[ \frac{1}{2}, 1 \right]$ and $\varphi : X \rightarrow X$ defined by

$$
\varphi(x) = \begin{cases} 
2x & (\frac{1}{2} \leq x \leq \frac{1}{2}) \\
1 & (\frac{1}{2} \leq x \leq 1)
\end{cases}
$$

Then

(i) $(X, d)$ is connected, compact metric space,

(ii) $K$ is a compact set in the metric space $(X, d)$,

(iii) $\varphi(X) \subseteq K$ and $\varphi$ is a supercontraction from $(X, d)$ to $(X, d)$,

(iv) $\varphi$ induces a nontrivial unital compact endomorphism of $\text{Lip}(X, K, d)$.

We now give a necessary condition for which a unital homomorphism $T : A_1 \rightarrow A_2$ to be compact.

**Theorem 2.5.** If $T$ is a unital compact homomorphism from $A_1$ into $A_2$, then there exists a continuous mapping $\varphi$ from $(X_2, d_2^2)$ to $(X_1, d_1^1)$ such that $\varphi|_K$ is a supercontraction from $(K_2, d_2^2)$ to $(X_1, d_1^1)$ and $\varphi$ induces $T$.

**Corollary 2.6.** Let $(X, d)$ be a compact metric space and let $K$ be a compact subset of $X$. Suppose that $\alpha \in (0, 1]$ and $A = \text{Lip}(X, K, d^\alpha)$. If $T$ is a unital compact endomorphism of $A$, then there exists a continuous mapping $\varphi$ from $(X, d)$ to $(X, d)$ such that $\varphi|_K$ is a supercontraction from $(K, d^\alpha)$ to $(X, d^\alpha)$ and $\varphi$ induces $T$.

**References**


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Duals of \( *-g \)-frames in Hilbert \( C^* \)-module

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Abstract
Certain facts about frames are extended for generalized frames \((g-\) frames) in Hilbert \( C^* \)-modules. The \( C^* \)-algebraic version of frames and generalized frames are considered in some papers. The paper presents some characterizations of dual \( *-g \)-frames for Hilbert \( C^* \)-modules with respect to \( *-g \)-Bessel sequences.

Keywords: dual \( *-g \)-frame, \( *-g \)-Bessel sequence, \( *-g \)-frame operator, Hilbert \( C^* \)-module.

Mathematics Subject Classification: Primary: 42C15; Secondary: 47C15, 46L99.

1 Introduction
Frames were first introduced in 1952 by Duffin and Schaeffer [2] in the study of nonharmonic fourier series. Frames possess many nice properties which make them very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields. The theory of frames has been rapidly and various generalizations of frames in Hilbert spaces and Hilbert \( C^* \)-modules.

In this paper, we characterize dual of all of \( *-g \)-frames with respect to \( *-g \)-Bessel sequences in Hilbert \( C^* \)-modules.

Let us recall some definitions and basic properties of \( C^* \)-algebras and Hilbert \( C^* \)-modules that we need in the rest of the paper. For more details, we refer the interested reader to [2, 4].

Let \( \mathcal{A} \) be a \( C^* \)-algebra. A pre-Hilbert \( C^* \)-module \( \mathcal{H} \) is a Hilbert \( C^* \)-module or, simply, a Hilbert \( \mathcal{A} \)-module if it is complete with respect to the norm \( \| f \| = \| \langle f, f \rangle \|_{\mathcal{A}} \). Let \((\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})\) and \((\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})\) be Hilbert \( \mathcal{A} \)-modules. A map \( T : \mathcal{H} \rightarrow \mathcal{K} \) is said to be adjointable if there exists a map \( T^* : \mathcal{K} \rightarrow \mathcal{H} \) satisfying \( \langle Tf, g \rangle_{\mathcal{K}} = \langle f, T^*g \rangle_{\mathcal{H}} \) whenever \( f \in \mathcal{H} \), and \( g \in \mathcal{K} \). The map \( T^* \) is called the adjoint of \( T \). The class of all adjointable maps from \( \mathcal{H} \) into \( \mathcal{K} \) is denoted by \( B_{\mathcal{A}}(\mathcal{H}, \mathcal{K}) \) and the class of all bounded \( \mathcal{A} \)-module maps from \( \mathcal{H} \) into \( \mathcal{K} \) is denoted by \( B_b(\mathcal{H}, \mathcal{K}) \) where a \( \mathcal{A} \)-module map is a \( \mathcal{C}^* \) and \( \mathcal{A} \)-linear map. It is known that \( B_{\mathcal{A}}(\mathcal{H}, \mathcal{K}) \subseteq B_b(\mathcal{H}, \mathcal{K}) \).

Throughout the paper, we fix the notations \( \mathcal{A} \) and \( J \) for a given unital \( C^* \)-algebra and a finite or countably infinite index set, respectively. Also, the Hilbert \( \mathcal{A} \)-modules \( \mathcal{H} \) and \( \bigoplus_{j \in J} \mathcal{K}_j \), for \( j \in J \), are assumed to be finitely or countably generated and ordered pairs \( \{(\Lambda_j, \mathcal{K}_j) : j \in J\} \) consisting of Hilbert \( \mathcal{A} \)-modules \( \bigoplus_{j \in J} \mathcal{K}_j \) and operators \( \Lambda_j \in B_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_j) \). The mapping \( \pi_j \) sending \( (f_j)_{j \in J} \in \bigoplus_{j \in J} \mathcal{K}_j \) to \( f_j \) is called the \( j^{th} \) projection operator from \( \bigoplus_{j \in J} \mathcal{K}_j \) onto \( \mathcal{K}_j \).

2 Main Result
Before, authors [3] extended the concept of \( g \)-frames from Hilbert spaces to Hilbert \( C^* \)-modules. By a \( g \)-frames for \( \mathcal{H} \) we mean a family of ordered pairs \( \{(\Lambda_j, \mathcal{K}_j) : j \in J\} \) satisfying

\[
A\langle f, f \rangle \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq B\langle f, f \rangle
\]  

(1)
for all $f \in \mathcal{H}$ and some positive constants $A, B$ independent of $f$.

The $*g$-frames are $C^*$-algebra version of $g$-frames, actually we need strictly positive elements of $C^*$-algebra $A$ instead of positive real numbers.

A $*g$-frame for $\mathcal{H}$ is a collection of ordered pairs \( \{ (\Lambda_j, K_j) : j \in J \} \) such that

\[
A(f, f)^* A \leq \sum_{j \in J} (\Lambda_j f, \Lambda_j f) \leq B(f, f) B^*,
\]

(2)

for all $f \in \mathcal{H}$ and strictly nonzero elements $A$ and $B$ in $A$, [2]. (Throughout the paper, series like (2) are assumed to be convergent in the norm sense.)

The numbers $A$ and $B$ are called lower and upper $*g$-frame bounds, respectively. The sequence \( \{ (\Lambda_j, K_j) : j \in J \} \) is called to be a $*g$-Bessel sequence for $\mathcal{H}$ if it has the upper bound condition in (2) if \( \{ (\Lambda_j, K_j) : j \in J \} \) is a $*g$-frame for $\mathcal{H}$ with an upper bound $B$, then $\{ \Lambda_j \}_{j \in J}$ is uniformly bounded by $\|B\|$, [1].

To throw more light on the subject and understand the use of the concepts, we include a example of nontrivial $*g$-frame.

**Example 2.1.** Let $A = \ell^\infty$ and let $\mathcal{H} = C_0$, the Hilbert $A$-module of the set of all null sequences equipped with the $A$-inner product and the $A$-module action

\[
((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) = (x_i y_i)_{i \in \mathbb{N}}, \quad (a_i)_{i \in \mathbb{N}}(x_i)_{i \in \mathbb{N}} = (a_i x_i)_{i \in \mathbb{N}}, \forall (x_i)_{i \in \mathbb{N}} \in \mathcal{H}, \quad (a_i)_{i \in \mathbb{N}} \in A.
\]

Let $j \in \mathbb{N}$ and $(a_i)_{i \in A} = (1 + \frac{1}{i})_{i \in \mathbb{N}}$. Define $\Lambda_j \in B_*(\mathcal{H})$ by $\Lambda_j(x_i)_{i \in \mathbb{N}} = (\delta_j x_i)_{i \in \mathbb{N}}$. We observe that

\[
\sum_{j \in \mathbb{N}} (\Lambda_j x, \Lambda_j x) = ((1 + \frac{1}{i})^2 x_i, x_i)_{i \in \mathbb{N}} = (1 + \frac{1}{i})_{i \in \mathbb{N}} (x, x) (1 + \frac{1}{i})_{i \in \mathbb{N}}, \quad \forall x = (x_i)_{i \in \mathbb{N}} \in \mathcal{H}.
\]

Thus $\{ (\Lambda_j, \mathcal{H}) \}_{j \in J}$ is a tight $*g$-frame with bounds $(1 + \frac{1}{i})_{i \in \mathbb{N}}$. (The element $(1 + \frac{1}{i})_{i \in \mathbb{N}}$ is strictly nonzero in $A$).

For a sequence $\{ (\Lambda_j, K_j) : j \in J \}$, the pre-$*g$-frame operator $\theta$ from $\mathcal{H}$ into $\oplus_{j \in J} K_j$ is defined by $\theta f = (\Lambda_j f)_{j \in J}$. Moreover, $\theta^*(k_j)_{j \in J} = \sum_{j \in J} \Lambda_j^* k_j$ for all $(k_j)_{j \in J} \in K_j$. The adjointable map $\theta^*$ is called the synthesis operator of $\{ \Lambda_j \}_{j \in J}$ and $*g$-frame operator $S$ is defined by $S = \theta^* \theta$ for the $*g$-frame $\{ (\Lambda_j, K_j)_{j \in J} \}$, [2].

The relation between the $*g$-Bessel sequences and their pre-frame operators plays an important role in characterization of duals of $*g$-frames.

**Proposition 2.2 (1).** Let $\theta$ be an adjointable map from $\mathcal{H}$ into $\oplus_{j \in J} K_j$. Define $\Lambda_j f = \pi_j \theta f$ for $f \in \mathcal{H}$ and $j \in J$. Then $\Lambda_j$ is an adjointable map for all $j \in J$ and $\{ (\Lambda_j, K_j)_{j \in J} \}$ is a $*g$-Bessel sequence with the upper bound $\|\theta\|$.

**Definition 2.3.** A $*g$-frame $\{ (\Omega_j, K_j)_{j \in J} \}$ is a dual $*g$-frame for a given $*g$-frame $\{ (\Lambda_j, K_j)_{j \in J} \}$ if $\sum_{j \in J} \Lambda_j^* \Omega_j = I$.

In particular, the $*g$-frame $\{ (\Lambda_j, K_j)_{j \in J} \}$ is called the canonical dual $*g$-frame.

We now give a characterization of dual $*g$-frames in terms of right inverses of the synthesis operators.

**Theorem 2.4.** Let $\{ (\Lambda_j, K_j)_{j \in J} \}$ be a $*g$-frame in $\mathcal{H}$ with the pre-$*g$-frame operator $\theta$, the $*g$-frame operator $S$ and the canonical dual $*g$-frame $\{ (\Lambda_j, K_j)_{j \in J} \}$. Let $\{ (\Omega_j, K_j)_{j \in J} \}$ be an arbitrary dual $*g$-frame of $\{ (\Lambda_j, K_j)_{j \in J} \}$ with the pre-$*g$-frame operator $\eta$. Then the following assertions are true.

1. $\theta^* \eta = I$. 

2. \( \Omega_j = \pi_j \eta, \) for all \( j \in J. \)

3. If \( \eta^* : \mathcal{H} \to \mathcal{K}_j \) is any adjointable right inverse of \( \theta^* \), then \( \{ (\pi_j \eta^*, \mathcal{K}_j) \}_{j \in J} \) is a dual \(*\)-g-frame of \( \{ (\Lambda_j, \mathcal{K}_j) \}_{j \in J} \) with the pre-\(*\)-g-frame operator \( \eta^* \).

4. Every adjointable right inverse \( \eta^* \) of \( \theta^* \) is of the form

\[
\eta^* = \theta S^{-1} + (I - \theta S^{-1} \theta^*) \xi,
\]

for some adjointable map \( \xi : \mathcal{H} \to \mathcal{K}_j \), and vice versa.

5. The \(*\)-g-frame operator \( S_\Omega \) of \( \{ (\Omega_j, \mathcal{K}_j) \}_{j \in J} \) is equal to \( S^{-1} + \eta^*(I - \theta S^{-1} \theta^*) \eta \).

6. There exists a \(*\)-Bessel sequence \( \{ (\Delta_j, \mathcal{K}_j) \}_{j \in J} \) in \( \mathcal{H} \) whose pre-\(*\)-g-frame operator is \( \eta \) and yields

\[
\Omega_j = \tilde{\Lambda}_j + \Delta_j - \sum_{k \in J} \tilde{\Lambda}_j \Lambda_k \Delta_k,
\]

for all \( j \in J. \)

**Proof.** (1) For \( j \in J \) and \( f, g \in \mathcal{H} \),

\[
\langle \theta^* \eta f, g \rangle = \langle \eta f, \theta g \rangle = \langle (\Omega_j f), \Omega_j g \rangle = \sum_{j \in J} \langle \Lambda_j \Omega_j f, \Lambda_j \Omega_j g \rangle = \sum_{j \in J} \langle \Lambda_j^* \Omega_j f, g \rangle = \langle \sum_{j \in J} \Lambda_j^* \Omega_j f, g \rangle = \langle f, g \rangle.
\]

(2) By the definition of pre-\(*\)-g-frame operator \( \eta \), we have \( \eta f = (\Omega_j f)_{j \in J} \), for all \( f \in \mathcal{H} \). Then \( \Omega_j = \pi_j \eta, \) for \( j \in J. \)

(3) Since \( \eta^* \) is adjointable, it follows from Proposition 2.2 that \( \{ (\pi_j, \eta^*) \}_{j \in J} \) is a \(*\)-Bessel sequence in \( \mathcal{H} \). Also, since \( \eta^* \theta = I, \eta^* \) is surjective and for \( f \in \mathcal{H} \),

\[
|| (\eta^* \eta)^{-1} ||^{-1} \langle f, f \rangle \leq \langle f, f \rangle = \sum_{j \in J} \langle \pi_j f, \pi_j f \rangle,
\]

and we have \( \{ (\pi_j, \eta^* \mathcal{K}_j) \}_{j \in J} \) is a \(*\)-g-frame for \( \mathcal{H} \) with pre-\(*\)-g-frame operator \( \eta^* \). Moreover, from \( \eta^* \theta = I \) we obtain \( f = \sum_{j \in J} \langle \pi_j \eta^* \rangle \Lambda_j f \), for \( f \in \mathcal{H} \). It means that \( \{ (\pi_j, \eta^*) \}_{j \in J} \) is a dual \(*\)-g-frame for \( \{ (\Lambda_j, \mathcal{K}_j) \}_{j \in J} \).

(4) If \( \eta^* \) is such a right inverse, then by taking \( \xi = \eta^* \), we have

\[
\theta S^{-1} + (I - \theta S^{-1} \theta^*) \eta \theta S^{-1} + \eta^* - \theta S^{-1} \theta^* \eta = \eta^*.
\]

The converse is straightforward.

(5) The pre-\(*\)-g-frame operator \( \eta \) is right inverse of \( \theta^* \) and so \( \theta \) is right inverse of \( \eta^* \). Hence, by the part (4) of the theorem

\[
S_\Omega = \eta^* \eta = \eta^* \theta S^{-1} + \eta^* (I - \theta S^{-1} \theta^*) \eta = S^{-1} + \eta^*(I - \theta S^{-1} \theta^*) \eta.
\]

(6) Let \( \{ (\Delta_j, \mathcal{K}_j) \}_{j \in J} \) be a \(*\)-Bessel sequence in \( \mathcal{H} \) with the pre-\(*\)-g-frame operator \( \eta \). For \( j \in J, \) set

\[
\Omega_j = \tilde{\Lambda}_j + \Delta_j - \sum_{k \in J} \tilde{\Lambda}_j \Lambda_k \Delta_k.
\]

Let \( S \) and \( \theta \) be the \(*\)-g-frame operator and the pre-\(*\)-g-frame operator of \( \{ (\Lambda_j, \mathcal{K}_j) \}_{j \in J} \), respectively. Define the linear operator

\[
\xi : \mathcal{H} \to \mathcal{K}_j, \text{ by } \xi f = (\Omega_j f)_{j \in J}.
\]

Clearly, \( \xi \) is adjointable. For every \( j \in J \), we have

\[
\pi_j \xi = \Omega_j = \Lambda_j S^{-1} + \Delta_j - \sum_{k \in J} \Lambda_k \Delta_k
\]

\[
= \pi_j (\theta S^{-1} + \eta - \theta S^{-1} \theta^* \eta).
\]

Then \( \xi = \theta S^{-1} + \eta - \theta S^{-1} \theta^* \eta \). By parts (3) and (4) of the theorem, \( \{ (\Omega_j, \mathcal{H}) \}_{j \in J} \) becomes a dual \(*\)-g-frame of \( \{ (\Lambda_j, \mathcal{H}) \}_{j \in J} \).
References


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Involutions on topological centers of Banach \(*\)-algebras

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Abstract
In the present work, we give a necessary and sufficient condition on a Banach \(*\)-algebra for that the first (second) topological center of its second dual is a Banach \(*\)-algebra with the natural extension of the original involution.

Keywords: Banach \(*\)-algebra, involution, topological center.

Mathematics Subject Classification: 46H05.

1 Introduction
Let \( \mathcal{A} \) be a Banach \(*\)-algebra, and recall that there are two natural products on the second dual \( \mathcal{A}^{**} \) of \( \mathcal{A} \), called the first and second Arens products; they are denoted by \( \square \) and \( \diamondsuit \), respectively. In fact, the first Arens product is defined by

\[
\langle fa, b \rangle = \langle f, ab \rangle, \\
\langle Ff, a \rangle = \langle F, fa \rangle, \\
\langle F \diamondsuit H, f \rangle = \langle F, Hf \rangle,
\]

and the second Arens product is defined by

\[
\langle af, b \rangle = \langle f, ba \rangle, \\
\langle fF, a \rangle = \langle F, af \rangle, \\
\langle F \diamond H, f \rangle = \langle H, fF \rangle
\]

for all \( a, b \in \mathcal{A}, \ f \in \mathcal{A}^* \) and \( F, H \in \mathcal{A}^{**} \). Under either Arens product, \( \mathcal{A}^{**} \) becomes a Banach algebra. The map \( F \mapsto F \square H \) is \( w^*-w^* \)-continuous for all \( H \in \mathcal{A}^{**} \), but the map \( H \mapsto F \diamondsuit H \) is not \( w^*-w^* \)-continuous for all \( F \in \mathcal{A}^{**} \) except for the case where \( F \in \mathcal{A} \); similar statements hold for the second Arens product. The first and second topological centers of \( \mathcal{A}^{**} \) are defined by

\[
Z_1(\mathcal{A}^{**}) = \{ F \in \mathcal{A}^{**} : H \mapsto F \square H \text{ is } w^*-w^*\text{-continuous } \}, \\
Z_2(\mathcal{A}^{**}) = \{ H \in \mathcal{A}^{**} : F \mapsto H \diamondsuit F \text{ is } w^*-w^*\text{-continuous } \},
\]

respectively. Then \( Z_i(\mathcal{A}^{**}) \) is a closed subalgebra of \( \mathcal{A}^{**} \) with

\[
\mathcal{A} \subseteq Z_i(\mathcal{A}^{**}) \subseteq \mathcal{A}^{**}
\]

for \( i = 1, 2 \); furthermore,

\[
Z_1(\mathcal{A}^{**}) = \{ F \in \mathcal{A}^{**} : F \square H = F \diamondsuit H \text{ for all } H \in \mathcal{A}^{**} \}, \\
Z_2(\mathcal{A}^{**}) = \{ H \in \mathcal{A}^{**} : F \square H = F \diamondsuit H \text{ for all } F \in \mathcal{A}^{**} \}.
\]
Recall that $\mathfrak{A}$ is called Arens regular if

$$Z_4(\mathfrak{A}^{**}) = \mathfrak{A}^{**};$$

or equivalently, $Z_2(\mathfrak{A}^{**}) = \mathfrak{A}^{**}$. See Duncan and Hosseiniun [4] for a survey on the subject, and Civin and Yood [1] as the first systematic study of Arens products; see also Dales [2] and Dales and Lau [4] for more details.

Several authors have studied the existence of involutions on the second dual of a Banach $*$-algebra. For example, Grosser [6] proved that if $\mathfrak{A}$ has a right bounded approximate identity and $\mathfrak{A}^{**}$ is a Banach $*$-algebra, then $\mathfrak{A}^*\mathfrak{A} = \mathfrak{A}^*$. Also, Civin and Yood [1] showed that if a Banach $*$-algebra is Arens regular, then its involution can be extended to an involution on the second dual; see also Farhadi and Ghahramani [5].

Our aim in this work is to give a condition equivalent to that $Z_4(\mathfrak{A}^{**})$ is a Banach $*$-algebra with the natural extension of the involution of $\mathfrak{A}$.

## 2 Main Result

We commence with our main result. First, let us recall that the first and second topological centers of a Banach $*$-algebra do not coincide in general, although coincide for a large family of Banach $*$-algebras.

**Theorem 2.1.** Let $\mathfrak{A}$ be a Banach $*$-algebra. Then the following assertions are equivalent.

(a) $Z_4(\mathfrak{A}^{**})$ is a Banach $*$-algebra with the natural extension of the involution of $\mathfrak{A}$.

(b) $Z_2(\mathfrak{A}^{**})$ is a Banach $*$-algebra with the natural extension of the involution of $\mathfrak{A}$.

(c) $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**})$.

**Proof.** First, we show the natural extension of the involution of $\mathfrak{A}$ is an isometric conjugate linear mapping $*$ on $\mathfrak{A}^{**}$ such that $(F^*H)^* = H^*F^*$ for all $F,H \in \mathfrak{A}^{**}$. For each $f$ in $\mathfrak{A}^*$, the functional $f^* \in \mathfrak{A}^*$ is defined by $f^*(a) = \overline{f(a)}$ for all $a \in \mathfrak{A}$. Now, for $H \in \mathfrak{A}^{**}$, define $H^* \in \mathfrak{A}^{**}$ by

$$H^*(f) = \overline{H(f)}$$

for all $f \in \mathfrak{A}^*$, and note that $(fH^*)^* = Hz^*$; in fact, for $a,b \in \mathfrak{A}$ we have

$$(a^*f)^*(b) = (a^*f)(b^*) = \overline{f(b^*)a} = f^*(ab) = (f^*a)(b),$$

and so $(a^*f)^* = f^*a$ whence

$$(fH^*)^*(a) = (fH^{**})(a^*) = H^*(a^*f)$$

$$= H((a^*f)^*) = H(f^*a)$$

$$= (Hf^*)(a).$$

Then

$$(F^*H)^*(f) = (F^*H)(f^*) = F(h^*)$$

$$= F((fH^*)^*) = F^*(fH^*)$$

$$= (H^*F^*)(f),$$

for all $F,H \in \mathfrak{A}^{**}$ and $f \in \mathfrak{A}^*$. Therefore the map $H \mapsto H^*$ is a conjugate linear mapping on $\mathfrak{A}^{**}$ extending the involution of $\mathfrak{A}$; moreover,

$$\|f^*\| = \sup\{|f^*(a)| : a \in \mathfrak{A}, \|a\| \leq 1\}$$

$$\leq \sup\{\|f(a^*)\| : a \in \mathfrak{A}, \|a\| \leq 1\}$$

$$\leq \sup\{\|f\| \|a^*\| : a \in \mathfrak{A}, \|a\| \leq 1\}$$

$$= \|f\|.$$
Thus \( \| f^* \| = \| f \| \) and so we can write
\[
\| F^* \| = \sup \{|F^*(f)| : f \in \mathfrak{A}^*, \| f \| \leq 1\}
= \sup \{|F(f^*)| : f \in \mathfrak{A}^*, \| f \| \leq 1\}
\leq \sup \{\| F \| \| f^* \| : f \in \mathfrak{A}^*, \| f \| \leq 1\}
= \| F \|.
\]

Therefore the map \( H \mapsto H^* \) is also isometric. Now we prove that (a) and (c) are equivalent; that (b) and (c) are equivalent follows similarly.

For this end, suppose that (a) holds and let \( F \in Z_1(\mathfrak{A}^{**}) \). Then \( F^* \in Z_1(\mathfrak{A}^{**}) \) and for each \( H \in \mathfrak{A}^{**} \) we have \( F^* \circ H = F \circ H^* \); thus \( (F^* \circ H)^* = (F \circ H^*)^* \) and therefore \( H \circ F = H^* \circ F \). This means that \( F \in Z_2(\mathfrak{A}^{**}) \), and so \( Z_1(\mathfrak{A}^{**}) \subseteq Z_2(\mathfrak{A}^{**}) \). Similarly \( Z_2(\mathfrak{A}^{**}) \subseteq Z_1(\mathfrak{A}^{**}) \).

Conversely, suppose that (c) holds and let \( F \in Z_1(\mathfrak{A}^{**}) \). Then \( F \in Z_2(\mathfrak{A}^{**}) \) and so \( H^* \circ F = H^* \circ F \) for all \( H \in \mathfrak{A}^{**} \). Thus
\[
F^* \circ H = (H^* \circ F)^* = (H^* \circ F)^* = F^* \circ H
\]
for all \( H \in \mathfrak{A}^{**} \). Therefore \( F^* \in Z_1(\mathfrak{A}^{**}) \) and so (a) holds. \( \square \)

As a special consequence of Theorem 2.1, we have the following result was proved by Civin and Yood [1]; see also Farhadi and Ghahramani [5], Lemma 2.2.

**Corollary 2.2.** Let \( \mathfrak{A} \) be a Banach \( * \)-algebra. If \( \mathfrak{A} \) is Arens regular, then \( \mathfrak{A}^{**} \) is a Banach \( * \)-algebra with the natural extension of the involution of \( \mathfrak{A} \).

We conclude the work by some two examples.

**Example 2.3.** Let \( \mathfrak{A} \) be a Banach algebra, and suppose that there is a linear involution \( a \mapsto \overline{a} \) on \( \mathfrak{A} \) such that for \( a, b \in \mathfrak{A} \),
\[
\overline{ab} = \overline{a} \overline{b}.
\]

Set \( \mathfrak{C} = \mathfrak{A} \oplus \mathfrak{A}^{op} \), where \( \mathfrak{A}^{op} \) is the opposite algebra to \( \mathfrak{A} \), and, for \( a, b \in \mathfrak{A} \), define
\[
(a, b)^* = (\overline{b}, \overline{a}).
\]

Certainly the algebra \( \mathfrak{C} \) is a Banach \( * \)-algebra. It is clear that
\[
Z_1(\mathfrak{C}^{**}) = Z_1(\mathfrak{A}^{**}) \oplus Z_2(\mathfrak{A}^{**}) \text{ and } Z_2(\mathfrak{C}^{**}) = Z_2(\mathfrak{A}^{**}) \oplus Z_1(\mathfrak{A}^{**}).
\]

Thus, in the case where \( Z_1(\mathfrak{A}^{**}) \neq Z_2(\mathfrak{A}^{**}) \), we shall obtain a Banach \( * \)-algebra \( \mathfrak{C} \) such that \( Z_1(\mathfrak{C}^{**}) \neq Z_2(\mathfrak{C}^{**}) \); see [3], Example 9.13.

**Example 2.4.** Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be two commutative Banach \( * \)-algebras. Then the Banach algebra \( \mathfrak{C} = \mathfrak{A} \oplus \mathfrak{B} \) with the involution
\[
(a, b)^* = (a^*, b^*)
\]
for all \( a \in \mathfrak{A} \) and \( b \in \mathfrak{B} \) is a Banach \( * \)-algebra with \( Z_1(\mathfrak{C}^{**}) = Z_2(\mathfrak{C}^{**}) \). So it follows from Theorem 1 that \( Z_i(\mathfrak{C}^{**}) \) for \( i = 1, 2 \) is a Banach \( * \)-algebra with the natural extension of the involution of \( \mathfrak{C} \).

Note that \( \mathfrak{C} \) is neither Arens regular nor strongly Arens irregular if \( \mathfrak{A} \) is not Arens regular and \( \mathfrak{B} \) is not strongly Arens irregular; in fact, by Example 7.7 of [3], we have
\[
\mathfrak{C} \nsubseteq Z_1(\mathfrak{C}^{**}) = Z_2(\mathfrak{C}^{**}) \nsubseteq \mathfrak{C}^{**}.
\]
References


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System of imprimitivity on locally compact groupoids

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Abstract
We define a system of imprimitivity on a locally compact groupoid \( G \), which is an ordered triple \( \Sigma = (\pi, X, P) \) consisting of a representation \( \pi \) of \( G \) on a Hilbert bundle \( \mathcal{H} = (G^0, \{ H_u \}, \mu) \), a \( G \)-space \( X \) and a \( \mathcal{H} \)-projection-valued measure \( P \) on \( X \) with some suitable properties. The aim of this paper is to show that, when \( \Sigma = (\pi, X, P) \) is a system of imprimitivity on a locally compact groupoid \( G \), then we can define a representation from \( C_c(X) \) to \( B(L^2(G^0, \{ H_u \}, \mu)) \).

Keywords: Locally compact groupoids, System of imprimitivity

Mathematics Subject Classification: 22A22

1 Introduction

A groupoid is a set \( G \) endowed with a product map \( (x, y) \mapsto xy : G^2 \to G \) where \( G^2 \) is a subset of \( G \times G \) called the set of composable pairs, and an inverse map \( x \mapsto x^{-1} : G \to G \) such that the following relations are satisfied:

i) \((x^{-1})^{-1} = x,\)

ii)\((x, y), (y, z) \in G^2 \Rightarrow (xy, z), (x, yz) \in G^2 \) and \((xy)z = x(yz),\)

iii)\((x^{-1}, x) \in G^2 \) and if \((x, y) \in G^2 \), then \( x^{-1}(xy) = y,\)

iv) \((x, x^{-1}) \in G^2 \) and if \((z, x) \in G^2 \), then \((zx)x^{-1} = z.\)

If the reader is interested in groupoids and wants to know more about it, the books \[2\], \[3\] and \[4\] are suggested.

If \( x \in G, d(x) = x^{-1}x \) is the domain of \( x \) and \( r(x) = xx^{-1} \) is its range. \( G^0 = d(G) = r(G) \)

is the unit space of \( G \), its elements are units in the sense that \( xd(x) = x \) and \( r(x)x = x \). For \( u, v \in G^0, G^u = r^{-1}(\{ u \}), G_v = d^{-1}(\{ v \}) \). A topological groupoid consists of a groupoid \( G \) and a topology, for which the inverse and the product map are continuous. In this paper \( G \) is a locally compact Hausdorff groupoid. Also \( X \) is a topological space equipped with a \( \sigma \)-algebra \( \mathcal{M} \) of its Borel subsets.

Definition 1.1. \[1\] Let \( G \) be a groupoid and \( X \) be a topological space with a continuous open map \( \rho \) from \( X \) onto \( G^0 \). Form \( G \star X = \{(x, s) : d(x) = \rho(s)\} \). To saying that \( G \) acts (continuously) on the right on \( X \) means that there is a continuous map from \( G \star X \) to \( X \), where the image of \((x, s)\) is denoted by \( x.s \), such that

(a) for all \((x, s) \in G \star X \), \( \rho(x.s) = r(x),\)

(b) for all \((x_1, x_2) \in G^2, (x_2, s) \in G \star X \), we have

\((x_1x_2, s), (x_1, x_2, s) \in G \star X \) and \( x_1(x_2.s) = (x_1x_2).s \)

(c) for all \( s \in X, \rho(s).s = s.\)

The space \( X \) in definition 1.1 is called a \( G \)-space.

For all \( u \in G^0 \) we put \( X^u = \rho^{-1}(u) \) and for \( E \subset X, E^u = E \cap X^u \).
Definition 1.2. Let $X$ be a $G$–space equipped with the $\sigma$–algebra $\mathcal{M}$ of Borel subset of $X$ and $\mathcal{H} = (G^0, \{H_u\}, \mu)$ is a Hilbert bundle in the sense of [3], where $\mu$ is a probability measure on $G^0$. An $\mathcal{H}$–projection valued measure is a map $P$ on $G^0$ such that

- for each $u \in G^0$, $P^u := P(u)$ is a $H_u$–projection in the usual sense on $\mathcal{M}_u$, the algebra of Borel subset of $X^u$,
- for all $f \in B(X)$ and $\xi, \eta \in L^2(G^0, \{H_u\}, \mu)$, the function $f^0$ defined by $f^0(u) = \int_{X^u} f(s) dP^u_\xi,\eta(s)$ is $\mu$ measurable, where $P^u_\xi,\eta(E) = \langle P^u(E \cap X^u) \xi(u), \eta(u) \rangle$ for all $E \in \mathcal{M}$.

Remark 1.3. If $E \in \mathcal{M}$, then $P(E) : L^2(G^0, \{H_u\}, \mu) \to L^2(G^0, \{H_u\}, \mu)$ defined by $(P(E)(\xi))(u) = P^u(E \cap X^u)\xi(u)$ is a projection, in addition the map $P$ on $\mathcal{M}$ defined by $E \mapsto P(E)$ is a $L^2(G^0, \{H_u\}, \mu)$ projection valued measure.

Definition 1.4. Let $G$ be a locally compact groupoid. A system of imprimitivity on $G$ is an ordered triple $(\pi, X, P)$ consisting of

- a representation $\pi$ of $G$ on a Hilbert bundle $\mathcal{H}$,
- a $G$–space $X$ and,
- a $\mathcal{H}$–projection-valued measure $P$,

such that

$$\pi(x)P^d(x)(E)\pi(x)^{-1} = P^{r(x)}(x,E)$$

for all $x \in G$ and $E \in \mathcal{M}$.

2 Main Result

If $X$ is a $G$–space, $f \in C_c(X)$, $\xi \in L^2(G^0, \{H_u\}, \mu)$, then

$$|\int_{G^0} \int_{X^u} f(s) dP^u_\xi(s) d\mu(u)| \leq \|f\|_{\sup} \int_{G^0} \|P^u_\xi(s)\|_{\L^2} d\mu(u) = \|f\|_{\sup} \int_{G^0} \|\xi(u)\|_{\L^2}^2 d\mu(u) = \|f\|_{\sup} \|\xi\|_{\L^2}^2.$$  

Thus, by polarization identity, if $\xi, \eta \in H = L^2(G^0, \{H_u\}, \mu)$,

$$|\int_{G^0} \int_{X^u} f(s) dP^u_\xi(s) d\mu(u)| \leq 4\|f\|_{\sup} \|\xi||\eta||.$$  

Hence for $f \in C_c(X)$ we define a bounded operator $T(f) = \int_{G^0} \int_{X^u} f(s) dP(s) d\mu(u)$ on $L^2(G^0, \{H_u\}, \mu)$ by

$$\langle T(f) \xi, \eta \rangle = \int_{G^0} \int_{X^u} f(s) dP^u_\xi(s) d\mu(u) \quad \xi, \eta \in L^2(G^0, \{H_u\}, \mu) \quad (\ast).$$

If $f = \sum c_j \chi_{E_j}$ is a simple function, then by Remark 1.3,

$$\int_{G^0} \int_{X^u} f(s) dP^u_\xi(s) d\mu(u) = \int_{G^0} \sum c_j P^u_\xi(E_j \cap X^u) d\mu(u) = \int_{G^0} \sum c_j \langle P^u(E_j \cap X^u) \xi(u), \eta(u) \rangle d\mu(u) = \int_{G^0} \langle \sum c_j P^u(E_j \cap X^u) \xi(u), \eta(u) \rangle d\mu(u) = \langle \sum c_j P(E_j) \xi, \eta \rangle,$$

so by (\ast), $T(f) = \sum c_j P(E_j)$. Moreover, since every $f \in C_c(X)$ is a uniform limit of a sequence $\{f_n\}$ simple functions, and

$$\|\int_{G^0} \int_{X^u} f(s) dP(s) d\mu(u) - \int_{G^0} \int_{X^u} f_n(s) dP(s) d\mu(u)\| \leq \|f - f_n\|_{\sup},$$

we can obtain $\int_{G^0} \int_{X^u} f(s) dP(s) d\mu(u)$ as a limit (in the norm topology of $B(L^2(G^0, \{H_u\}, \mu))$ of “Riemann sums” as ordinary integrals. Therefore we have the following proposition.
Proposition 2.1. Let $\mathcal{H} = (G^0, \{H_u\}, \mu)$ be a Hilbert bundle and $P$ be a $\mathcal{H}$-projection-valued measure on $(X, \mathcal{M})$, then the map $f \to T(f) = \int_{G^0} \int_X f(s)dP(s)d\mu(u)$ is a $*$-homomorphism from $C_c(X)$ to $B(L^2(G^0, \{H_u\}, \mu))$.

Proof. The map $f \to T(f) = \int_{G^0} \int_X f(s)dP(s)d\mu(u)$ is clearly linear and $\|T(f)\| \leq 4\|f\|_{\sup}$. If $f = \sum_1^n c_j \chi_{E_j}$ and $g = \sum_1^m d_k \chi_{F_k}$ are two simple function, then $fg = \sum_{j,k} c_j d_k \chi_{E_j \cap F_k}$, so by Remark 1.3,

$$T(fg) = \sum_{j,k} c_j d_k P(E_j \cap F_k) = \sum_{j,k} c_j d_k P(E_j) P(F_k) = T(f)T(g).$$

By passing to uniform limits, we obtains $T(fg) = T(f)T(g)$ for all $f, g \in C_c(X)$. Similarly, $T(\overline{f}) = (T(f))^*$.

References


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Some approximate fixed point results for generalized \(\alpha\)-contractive mappings

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Abstract
In this talk, by using the main idea of the work of Aleomraninejad, Rezapour and Shahzad ([3]), we introduce the concept of generalized \(\alpha\)-contractive mappings and give some results about approximate fixed points of the mappings on metric spaces.

Keywords: \(\alpha\)-contractive mapping, Approximate fixed point, Fixed point, Ordered metric space.

Mathematics Subject Classification: 47H09, 47H10.

1 Introduction
Let \((X, d)\) be a metric space, \(T\) a selfmap on \(X\), \(\alpha : X \times X \to [0, \infty)\) a mapping and \(\varepsilon > 0\). We say that \(T\) is \(\alpha\)-admissible whenever \(\alpha(x, y) \geq 1\) implies \(\alpha(Tx, Ty) \geq 1\) ([1]). We say that \(x_0 \in X\) is an \(\varepsilon\)-fixed point of \(T\) whenever \(d(Tx_0, x_0) < \varepsilon\). We say that \(T\) has an approximate fixed point whenever \(T\) has an \(\varepsilon\)-fixed point for all \(\varepsilon > 0\). You know, there are many selfmaps which have approximate fixed points while have no fixed points. Denote by \(\mathcal{R}\) the set of all continuous mappings \(g : [0, \infty)^5 \to [0, \infty)\) satisfying the following conditions:

a) \(g(1, 1, 1, 0, 2) = g(1, 1, 1, 0, 2) = h \in (0, 1)\),
b) \(g(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5) \leq \alpha g(x_1, x_2, x_3, x_4, x_5)\) for all \((x_1, x_2, x_3, x_4, x_5) \in [0, \infty)^5\) and all \(\alpha \geq 0\),
c) if \(x_i, y_i \in [0, \infty)\) and \(x_i < y_i\) for \(i = 1, \ldots, 4\), then

\[
g(x_1, x_2, x_3, x_4, 0) < g(y_1, y_2, y_3, y_4, 0)\quad \text{and}\quad g(x_1, x_2, x_3, 0, x_4) < g(y_1, y_2, y_3, 0, y_4).
\]

We say that the selfmap \(T\) is a generalized \(\alpha\)-contractive mapping whenever there exists \(g \in \mathcal{R}\) such that

\[
\alpha(x, y)d(Tx, Ty) \leq g(d(x, y), d(y, Tx), d(x, Tx), d(x, Ty), d(y, Tx))
\]

for all \(x, y \in X\). We appeal the following result.

Proposition 1.1. ([3]) If \(g \in \mathcal{R}\) and \(u, v \in [0, \infty)\) are such that

\[
u \leq \max\{g(v, v, u, v + u, o), g(v, v, u, o, v + u), g(v, u, v + u, o), g(v, u, v, o, v + u)\},
\]

then \(u \leq hv\).
2 Main Result

Now, we are ready to state and prove our main results.

**Theorem 2.1.** Let \((X,d)\) be a metric space, \(T\) an \(\alpha\)-admissible and generalized \(\alpha\)-contractive selfmap on \(X\) such that \(\alpha(x_0,Tx_0) \geq 1\) for some \(x_0 \in X\). Then \(T\) has an approximate fixed point.

**Corollary 2.2.** Let \((X,d)\) be a complete metric space, \(T\) an \(\alpha\)-admissible, continuous and generalized \(\alpha\)-contractive selfmap such that \(\alpha(x_0,Tx_0) \geq 1\) for some \(x_0 \in X\). Then \(T\) has a fixed point.

Let \((X,d)\) be a metric space, \(T\) a selfmap on \(X\) and \(\alpha : X \times X \to [0,\infty)\) a mapping. We say that \(T\) is \(\alpha\)-Kannan mapping whenever there exists \(\beta \in (0,\frac{1}{2})\) such that \(\alpha(x,y)d(Tx,Ty) \leq \beta(d(x,Tx) + d(y,Ty))\) for all \(x,y \in X\).

**Corollary 2.3.** Let \((X,d)\) be a metric space, \(T\) an \(\alpha\)-admissible and \(\alpha\)-Kannan selfmap on \(X\) such that \(\alpha(x_0,Tx_0) \geq 1\) for some \(x_0 \in X\). Then \(T\) has an approximate fixed point.

It has been published many papers about fixed point of Kannan selfmaps on metric spaces, hyper-metric spaces, fuzzy metric spaces and especially ordered metric spaces. The following result generalizes many results in the literature. For example, let \((X,d)\) be an ordered metric space. Define \(\alpha(x,y) = 1\) whenever \(x \leq y\) or \(y \leq x\) and otherwise \(\alpha(x,y) = 0\). In this case, the following result generalize the main result of [4] about fixed point of Kannan maps on partial metric spaces.

**Corollary 2.4.** Let \((X,d)\) be a complete metric space, \(T\) an \(\alpha\)-admissible, continuous and \(\alpha\)-Kannan selfmap such that \(\alpha(x_0,Tx_0) \geq 1\) for some \(x_0 \in X\). Then \(T\) has a fixed point.

Let \((X,d)\) be a metric space, \(T\) a selfmap on \(X\) and \(\alpha : X \times X \to [0,\infty)\) a mapping. We say that \(T\) is \(\alpha\)-Zamfirescu mapping whenever there exists \(\beta \in [0,1]\) such that \(\alpha(x,y)d(Tx,Ty) \leq \beta M_T(x,y)\) for all \(x,y \in X\), where

\[ M_T(x,y) = \max\{d(x,y), \frac{1}{2}[d(x,Ty) + d(y,Tx)], \frac{1}{2}[d(x,Tx) + d(y,Ty)]\}. \]

**Corollary 2.5.** Suppose that \((X,d)\) is a metric space and \(T\) is an \(\alpha\)-admissible and \(\alpha\)-Zamfirescu selfmap on \(X\) such that \(\alpha(x_0,Tx_0) \geq 1\) for some \(x_0 \in X\). Then \(T\) has an approximate fixed point.

Again, The following result generalizes some results in the literature about fixed point of Zamfirescu selfmaps on ordered metric spaces.

**Corollary 2.6.** Let \((X,d)\) be a complete metric space, \(T\) an \(\alpha\)-admissible, continuous and \(\alpha\)-Zamfirescu map such that \(\alpha(x_0,Tx_0) \geq 1\) for some \(x_0 \in X\). Then \(T\) has a fixed point.

References


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Properties of $p$-valent meromorphic functions included Ruscheweyh-Salagean operators

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Abstract
Let $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \geq 0$ and $z \in D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose $D_n^\lambda f(z) = (1 - \lambda)S^n f(z) + \lambda R^n f(z)$, where $S^n$ and $R^n$ are Salagean and Ruscheweyh operators respectively. By using jack’s lemma we prove some inequalities for $p$-valent meromorphic functions in $D^* = D \setminus \{0\}$, then we give some corollaries of the main results.

Keywords: $p$-valent meromorphic functions, Salagean-Ruscheweyh operator.

Mathematics Subject Classification: 30C45

1 Introduction

For $p \in \mathbb{N}$, we denote by $M_p(D)$ the class of meromorphic functions $f(z)$ of the form

$$f(z) = \frac{1}{z^p} + \sum_{m=p}^{\infty} a_m z^m; \quad (z \in D^*)$$

also let $M(D)$ be the class of all meromorphic functions in $D$. Suppose $S^n$ is Salagean differential operator (see [5]) on $M(D)$ defined by

$$S^0 f(z) = f(z), \quad S^1 f(z) = zf'(z)$$

$$S^n f(z) = S(S^{n-1} f(z)); \quad (n \in \mathbb{N}, z \in D).$$

It is easy to see that for $f \in M_p(D)$ we have

$$S^n f(z) = \frac{(-1)^p p^n}{z^p} + \sum_{m=p}^{\infty} m^n a_m z^m. \quad (2)$$

Also, $R^n$ is Ruscheweyh differential operator (see [4]) on $M(D)$ defined by

$$R^0 f(z) = f(z), \quad R^1 f(z) = zf'(z)$$

$$(n + 1)R^{n+1} f(z) - nR^n f(z) = z(R^n f(z))'; \quad (n \in \mathbb{N}, z \in D).$$

After a simple calculations we see that

$$R^n f(z) = \frac{z(z^{n-1} f(z))^n}{n!} \quad (n \in \mathbb{N}, z \in D). \quad (3)$$
and so for \( p \geq n \) and \( f \in M_p(\mathbb{D}) \) we obtain

\[
R^n f(z) = \frac{(-1)^n(p)}{z^p} + \sum_{m=p}^{\infty} \binom{n+m-1}{n} a_m z^m,
\]

(4)

where \( \binom{n}{p} = \frac{n!}{(p-n)! n!} \).

Now for \( f \in M(\mathbb{D}) \), consider the linear operator \( D_\lambda^n \) on \( M(\mathbb{D}) \) by

\[
D_\lambda^n f(z) = (1 - \lambda)S^n f(z) + \lambda R^n f(z); \quad (n \in \mathbb{N}, z \in \mathbb{D}),
\]

(5)

(see [3]).

So by using (2) and (4) we have

\[
D_\lambda^n f(z) = \frac{(-1)^n}{z^p} \left[ \frac{\lambda}{n} + p^n(1 - \lambda) \right] + \sum_{m=p}^{\infty} \left[ \lambda \binom{n+m-1}{n} + (1 - \lambda) m^n \right] a_m z^m,
\]

where \( 0 \leq \lambda < \lambda < p \) and \( z \in \mathbb{D}^* \).

We say that \( f(z) \) is \( p \)-valently starlike of order \( \alpha \) (\( 0 \leq \alpha < p \)), if and only if for \( z \in \mathbb{D}^* \)

\[
- \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha.
\]

(6)

Also, \( f(z) \) is \( p \)-valently convex of order \( \alpha \) (\( 0 \leq \alpha < p \)), if and only if

\[
- \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha.
\]

(7)

**Definition 1.1.** A function \( f \in M_p(\mathbb{D}) \) is said to be in the class \( M_p(n, \lambda) \) if it satisfies the inequality

\[
\left| \frac{(-1)^n z^p D_\lambda^n f(z)}{\lambda \binom{p}{n} + p^n(1 - \lambda)} - 1 \right| < 1,
\]

(8)

and a function \( f \in M_p(\mathbb{D}) \) is said to be in the class \( N_p(n, \lambda) \) if it satisfies the inequality

\[
\left| \frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} + p \right| < p.
\]

(9)

To prove the main results, we use the following lemma that is known as jack’s lemma.

**Lemma 1.2.** Let \( z_0 \in \mathbb{D}, r_0 = |z_0| \) and \( \mathbb{D}_{r_0} = \{ z : |z| < r_0 \} \). Let

\[
w(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots
\]

be continuous on \( \overline{\mathbb{D}}_{r_0} \) and analytic on \( \mathbb{D}_{r_0} \cup \{ z_0 \} \) with \( w(z) \neq 0 \) and \( n \geq 1 \). If

\[
|w(z_0)| = \max\{|w(z)| : z \in \mathbb{D}_{r_0}\}
\]

then there exists an \( m \geq n \) such that \( z_0 f'(z_0) = mf(z_0) \).

**2 Main Results**

In the first theorem we give a sufficient condition for \( f \in M_p(\mathbb{D}) \) to be in the class \( M_p(n, \lambda) \).
\textbf{Theorem 2.1.} Suppose \( f(z) \in M_p(\mathbb{D}) \) satisfies the inequality
\[
\Re \left\{ \frac{z(D_{\alpha}^n f(z))'}{D_{\alpha}^n f(z)} + p \right\} < \frac{1}{2p}, \tag{10}
\]
then \( f \in M_p(n, \lambda). \)

\textit{Proof.} The proof is technical so, we omit it. \( \square \)

\textbf{Theorem 2.2.} Suppose \( f \in M_p(\mathbb{D}) \). If \( f(z) \) satisfies the differential inequality
\[
\Re \left\{ \frac{z(D_{\alpha}^n f(z))'}{D_{\alpha}^n f(z)} - (1 + \frac{z(D_{\alpha}^n f(z))''}{(D_{\alpha}^n f(z))'}) \right\} > \frac{-p}{2p + 1}, \tag{11}
\]
then \( f \in N_p(n, \lambda). \)

\textit{Proof.} Let \( f \in M_p(\mathbb{D}) \). Consider the function \( w(z) \) by
\[
\frac{z(D_{\alpha}^n f(z))'}{D_{\alpha}^n f(z)} = p(w(z) - 1); \quad (z \in \mathbb{D}). \tag{12}
\]
\( w(z) \) is analytic in \( \mathbb{D} \), because \( f \in M_p(\mathbb{D}) \) and \( w(0) = 0 \). In addition, by differentiation of both sides of \( (12) \) we obtain
\[
p(w(z) - 1) + \frac{zw'(z)}{w(z) - 1} = 1 + \frac{z(D_{\alpha}^n f(z))''}{(D_{\alpha}^n f(z))'}. \tag{13}
\]
Now, for getting a contradiction, let there exists a point \( z_0 \in \mathbb{D}^+ \) such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.
\]
So for \( \theta_0 \in \mathbb{R} \) and \( \theta_0 \neq 0 \) we may write \( w(z_0) = e^{i\theta_0} \). Then by taking \( z = z_0 \) and using \( (12), (13) \) and jack’s lemma we conclude that
\[
\Re \left\{ \frac{z_0(D_{\alpha}^n f(z_0))'}{D_{\alpha}^n f(z_0)} - (1 + \frac{z_0(D_{\alpha}^n f(z_0))''}{(D_{\alpha}^n f(z_0))'}) \right\} = \Re \left\{ \frac{z_0w'(z_0)}{1 - w(z_0)} \right\} = \Re \left\{ \frac{mw(z_0)}{1 - w(z_0)} \right\} = \Re \left\{ \frac{e^{i\theta_0}}{1 - e^{i\theta_0}} \right\} = -\frac{m}{2} \leq -\frac{1}{2},
\]
which contradicts with \( (11) \). So we should have \( |w(z)| < 1 \) for all \( z \in \mathbb{D} \). Hence we obtain
\[
|w(z)| = \left| \frac{z(D_{\alpha}^n f(z))'}{pD_{\alpha}^n f(z)} + 1 \right| < 1,
\]
or \( f \in N_p(n, \lambda). \) \( \square \)

By taking \( n = 0 \) in theorems 2.1 and 2.2, we obtain the following results.

\textbf{Corollary 2.3.} If \( f \in M_p(\mathbb{D}) \) satisfies the inequality
\[
\Re \left\{ \frac{zf'}{f} + p \right\} < \frac{1}{2p},
\]
then \( |z^p f(z) - 1| < 1. \)
**Corollary 2.4.** If $f \in M_p(\mathbb{D})$ satisfies the inequality
\[
\Re \left\{ \frac{zf'}{f} - (1 + \frac{zf''}{f'}) \right\} > \frac{-p}{2p + 1},
\]
then $|zf' + p| < p$, or equivalently $f(z)$ is $p$-valent starlike with respect to the origin.

By taking $n = 1$ in theorems 2.1 and 2.2, we obtain the following corollaries.

**Corollary 2.5.** If $f \in M_p(\mathbb{D})$ satisfies the inequality
\[
\Re \left\{ \frac{zf''}{f'} + 1 + p \right\} < \frac{1}{2p},
\]
then $|z^{p+1}f(z) + p| < p$.

**Corollary 2.6.** If $f \in M_p(\mathbb{D})$ satisfies the inequality
\[
\Re \left\{ z \left( \frac{2f''}{f'} + z\frac{f'''}{f''} - \frac{f'''}{f'} \right) \right\} < \frac{p}{2p + 1},
\]
then
\[
\left| \frac{zf''}{f'} + p + 1 \right| < p,
\]
or equivalently $f(z)$ is $p$-valent convex with respect to the origin.

**References**


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The set of linear preservers of majorization

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Abstract

Let \( \mathbb{R}^n \) be the vector space of \( n \times 1 \) real vectors. For two vectors \( x = (x_1, \ldots, x_n)^t \) and \( y = (y_1, \ldots, y_n)^t \) in \( \mathbb{R}^n \), it is said that \( x \) is majorized by \( y \), if
\[
\sum_{k=1}^{i} y_{[i]} \leq \sum_{k=1}^{i} x_{[i]} \quad (k = 1, \ldots, n - 1)
\]
and
\[
\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i,
\]
where \( x_{[i]} \) and \( y_{[i]} \) are the \( i \)th largest elements of \( x \) and \( y \), respectively. For a relation \( \sim \) on \( \mathbb{R}^n \), the set of all linear preservers of \( \sim \) is denoted by \( P_n^\sim \). In this note we consider some relations \( \sim \) on \( \mathbb{R}^n \) and we investigate the corresponding \( P_n^\sim \). In the special case, if \( \sim \) is the vector majorization on \( \mathbb{R}^n \), \( P_n^\sim \) contains an algebra of dimension \( n \).

Keywords: Vector majorization, Linear preservers.

Mathematics Subject Classification: 15A03, 15A45

1 Introduction

Let \( \sim \) be a relation on \( \mathbb{R}^n \). A linear function \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is said to be a linear preserver (or strong linear preserver) of \( \sim \) if
\[
T(x) \sim T(y) \quad \text{whenever} \quad x \sim y \quad (\text{or} \quad T(x) \sim T(y) \quad \text{if and only if} \quad x \sim y).
\]
The linear preservers of vector majorization (denote by \( \prec \)) are characterized in [1] as follows:

**Proposition 1.1.** Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a linear function. Then \( T \) preserves \( \prec \) if and only if one of the following holds:

(a) \( T(x) = \text{tr}(x)a \), for some \( a \in \mathbb{R}^n \).

(b) \( T(x) = \alpha Px + \beta Jx \), for some \( \alpha, \beta \in \mathbb{R} \) and an \( n \times n \) permutation matrix \( P \), where \( J \) is the \( n \times n \) matrix all of whose entries are equal to one.

A matrix \( R \in M_n \) is a generalized row stochastic matrix (g-row stochastic, for short) if \( Re = e \), where \( e = (1, 1, \ldots, 1)^t \). The set of all \( n \times n \) g-row stochastic matrices is denoted by \( \text{GR}_n \). For two vectors \( x, y \in \mathbb{R}^n \), \( x \) is said to be lgw-majorized by \( y \) if there exists an \( n \times n \) g-row stochastic matrix \( R \) such that \( x = Ry \) (denoted \( x \prec_{gw} y \)). The linear preservers of gw-majorization are characterized in [2] as follows:

**Proposition 1.2.** A linear operator \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) preserves gw-majorization if and only if one of the following assertions holds:

(i) There exist an invertible matrix \( R \in \text{GR}_n \) and \( \alpha \in \mathbb{R} \) such that \( Tx = \alpha Rx \) for every \( x \in \mathbb{R}^n \).

(ii) there exists an invertible matrix \( D \in \text{GR}_n \), such that
\[
Tx = (D - \frac{1}{n} J)x \quad \text{for every} \quad x \in \mathbb{R}^n.
\]

In the next section we consider the relations \( \prec \) and \( \prec_{gw} \) on \( \mathbb{R}^n \) and we investigate the sets \( P_n^\prec \) and \( P_n^\prec_{gw} \).
2 The set of linear preservers of $\prec$ and $\prec_{gw}$

Now, by using Proposition 1.1, we can obtain the following results.

**Lemma 2.1.** The set $\mathcal{P}_n^2$ on $\mathbb{R}^n$ contains an algebra of dimension $n$.

**Proof.** Let $\mathcal{A}_n^2$ be the set of linear functions satisfying the condition (a) of Proposition 1.1. We show that $\mathcal{A}_n^2$ forms an algebra of dimension $n$. It is clear that $S \circ T \in \mathcal{P}_n^2$ whenever $S, T \in \mathcal{P}_n^2$. If $S, T \in \mathcal{A}_n^2$ and $\alpha, \beta \in \mathbb{R}$, it is easy to see that $\alpha T + \beta S \in \mathcal{A}_n^2$, and hence $\mathcal{A}_n^2$ is an algebra. Corresponding to the standard basis of $\mathbb{R}^n$, one may find a basis for $\mathcal{P}_n^2$ with $n$ elements, as desired. □

The following example shows that for $n \geq 2$, $\mathcal{P}_n^2$ is not an algebra.

**Example 2.2.** Let $T, S : \mathbb{R}^n \to \mathbb{R}^n$ be linear functions which are defined as follows:

$$Tx = x \quad \text{and} \quad Sx = \text{tr}(x)(1, 2, 0, \ldots, 0)^T,$$

for all $x \in \mathbb{R}^n$.

Now, let $x = (2, 1, 0, \ldots, 0)^T$, $y = (1, 2, 0, \ldots, 0, 0)^T$. It is clear that $y \prec x$ but $Sy + Tx = (4, 8, 0, \ldots, 0) \neq (5, 7, 0, \ldots, 0)^T = Tx + Sx$. Therefore $T + S \notin \mathcal{P}_n^2$ whenever $T, S \in \mathcal{P}_n^2$.

The next proposition shows that $\mathcal{P}_n^2$ is a union of two algebras of dimension 2.

**Proposition 2.3.** There exist algebras $\mathcal{A}_n^2$ and $\mathcal{B}_n^2$ such that $\dim \mathcal{A}_n^2 = \dim \mathcal{B}_n^2 = 2$, $\dim (\mathcal{A}_n^2 \cap \mathcal{B}_n^2) = 1$ and $\mathcal{P}_n^2 = \mathcal{A}_n^2 \cup \mathcal{B}_n^2$.

**Proof.** Let $\mathcal{A}_n^2$ be as in Lemma 2.1. Then $\mathcal{A}_n^2$ is an algebra of dimension 2. Put

$$\mathcal{B}_n^2 = \{ T \in \mathcal{P}_n^2 : T \text{ is of the form (b) of Proposition 1.1} \}.$$

Now, we show that $\mathcal{B}_n^2$ is an algebra with dimension 2. Since the only $2 \times 2$ permutations are the identity and backward identity matrices, every $T, S \in \mathcal{B}_n^2$ are as follows:

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix} + \beta \begin{pmatrix} x + y \\ x + y \end{pmatrix}, \quad S \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} y \\ x \end{pmatrix} + \beta \begin{pmatrix} x + y \\ x + y \end{pmatrix}.$$

By a simple calculation it is easy to see that $T + S \in \mathcal{B}_n^2$. Also $(\mathcal{A}_n^2 \cap \mathcal{B}_n^2) = \{ T \in \mathcal{P}_n^2 : Tx = \alpha Jx \}$ which has dimension 1. □

The Lemma 2.1, shows that the linear preservers of $\prec$ satisfying (b) form an algebra. The following example shows this is not true about the part (ii) of Proposition 1.2.

**Example 2.4.** Let $T, S : \mathbb{R}^2 \to \mathbb{R}^2$ to be linear operators which are defined as follows:

$$Tx = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x \quad \text{and} \quad Sx = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} x.$$

It is easy to see that $T, S$ satisfying the condition (ii) of Proposition 1.2. If $T + S$ is of form (ii) then $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} = D + \frac{1}{2} J$ for some invertible matrix $D$. This implies that $D = \begin{pmatrix} 1.5 & -0.5 \\ 1.5 & -0.5 \end{pmatrix}$, which is a contradiction. Therefore the set of linear functions satisfying the conion (ii) of Proposition 1.2 is not an algebra.

Also it is clear that the set of linear functions satisfying the conion (i) of Proposition 1.2 is not an algebra.
References


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What can be said about the extended geodesic and $CAT(0)$ spaces

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Abstract

In this paper we introduce the extended geodesic and the extended $CAT(0)$ space, and we verify some results on this spaces. Finally we state some problems for reverifying on extended $CAT(0)$ space.

Keywords: $CAT(0)$ space, Fixed point.

Mathematics Subject Classification: 05C05; 54H25.

1 Introduction

Let $(X, d)$ and $(Y, D)$ be metric spaces such that for some $y_0, y'_0 \in Y$. We have $D(y_0, y'_0) = \ell$. An extended geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, an e-geodesic from $x$ to $y$) is a map $c$ from $Y$ to $X$ such that $c(y_0) = x, c(y'_0) = y$, and $d(c(y_1), c(y_2)) = D(y_1, y_2)$ for all $y_1, y_2 \in Y$. In particular, $c$ is an isometry and $d(x, y) = d(c(y_0), c(y'_0)) = D(y_0, y'_0) = \ell$. The image $\alpha$ of $c$ is called an e-geodesic (or metric) segment joining $x$ and $y$. When it is unique, this e-geodesic is denoted by $[x, y]$. The space $(X, d)$ is said to be an e-geodesic space if every two points of $X$ are joined by an e-geodesic, and $X$ is said to be uniquely e-geodesic if there is exactly one e-geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $C \subseteq X$ is said to be convex if $C$ includes every e-geodesic segment joining any two of its points.

An e-geodesic triangle $\triangle(x_1, x_2, x_3)$ in an e-geodesic metric space $(X, d)$ consists of three points in $X$ (the vertices of $\triangle$) and an e-geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for an e-geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in metric space $Y$ such that

$$D(\overline{x}_i, \overline{x}_j) = d(x_i, y_j)$$

for $i, j \in \{1, 2, 3\}$.

An e-geodesic metric space is said to be an extended $CAT(0)$ space if all e-geodesic triangles of appropriate size satisfy the following comparison axiom:

Let $\triangle$ be an e-geodesic triangle in $X$ and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the extended $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq D(\overline{x}, \overline{y}).$$

Let $\{x_n\}$ be a bounded sequence in $X$ and $K$ be a nonempty bounded subset of $X$. We associate this sequence with the number $r = r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}$, where $r(x, \{x_n\}) = \limsup_{n \to \infty} d(x_n, x)$, and the set

$$A = A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r\}.$$
The number $r$ is known as the \textit{asymptotic radius} of $\{x_n\}$ relative to $K$. Similarly, set $A$ is called the \textit{asymptotic center} of $\{x_n\}$ relative to $K$.

A sequence $\{x_n\}$ in a extended $\text{CAT}(0)$ space $X$ said to be $\Delta$-convergent to $x \in X$ if $x$ is the unique asymptotic center of every subsequence of $\{x_n\}$.

Notice that given $\{x_n\} \subseteq X$ such that $\{x_n\}$ is $\Delta$-convergent to $x$ and given $y \in X$ with $x \neq y$,

$$\limsup_{n \to \infty} d(x_n, x) < \limsup_{n \to \infty} d(x_n, y).$$

2 \hspace{1pt} \textbf{Main Result}

\textbf{Theorem 2.1.} Let $(X,d)$ be an extended $\text{CAT}(0)$ space. Then

1. Let $x,y \in X$, $x \neq y$, $z,w \in [x,y]$ and For each $t \in [0,1]$ such that $d(x,z) = d(x,w)$ and $\partial B_d(x,\ell t) \cap \partial B_d(y,(1-t)\ell)$ be a singleton set. Then $z = w$.

2. Let $x,y \in X$. For each $t \in [0,1]$ there exists a unique point $z \in X$ such that

$$d(z,x) = t \ell, \quad d(y,z) = (1-t)\ell.$$ 

\textbf{Proof.} Let $c$ be the e-geodesic path joining $x$ and $y$ and let $d(x,y) = \ell$.

(1) Since $z,w \in [x,y]$, there are $y_1 \in \partial B_d(y_0,t_1\ell) \subseteq Y$ and $y_2 \in \partial B_d(y_0,t_2\ell) \subseteq Y$ such that $c(y_1) = z$ and $c(y_2) = w$. Thus, $d(x,z) = d(c(y_1),c(y_1)) = D(y_0,y_1) = t_1\ell$, and similarly $d(x,w) = D(y_0,y_2) = t_2\ell$. Since $d(x,z) = d(x,w)$, we have $t_1\ell = t_2\ell$ and or $t_1 = t_2$. Therefore

$$y_1, y_2 \in \partial B_d(y_0,\ell) \cap \partial B_d(y_0,(1-t)\ell) = \partial B_d(x,\ell) \cap \partial B_d(y,(1-t)\ell),$$

which is singleton so $z = w$.

To prove (2) put $z \in \partial B_d(x,\ell) \cap \partial B_d(y,(1-t)\ell) \subseteq X$ so

$$d(z,x) = t \ell, \quad d(y,z) = (1-t)\ell.$$ 

And also $d(x,z) + d(y,z) = \ell = d(x,y)$.

The uniqueness of $z$ follows from (1). \hfill $\square$

For convenience, from now on we will use the notation $(1-t)x \oplus ty$ for the unique point $z$ satisfying (2). By using this notation, the following is easy to verify.

\textbf{Remark 2.2.} Let $X$ be an extended $\text{CAT}(0)$ space and let $x,y \in X$ such that $x \neq y$ and $s,t \in [0,1]$. Then

$$(1-t)x \oplus ty = (1-s)x \oplus sy \iff t = s.$$ 

\textbf{Lemma 2.3.} Let $X$ be an extended $\text{CAT}(0)$ space and let $x,y \in X$ such that $x \neq y$. Then

1. $[x,y] = \{(1-t)x \oplus ty \mid t \in [0,1]\}$.

2. $d(x,z) + d(y,z) = d(x,y) \iff z \in [x,y]$.

3. The mapping $f : [0,1] \to [x,y]$, $f(t) = (1-t)x \oplus ty$ is continuous and bijective.

The following corollary is a direct result of above theorem when we let $Y$ be a subspace of normed space $(N,\|\|)$ on field $\mathbb{R}^+$ with an element $e \in Y$ such that $\|e\| = \ell = d(x,y)$ for $x,y \in (X,d)$.

\textbf{Corollary 2.4.} Let $(X,d)$ be an extended $\text{CAT}(0)$ space. Then

1. Let $x,y \in X$, $x \neq y$ and $z,w \in [x,y]$ such that $d(x,z) = d(x,w)$. Then $z = w$. 

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2. Let \( x, y \in X \). For each \( t \in \mathbb{R}^+ \) there exists an unique point \( z \in X \) such that
\[
d(z, x) = t\ell, \quad d(y, z) = (1 - t)\ell.
\]

**Proof.** Let \( c \) be the geodesic path joining \( x \) and \( y \) and let \( d(x, y) = \ell \).
(1) Since \( z, w \in [x, y] \), there are \( y_1, y_2 \in Y \) such that \( c(y_1) = z \) and \( c(y_2) = w \) and \( y_1 = t_1e, y_2 = t_2e \) for some \( t_1, t_2 \in \mathbb{R}^+ \). Thus, \( d(x, z) = d(c(0), c(y_1)) = \|0 - y_1\| = \|y_1\|, \) and similarly \( d(x, w) = \|y_2\| \).
Since \( d(x, z) = d(x, w) \), we have \( \|y_1\| = \|y_2\| \) and or \( \|t_1e\| = \|t_2e\| \) so \( t_1\ell = t_2\ell \). Therefore \( t_1 = t_2 \).
That is \( z = w \).
To prove (2) put \( z := te \) so
\[
d(z, x) = d(c(te), c(0)) = \|te - 0\| = t\ell, \quad d(y, z) = d(c(e), c(te)) = \|e - te\| = (1 - t)\ell.
\]
And also \( d(x, z) + d(y, z) = \ell = \|e\| = d(x, y) \).
The uniqueness of \( z \) follows from (1). \( \square \)

**Example 2.5.** Let \( Y := [0, \ell] \) we obtain some pervious Theorems and corollaries in \( \text{CAT}(0) \) space.
For more details please refer to \([1, 2, 3, 4, 5]\).

## 3 Problems


What can be said about strong convergence of a general iteration scheme for a finite family of asymptotically quasi-nonexpansive maps in convex metric spaces in the Extended CAT(0).

What can be said about best approximation, invariant approximation and best proximity pair problems for multivalued mappings that are condensing or nonexpansive in the Extended CAT(0). What can be said about the asymptotic center. Is this correct that \( A = A(K, \{x_n\}) \) of \( \{x_n\} \) consists of exactly one point whenever \( K \) is closed and convex in the extended CAT(0) space.

As you know every \( \text{CAT}(0) \) space \( X \) satisfies the Opial property. What can be said about the Opial property in the extended \( \text{CAT}(0) \) space.

For other results please refer to \([1, 2, 3, 4, 5]\). And many problems there exist in this space, likewise as following problems:

**Lemma 3.1.** Let \((X, d)\) be an extended \( \text{CAT}(0) \) space. Then
\[
d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z),
\]
for \( x, y, z \in X \) and \( t \in [0, 1] \).

**Lemma 3.2.** Let \((X, d)\) be an extended \( \text{CAT}(0) \) space. Then
\[
d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z) - t(1 - t)d(x, y)^2,
\]
for all \( x, y, z \in X \) and \( t \in [0, 1] \).

**Theorem 3.3.** Let \( n \in \mathbb{N} \). For each \((z_1, z_2, \cdots, z_n) \in X^n\), there exists \( S \in \mathbb{R}^+ \) such that
\[
\forall \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \ell^n := [0, 1]^n \text{ such that } \|\alpha\| = 1.
\]
Then there exists \( z \in \Delta(z_1, z_2, \cdots, z_n) \) such that \( d(z, z_i) = \alpha_iS, \) for all \( i = 1, 2, \cdots, n \) and
\[
\sum_{i=1}^n d(z, z_i) = S.
\]
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Some notes on locally convex quotient lattice cones

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Abstract
We consider some conditions which guarantee that a locally convex quotient cone to be a lattice and investigate this conditions on some examples.

Keywords: Locally convex Lattice cones, Quotients.

Mathematics Subject Classification: 46A20

1 Introduction

A cone is a set $P$ endowed with an addition $(a, b) \rightarrow a + b$ and a scalar multiplication $(\alpha, a) \rightarrow \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, and there is a neutral element $0 \in P$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha \beta)a$, $(\alpha + \beta)a = \alpha a + \beta a$, $\alpha(a + b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in P$ and $\alpha, \beta \geq 0$. An ordered cone $P$ carries a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in P$ and $\alpha \geq 0$. The theory of locally convex cones as developed in [1] uses order theoretical concepts to introduce a quasiuniform topological structure on an ordered cone. In a first approach, the resulting topological neighborhoods themselves will be considered to be elements of the cone. In this vein, a full locally convex cone $(P, V)$ is an ordered cone $P$ that contains an abstract neighborhood system $V$, that is a subset of positive elements which is directed downward, closed for addition and multiplication.

For cones $P$ and $Q$ a mapping $T : P \rightarrow Q$ is called a linear operator if $T(a + b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ holds for all $a, b \in P$ and $\alpha \geq 0$. If both $P$ and $Q$ are ordered, then $T$ is called monotone, if $a \leq b$ implies $T(a) \leq T(b)$. If both $(P, V)$ and $(Q, W)$ are locally convex cones, the operator $T$ is called (uniformly) continuous if for every $w \in W$ one can find $v \in V$ such that $T(a) \leq T(b) + w$ whenever $a \leq b + v$ for $a, b \in P$. A linear functional on $P$ is a linear operator $\mu : P \rightarrow \mathbb{R}$. The dual cone $P^{\ast}$ of a locally convex cone $(P, V)$ consists of all continuous linear functionals on $P$ and is the union of all polars $v^{\ast}$ of neighborhoods $v \in V$, where $\mu \in v^{\ast}$ means that $\mu(a) \leq \mu(b) + 1$, whenever $a \leq b + v$ for $a, b \in P$. For an element $a \in P$ we define the upper, lower and symmetric boundedness components of $a$ as

$$B(a) = \bigcap_{\epsilon > 0} \bigcup_{v \in V} v_{\epsilon}(a), \quad (a)B = \bigcap_{\epsilon > 0} \bigcup_{v \in V} (a)v_{\epsilon} \quad \text{and} \quad B^{\ast}(a) = B(a) \cap (a)B,$$
respectively. We shall say that a locally convex cone \((\mathcal{P}, \mathcal{V})\) is a locally convex \(\vee\)-semilattice cone if its order is antisymmetric and if for any two elements \(a, b \in \mathcal{P}\) their supremum \(a \vee b\) exists in \(\mathcal{P}\) and if

1. \((a + c) \vee (b + c) = a \vee b + c\) holds for all \(a, b, c \in \mathcal{P}\).
2. \(a \leq c + v\) and \(b \leq c + w\) for \(a, b, c \in \mathcal{P}\) and \(v, w \in \mathcal{V}\) imply that \(a \vee b \leq c + (v + w)\).

Likewise, \((\mathcal{P}, \mathcal{V})\) is a locally convex \(\wedge\)-semilattice cone if its order is antisymmetric and if for any two elements \(a, b \in \mathcal{P}\) their infimum \(a \wedge b\) exists in \(\mathcal{P}\) and if

1. \((a + c) \wedge (b + c) = a \wedge b + c\) holds for all \(a, b, c \in \mathcal{P}\).
2. \(c \leq a + v\) and \(c \leq b + w\) for \(a, b, c \in \mathcal{P}\) and \(v, w \in \mathcal{V}\) imply that \(c \leq a \wedge b + (v + w)\).

If both sets of the above conditions hold, then \((\mathcal{P}, \mathcal{V})\) is called a locally convex lattice cone ([3] or [5]). Let both \((\mathcal{P}, \mathcal{V})\) and \((\mathcal{Q}, \mathcal{W})\) be locally convex \(\vee\)- (or \(\wedge\)-) semilattice cones. A continuous linear operator \(T : \mathcal{P} \to \mathcal{Q}\) is called a \(\vee\)- (or \(\wedge\)-)semilattice homomorphism if it is compatible with the lattice operations in \(\mathcal{P}\) and \(\mathcal{Q}\), that is if \(T(a \vee b) = T(a) \vee T(b)\) (or \(T(a \wedge b) = T(a) \wedge T(b)\)). holds for all \(a, b \in \mathcal{P}\). If both \((\mathcal{P}, \mathcal{V})\) and \((\mathcal{Q}, \mathcal{W})\) are locally convex lattice cones and \(T : \mathcal{P} \to \mathcal{Q}\) is both a \(\vee\)- and a \(\wedge\)-semilattice homomorphism, then \(T\) is called a lattice homomorphism. Non-negative multiples of lattice homomorphism are again lattice homomorphisms, but sums are generally not.

## 2 Locally convex quotient lattice cones

We consider an equivalence relation \(\sim\) on a locally convex cone \((\mathcal{P}, \mathcal{V})\) which is compatible with the algebraic operations in \(\mathcal{P}\), that is \(a + c \sim b + c \text{ and } a \alpha b \sim a \alpha b\) whenever \(a \sim b\) for \(a, b, c \in \mathcal{P}\) and \(\alpha \geq 0\). By \([a]\) we denote the equivalence class of an element \(a \in \mathcal{P}\) and set \([\mathcal{P}] = \{[a] | a \in \mathcal{P}\}\).

The operations \([a] + [b] = [a + b]\) and \(\alpha[a] = [\alpha a]\) are well-defined for \(a, b \in \mathcal{P}\) and \(\alpha \geq 0\) and \([\mathcal{P}]\) becomes a cone with these operations. It is fairly obvious how to assign a suitable order \(\leq\) and a locally convex cone topology to \([\mathcal{P}]\) and if its order is antisymmetric and if for any two elements \(a, b \in \mathcal{P}\) their infimum \(a \wedge b\) exists in \(\mathcal{P}\) and if

1. \((a + c) \wedge (b + c) = a \wedge b + c\) holds for all \(a, b, c \in \mathcal{P}\).
2. \(c \leq a + v\) and \(c \leq b + w\) for \(a, b, c \in \mathcal{P}\) and \(v, w \in \mathcal{V}\) imply that \(c \leq a \wedge b + (v + w)\).

If both sets of the above conditions hold, then \((\mathcal{P}, \mathcal{V})\) is called a locally convex lattice cone ([3] or [5]). Let both \((\mathcal{P}, \mathcal{V})\) and \((\mathcal{Q}, \mathcal{W})\) be locally convex \(\vee\)- (or \(\wedge\)-) semilattice cones. A continuous linear operator \(T : \mathcal{P} \to \mathcal{Q}\) is called a \(\vee\)- (or \(\wedge\)-)semilattice homomorphism if it is compatible with the lattice operations in \(\mathcal{P}\) and \(\mathcal{Q}\), that is if \(T(a \vee b) = T(a) \vee T(b)\) (or \(T(a \wedge b) = T(a) \wedge T(b)\)). holds for all \(a, b \in \mathcal{P}\). If both \((\mathcal{P}, \mathcal{V})\) and \((\mathcal{Q}, \mathcal{W})\) are locally convex lattice cones and \(T : \mathcal{P} \to \mathcal{Q}\) is both a \(\vee\)- and a \(\wedge\)-semilattice homomorphism, then \(T\) is called a lattice homomorphism. Non-negative multiples of lattice homomorphism are again lattice homomorphisms, but sums are generally not.
for $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. Furthermore we suppose the equivalence relation $\sim$ compatibles with the lattice operations in $\mathcal{P}$, that is $[a \lor b] = [a] \lor [b]$ (or $[a \land b] = [a] \land [b]$) for $a, b \in \mathcal{P}$. In the following we consider the weak preorder on $[\mathcal{P}]$ and denote by $\preceq$. Thus we have $[a] \preceq [b]$ for $[a], [b] \in [\mathcal{P}]$, if and only if $\mu(a) \leq \mu(b)$ for all $\mu \in \mathcal{P}^*_\sim$.

**Theorem 2.1.** Let $(\mathcal{P}, \mathcal{V})$ be a locally convex $\lor \land$-semilattice cone and let $\sim$ be an equivalence relation on $\mathcal{P}$ such that

(i) the equivalence relations $\sim$ and $\sim_{\mathcal{P}^*_\sim}$ coincide on $\mathcal{P}$,

(ii) $[a \lor b] = [a] \lor [b]$ (or $[a \land b] = [a] \land [b]$) holds for $a, b \in \mathcal{P}$,

(iii) $\mu(a \lor b) = \mu(a) \lor \mu(b)$ (or $\mu(a \land b) = \mu(a) \land \mu(b)$) holds for all $\mu \in \mathcal{P}^*_\sim$ and $a, b \in \mathcal{P}$.

Then $(([\mathcal{P}], \mathcal{V}))$ is a locally convex $\lor \land$-semilattice cone and the canonical projection $\Pi : \mathcal{P} \to [\mathcal{P}]$, is a $\lor$ (or $\land$)-semilattice homomorphism.

**Example 2.2.** Let us consider the full locally convex lattice cone $(\mathbb{R}_+, \mathcal{V})$, where $\mathcal{V} = \{0\}$. The dual of $(\mathbb{R}_+, \mathcal{V})$ consists of positive reals and functionals $\overline{a}$ and $+\infty$ acting as

$$\overline{a}(a) = \begin{cases} +\infty & a=+\infty \\ 0 & \text{else} \end{cases} \quad \text{and} \quad +\infty(a) = \begin{cases} +\infty & a=0 \\ 0 \quad \text{else} \end{cases},$$

respectively.

The functional $+\infty$ is obviously contained in the polar of the neighborhood $0 \in \mathcal{V}$. For the symmetric boundedness components in $\mathbb{R}_+$, we have $B^+(0) = \{0\}$, $B^+(+\infty) = \{+\infty\}$ and $B^+(a) = (0, +\infty)$ for $a \in (0, +\infty)$. For $a, b \in \mathbb{R}_+$ we set $a \sim b$ if $B^+(a) = B^+(b)$, This yields $([\mathbb{R}_+], \mathcal{V})$ is a $\lor \land$-semilattice cone. We define the equivalence relation $\sim$ satisfies in the conditions of theorem 1.3. Indeed, if $a \sim_{\mathbb{R}_+} b$ for $a, b \in \mathbb{R}_+$, then $\overline{a}(a) = \overline{b}(b)$ and $+\infty(a) = +\infty(b)$ that implies $a, b \in (0, +\infty) \lor a, b \in (0, +\infty)$. This yields $[a \lor b] = [a] \lor [b]$ and $[a \land b] = [a] \land [b]$. The elements of $([\mathbb{R}_+], \mathcal{V})$ are lattice homomorphisms from $\mathbb{R}_+$ to $\overline{\mathbb{R}}$. Thus according to theorem 1.3, $([\mathbb{R}_+], \mathcal{V})$ is a locally convex lattice cone.

**Example 2.3.** Let $X$ be a topological space, and let $\mathcal{P}$ be the cone of all $\overline{\mathbb{R}}$-valued lower semicontinuous functions on $X$, where $\overline{\mathbb{R}}$ is endowed with the usual, that is the one-point compactification topology, $\mathcal{P}$ is endowed with the pointwise operations and order and neighborhoods $\nu \in \mathcal{V}$ for $\mathcal{P}$ are given by the strictly positive constant functions. It is proven in [5] that $(\mathcal{P}, \mathcal{V})$ forms a locally convex $\lor \land$-semilattice cone. We define the equivalence relation $\sim$ on $\mathcal{P}$ by $f \sim g$ for, $f, g \in \mathcal{P}$ if $I_f \subseteq I_g$, where $I_f = \{x \in X | f(x) < +\infty\}$. If $f \in \mathcal{P}^*$ and $x \in X$, then the mapping $\mu_x : \mathcal{P} \to \overline{\mathbb{R}}$ such that $\mu_x(f) = \mu(f(x))$ for all $f \in \mathcal{P}$, is an element of $\mathcal{P}^*$ and we obviously have $\{\mu_x|x \in X\} \subset \mathcal{P}^*_\sim$ where

$$\overline{\mu}_x(f) = \overline{\mu}(f(x)) = \begin{cases} f(x) < +\infty \\ +\infty \quad \text{else} \end{cases} \quad \text{for all } f \in \mathcal{P}.$$  

We claim that $\mathcal{P}^*_\sim$ is spanned by $\{\overline{\mu}_x| x \in X\}$. Otherwise, there is $\mu \in \mathcal{P}^*_\sim$ such that for all $n \in \mathbb{N}$, $\alpha_i > 0$ and $x_i \in X$ such $\{i \in \{1, \ldots, n\}, \text{ we have } \mu \neq \sum_{i=1}^n \alpha_i \overline{\mu}_{x_i}$. Hence, there is $f \in \mathcal{P}$ such that $\mu(f) \neq \sum_{i=1}^n \alpha_i \overline{\mu}_{x_i}(f)$. This yields $\mu(f) \neq 0$ and $\mu(f) < +\infty$. Now for every $\alpha > 1$ we have $\alpha f \sim f$ but $\mu(\alpha f) \neq \mu(f)$. This Contradiction proves that $\mathcal{P}^*_\sim$ is spanned by $\{\overline{\mu}_x|x \in X\}$. Now We show that the equivalence relation $\sim$ satisfies in the condition (i), (ii) and (iii). Let for $f, g \in \mathcal{P}$, $f \sim_{\mathcal{P}^*_\sim} g$. Then for every $x \in X$ we have $\overline{\mu}_x(f) = \overline{\mu}_x(g)$ that implies $f(x) < +\infty$ if and only if $g(x) < +\infty$. Hence $I_f = I_g$ that means $f \sim g$. For $[f], [g] \in [\mathcal{P}]$ we have

$$[f] \preceq [g] \iff \forall x \in X \quad \overline{\mu}_x(f) \leq \overline{\mu}_x(g)$$

$$\iff \forall x \in X \quad (g(x) < +\infty \implies f(x) < +\infty)$$

$$\iff I_g \subseteq I_f.$$  

(1)
Let \([h] \in [\mathcal{P}]\) is an upper bound for the subset \([\{f],[g]\}\) in \([\mathcal{P}]\). Then (1.3) shows that \(I_h \subseteq I_f \cap I_g\). Therefore, if \(h(x) < +\infty\), then \(f(x) < +\infty\) and \(g(x) < +\infty\) that imply \((f \vee g)(x) < +\infty\). This yields that \(I_h \subseteq I_{f \vee g}\) and \([f \vee g] \lesssim [h]\).

It is obvious that for every \(x \in X\), \(\mathfrak{D}_x\) is a \(\vee\)-lattice homomorphism. Now according to theorem 1.3, \((\mathcal{P},[\mathcal{V}])\) is a locally convex \(\vee\)-semilattice cone.

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An invariant subspace for composition operators

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Abstract
In this note the boundedness and boundedness from below of composition operators on the weighted locally convex space of measurable functions is studied by Radon-Nikodym derivation. Moreover, a subspace of these spaces is somehow constructed that composition operators leave it invariant.

Keywords: composition operators, conditional expectation, locally convex, bounded below, invariant subspace.

Mathematics Subject Classification: Primary 46A03; Secondary 47B38.

1 Introduction
Let $X$ be a locally compact Hausdorff space on which $(X, \Sigma, \mu)$ be a complete measure space with positive Radon measure (inner regular and locally finite) $\mu$ on a $\sigma$-algebra $\Sigma$ containing all Borel sets in $X$.

Let $\Omega$ be a locally convex Hausdorff space over a real (or complex) field. The space of all measurable functions from $X$ into $\Omega$ and the set of all continuous seminorms on $\Omega$ are denoted by $M(X, \Omega)$ and $cs(\Omega)$ respectively. A real-valued nonnegative measurable function on $X$ is called a weight. For $p \in [1, +\infty]$ the set of all weights on $X$ is denoted by $V^p(X)$. An $N_p$ family of $V^p$ is said to be the collection of all weights in $V^p$ provided that for each $u, v \in V^p$ and $\lambda > 0$ there exists $w \in V^p$ on which $\lambda u \leq w$ and $\lambda v \leq w$ pointwise on $X$. To find out the role of $p$ in the definition see [6]. Throughout this paper the $L^p$-space $L^p(X, \Sigma)$ is abbreviated to $L^p(\Sigma)$.

Now consider the following space

$$MV^p(X, \Omega) = \{ f \in M(X, \Omega) : vq(f) \in L^p(\Sigma) \text{ for all } v \in V^p, q \in cs(\Omega) \}.$$ 

The weighted topology $\omega_{v,p}$ on $MV^p(X, \Omega)$ is generated by the continuous seminorms of the form

$$\|f\|_{v,p} = \left( \int_X (vq(f))^p d\mu \right)^{1/p}.$$ 

The space $MV^p(X, \Omega)$ endowed with the such weighed topology $\omega_{v,p}$, is called the weighted locally convex space of measurable functions whereas $v$ and $q$ range over $V^p$ and $cs(\Omega)$ respectively. The closed convex neighbourhoods of the origin

$$B_{v,q} = \{ f \in MV^p(X, \Omega) : \|f\|_{v,q} \leq 1 \}$$

form its basis. For more details the reader is referred to [6].

For given $v \in V^p$ and $q \in cs(\Omega)$ the function $f \in MV^p(X, \Omega)$ is said to be $I_{v,q}$-vanished at infinity on $X$ if for each $\epsilon > 0$ there exists a compact subset $K \subseteq X$ such that $\int_K (vq(f))^p d\mu < \epsilon$, where the symbol "c" stands for the complement of $K$. The set of all functions $f \in MV^p(X, \Omega)$ such that for each $v \in V^p$ and $q \in cs(\Omega)$ they $I_{v,q}$-vanish at infinity on $X$, is denoted by $MV^p_0(X, \Omega)$. Clearly, $MV^p_0(X, \Omega)$ is a closed subspace of $MV^p(X, \Omega)$.
Definition 1.1. A measurable transformation $T : X \to X$ is said to be non-singular if $\mu(T^{-1}(F)) = 0$ whenever $\mu(F) = 0$. Corresponding to each non-singular measurable transformation $T$ on $X$ a composition operator $C_T : MV_p(X, \Omega) \to MV_p(X, \Omega)$ is defined by the mapping $f \mapsto f \circ T$.

Definition 1.2. A (closed) subspace $M$ of $MV_p(X, \Omega)$ is invariant for the composition operator $C_T$ whenever $C_T(M) \subseteq M$.

Most classical results pertaining composition operators on the weighted locally convex spaces of continuous functions have been studied in [8] in detail. Parallel to these results, we shall study some behaviours of composition operators from the viewpoint of some classical operator properties. Furthermore, in comparing the following theorem with Proposition 3.1 in [6] the advantage of using Radon-Nikodym derivation is made clear.

Definition 1.3. For a non-singular measurable transformation $T : X \to X$ with $T^{-1}\Sigma \subseteq \Sigma$ and nonnegative $f \in L^p(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique integrable and $T^{-1}\Sigma$-measurable function $E(f)$ such that for each $F \in T^{-1}\Sigma$ we have $\int_F f d\mu = \int_F E(f) d\mu$. The function $E(f)$ is called the conditional expectation of $f$ with respect to $T^{-1}\Sigma$.

Normally, this fact can be extended to complex-valued functions by passing conditional expectation to the real and imaginary parts. For a $\Sigma$-measurable function $f$ if $E(f)$ exists then $f$ is called conditionable. It is worth noting that all functions in $L^p(\Sigma)$ are conditionable. Hence a linear map $E : L^p(\Sigma) \to L^p(T^{-1}\Sigma)$ can be defined uniquely by the assignment $f \mapsto E(f)$. The mapping $E$ is an idempotent and contraction. Also in case $p = 2$, it is the orthogonal projection of $L^2(\Sigma)$ onto $L^2(T^{-1}\Sigma)$. Some of its useful properties used in this note are listed as follows:

- If $f$ is a $T^{-1}\Sigma$-measurable function, then $E(fg) = fE(g)$;
- $|E(f)|^p \leq E(|f|^p)$;
- If $f \geq 0$ then $E(f) \geq 0$ and if $f > 0$ then $E(f) > 0$.

For more details on the properties of conditional expectation see [7, 5].

2 Main Result

This section makes the essential use of Radon-Nikodym derivation and conditional expectation for studying the behavior of composition operators from the viewpoint of some classical operator properties. Furthermore, in comparing the following theorem with Proposition 3.1 in [6] the advantage of using Radon-Nikodym derivation is made clear.

Theorem 2.1. Suppose $U^p$ and $V^p$ be $N_p$ families on $X$ such that $U^p \subseteq V^p \circ T$. If $T : X \to X$ is a measurable transformation and $h \in L^\infty(\Sigma)$ then $C_T : MV_p(X, \Omega) \to MV_p(X, \Omega)$ is a bounded linear composition operator.

Proof. Let $f \in MV_p(X, \Omega)$ and $u \in U^p$ be arbitrary. Since $U^p \subseteq V^p \circ T$ there exists a $v \in V^p$
such that \( u \leq v \circ T \). So for each \( q \in cs(\Omega) \) we have

\[
\| C_T(f) \|_{u,q} = \| (f \circ T) \|_{u,q}
\]
\[
= \left( \int_X (uq(f \circ T))^p d\mu \right)^{\frac{1}{p}}
\]
\[
\leq \left( \int_X (v \circ Tq(f \circ T))^p d\mu \right)^{\frac{1}{p}}
\]
\[
= \left( \int_{T^{-1}X} (vq(f))^p d\mu \circ T^{-1} \right)^{\frac{1}{p}}
\]
\[
= \left( \int_X (vq(f))^p h d\mu \right)^{\frac{1}{p}}
\]
\[
\leq ||h||_{\frac{1}{p}} \left( \int_X (vq(f))^p d\mu \right)^{\frac{1}{p}}
\]
\[
= ||h||_{\frac{1}{p}} || f ||_{v,q}.
\]

Hence \( f \circ T \in MU^p(X,\Omega) \) and this proves that \( C_T \) is bounded. \( \square \)

In the following theorem, we are going to answer this question: how could the boundedness of a composition operator \( C_T \) is related to the system of weights of underlying spaces?

**Theorem 2.2.** Let \( V^p \) and \( U^p \) be \( N_p \) family on \( X \) such that \( V^p \geq 1 \) (for each \( v \in V^p, \ v \geq 1 \)). For each \( u \in U^p \), suppose that \( hE(u^p) \circ T^{-1} \in L^\infty(\Sigma) \). Then the measurable transformation \( T : X \to X \) induces a bounded linear composition operator \( C_T : MV^p(X,\Omega) \to MU^p(X,\Omega) \).

**Proof.** Let \( f \in MV^p(X,\Omega), \ u \in U^p \) and \( q \in cs(\Omega) \) be arbitrary elements. Then

\[
\| C_T(f) \|_{u,q}^p = \int_X (uq(f \circ T))^p d\mu
\]
\[
= \int_X E(u^p)(q(f \circ T))^p d\mu
\]
\[
\leq \int_X hE(u^p) \circ T^{-1}(q(f))^p d\mu
\]
\[
\leq ||hE(u^p) \circ T^{-1}||_\infty \int_X (q(f))^p d\mu
\]
\[
\leq ||hE(u^p) \circ T^{-1}||_\infty \int_X (vq(f))^p d\mu
\]
\[
= ||hE(u^p) \circ T^{-1}||_\infty || f ||_{v,q}^p.
\]

Therefore \( C_T \) is a bounded operator. \( \square \)

**Theorem 2.3.** For a surjective measurable transformation \( T : X \to X \) let \( U^p \geq V^p \circ T \) and \( h \geq \delta \) a.e for some \( \delta > 0 \). Then a composition operator \( C_T : MV^p(X,\Omega) \to MU^p(X,\Omega) \) is bounded below.

**Theorem 2.4.** Let \( V^p \) be \( N_p \) family on \( X \) such that \( V^p \leq V^p \circ T \). If \( h \in L^\infty(\Sigma) \) and \( T : X \to X \) is a measurable transformation such that \( T(K^c) \subseteq K^c \) for each compact subset \( K \) of \( X \). Then \( C_T : MV^p(X,\Omega) \to MV^p(X,\Omega) \) leaves the subspace \( MV^p_{0}(X,\Omega) \) invariant.
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Generalization of the related Carlson type inequality for fuzzy integrals

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Abstract
In this paper, a related inequality to Carlson type inequality is studied for fuzzy integrals. Several examples are given to illustrate the validity of these inequality. Finally, a conclusion is given in Section 2.

Keywords: Carlson’s inequality; Sugeno integral; Non-additive measure; Fuzzy integral inequality.

Mathematics Subject Classification: 03E72, 26E50, 28E10.

1 Introduction
In the sequel, we present some definitions and basic properties of the Sugeno integral that will be used in the next section.

In view of the fact that
\[
\int_A f \ast g \, d\mu \leq \left( \int_A f \, d\mu \right) \ast \left( \int_A g \, d\mu \right)
\]
holds for comonotone functions \( f, g \in \mathcal{F}_+(X) \) wherever \( \ast \geq \max \) (for a similar result, see [4])

\[\text{Definition 1.1.}\] Let \( \Sigma \) be a \( \sigma \)-algebra of subsets of \( X \) and let \( \mu : \Sigma \to [0, \infty) \) be a non-negative, extended real-valued set function, we say that \( \mu \) is a fuzzy measure iff:

(FM1) \( \mu(\emptyset) = 0 \);
(FM2) \( E, F \in \Sigma \) and \( E \subseteq F \) imply \( \mu(E) \leq \mu(F) \) (monotonicity);
(FM3) \( E_p \subseteq \Sigma, E_1 \subseteq E_2 \subseteq \ldots \) imply \( \lim \mu(E_p) = \mu(\bigcup_{n=1}^{\infty} E_n) \) (continuity from below);
(FM4) \( E_p \subseteq \Sigma, E_1 \supseteq E_2 \supseteq \ldots, \mu(E_1) < \infty \) imply \( \lim \mu(E_p) = \mu(\bigcap_{n=1}^{\infty} E_p) \) (continuity from above).

If \( f \) is a non-negative real-valued function on \( X \), we will denote \( F_\alpha = \{ x \in X \mid f(x) \geq \alpha \} = \{ f \geq \alpha \} \), the \( \alpha \)-level of \( f \), for \( \alpha > 0 \). \( F_0 = \{ x \in X \mid f(x) > 0 \} = \text{supp}(f) \) is the support of \( f \). We know that: \( \alpha \leq \beta \Rightarrow \{ f \geq \beta \} \subseteq \{ f \geq \alpha \} \).

If \( \mu \) is a following fuzzy measure on \( X \), we define the following:

\[ \mathcal{F}^\mu(X) = \{ f : X \to [0, \infty) \mid f \text{ is } \mu \text{-measurable} \}. \]

\[\text{Definition 1.2.}\] Let \( \mu \) be a fuzzy measure on \( (X, \Sigma) \). If \( f \in \mathcal{F}^\mu(X) \) and \( A \in \Sigma \), then the Sugeno integral (or fuzzy integral) of \( f \) on \( A \), with respect to the fuzzy measure \( \mu \), is defined (see [5]) as
\[
\int_A f \, d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)).
\]
Where ∨, ∧ denotes the operation sup and inf on \([0, \infty)\) respectively. In particular, if \(A = X\) then:

\[
\int_X f d\mu = \int f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)).
\]

(3)

**Remark 1.3.** Let \(F\) be distribution function associated to \(f\) on \(A\), that is,

\[
F(\alpha) = \mu(A \cap \{f \geq \alpha\}).
\]

From a numerical point of view, the Sugeno integral can be calculated by solving the equation \(F(\alpha) = \alpha\).

**Theorem 1.4.** (Fuzzy Carlson’s inequality [2]). Let \(f : [0, 1] \rightarrow [0, \infty)\) be a nondecreasing function and \(\mu\) the Lebesgue measure on \(\mathbb{R}\). Then the inequality

\[
\int_0^1 f(x) d\mu(x) \leq \sqrt{2} \left( \int_0^1 x^2 f^2(x) d\mu(x) \right)^{\frac{1}{2}} \left( \int_0^1 f^2(x) d\mu(x) \right)^{\frac{1}{2}}
\]

holds.

The following theorem shows the Hermite-Hadamard inequality for fuzzy integrals that appears in [1] and we use it for estimating fuzzy integrals.

**Theorem 1.5.** Let \(g : [0, 1] \rightarrow [0, \infty)\) be a convex function such that \(g(0) < g(1)\) and \(\mu\) be the Lebesgue measure on \(\mathbb{R}\). Then

\[
\int g d\mu \leq \min\{ \frac{g(1)}{1 + g(1) - g(0)}, 1 \}.
\]

(4)

2 Main Result

In this section, we prove a general related inequality to Carlson type inequality for Sugeno integrals.

**Theorem 2.1.** Let \(f : [0, 1] \rightarrow [0, \infty)\) be a strictly increasing function and \(\mu\) be the Lebesgue measure on \(\mathbb{R}\). And let \(* : [0, \infty)^2 \rightarrow [0, \infty)\) be continuous and nondecreasing in both arguments and bounded from below by maximum. Then, the inequality

\[
\frac{1}{2} \int_0^1 f(x) f(\alpha) d\mu(x) \leq \left( \int_0^1 x^2 f^2(x) d\mu(x) \right)^{\frac{1}{2}} \star \left( \int_0^1 f^2(x) d\mu(x) \right)^{\frac{1}{2}}.
\]

(5)

holds.

**Proof.** Since \(\int_0^1 x d\mu(x) = \frac{1}{2}\), then

\[
\frac{1}{4} \left( \int_0^1 f(x) d\mu(x) \right)^2 = \left( \int_0^1 x d\mu(x) \right)^2 \left( \int_0^1 f(x) d\mu(x) \right)^2.
\]

(6)

using Chebyshev’s inequality [3] we have

\[
\left( \int_0^1 x d\mu(x) \right)^2 \left( \int_0^1 f(x) d\mu(x) \right)^2 \leq \left( \int_0^1 x f(x) d\mu(x) \right)^2.
\]

(7)

Taking \(f = g\) in (1), we have

\[
\int_0^1 f \star f d\mu \leq \left( \int_0^1 f d\mu \right) \star \left( \int_0^1 f d\mu \right).
\]

(8)
This inequality implies that

$$\frac{1}{4} \left( \int_0^1 f(x) * f(x) \text{d}\mu(x) \right)^2 \leq \frac{1}{4} \left[ \left( \int_0^1 f(x) \text{d}\mu(x) \right)^2 \right] \left( \int_0^1 f(x) \text{d}\mu(x) \right)^2$$

(9)

Then we show that

$$\frac{1}{4} \left( \int_0^1 f(x) * f(x) \text{d}\mu(x) \right)^2 \leq \left( \int_0^1 x f(x) \text{d}\mu(x) \right)^2 \leq \left( \int_0^1 x^2 f^2(x) \text{d}\mu(x) \right)^\frac{1}{2}.$$ (12)

This inequality implies that

$$\frac{1}{2} \int_0^1 f(x) * f(x) \text{d}\mu(x) \leq \left( \int_0^1 x^2 f^2(x) \text{d}\mu(x) \right)^\frac{1}{2}.$$ (14)

This completes the proof.

**Example 2.2.** Suppose that \( f(x) = x^\frac{1}{2} \) for all \( x \in [0, 1] \), \( \mu \) be the Lebesgue measure on \( \mathbb{R} \) and \( * = \sup \). Then \( * \) satisfies in Theorem 2.1 conditions. We show that the following inequality holds.

$$\frac{1}{2} \int_0^1 \sup(x^\frac{1}{2}, x^\frac{1}{2}) \text{d}\mu(x) \leq \sup \left( \left( \int_0^1 x^2 \text{d}\mu(x) \right)^\frac{1}{2}, \left( \int_0^1 x \text{d}\mu(x) \right)^\frac{1}{2} \right).$$

We know that

$$\int_0^1 x^\frac{1}{2} \text{d}\mu(x) \approx 0.618.$$ Using (4) we can estimate \( \int_0^1 x^3 \text{d}\mu(x) \) i.e.

$$\left( \int_0^1 x^3 \text{d}\mu(x) \right)^\frac{1}{2} \leq 0.707.$$ Also a simple calculation reveals that

$$\left( \int_0^1 x \text{d}\mu(x) \right)^\frac{1}{2} \approx 0.707.$$ Hence

$$\frac{1}{2} \int_0^1 \sup(x^\frac{1}{2}, x^\frac{1}{2}) \text{d}\mu(x) \approx 0.309 \leq 0.707 \approx \sup \left( \left( \int_0^1 x^2 \text{d}\mu(x) \right)^\frac{1}{2}, \left( \int_0^1 x \text{d}\mu(x) \right)^\frac{1}{2} \right).$$

Which implies that inequality (5) holds.

**Example 2.3.** Let \( f(x) = x \) for all \( x \in [0, 1] \), \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Let \( * = S_D \) be the drastic sum:

$$S_D(x, y) = \begin{cases} 1 & \text{if } \min\{x, y\} > 0, \\ \max\{x, y\} & \text{if } \min\{x, y\} = 0. \end{cases}$$

Then we show that

$$\frac{1}{2} \int_0^1 S_D(x, x) \text{d}\mu(x) \leq S_D \left( \left( \int_0^1 x^4 \text{d}\mu(x) \right)^\frac{1}{2}, \left( \int_0^1 x^2 \text{d}\mu(x) \right)^\frac{1}{2} \right).$$
3 Conclusion

We have introduced a general related inequality of Carlson type inequality for the Sugeno integrals. For further investigation we propose to consider the Carlson’s inequality for the Choquet integral.

References


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Mann’s algorithm for strict pseudo-contractions in CAT(0) spaces

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Abstract

In this paper, we introduce the notion of strict pseudo-contractive mappings in the framework of CAT(0) spaces. Some properties of such mappings including demi-closed principle are investigated. Also, w-convergence of the well-known Mann iterative algorithm is established for strict pseudo-contractive mappings.

Keywords: CAT(0) space, Strict pseudo-contraction, Mann’s algorithm, Fixed point.

Mathematics Subject Classification: 47H09, 47H10

1 Introduction

A metric space $(X, d)$ is a CAT(0) space if it is geodesically connected and if every geodesic triangle in $X$ is at least as thin as its comparison triangle in the Euclidean plane. For other equivalent definitions and basic properties, we refer the reader to standard texts such as [1]. Let $x, y \in X$. We write $\lambda x \oplus (1 - \lambda) y$ for the unique point $z$ in the geodesic segment joining from $x$ to $y$ such that

$$d(z, x) = (1 - \lambda)d(x, y) \quad \text{and} \quad d(z, y) = \lambda d(x, y).$$

(1)

We also denote by $[x, y]$ the geodesic segment joining from $x$ to $y$, that is, $[x, y] = \{ \lambda x \oplus (1 - \lambda)y : \lambda \in [0, 1] \}$. A subset $C$ of a CAT(0) space is convex if $[x, y] \subseteq C$ for all $x, y \in C$.

Berg and Nikolaev in [2] have introduced the concept of quasilinearization in a metric space $X$. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2} \left( d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d) \right), \quad (a, b, c, d \in X).$$

(2)

We say that $X$ satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d)$$

(3)

for all $a, b, c, d \in X$. It known [2, Corollary 3 Hossein Dehghan and Jamal Rooin] that a geodesically connected metric space is CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality. In 2010, Kakavandi and Amini [3] introduced the concept of dual space for CAT(0) space $X$ as a metric space

$$X^* := \{ [\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X \},$$

where $[\overrightarrow{ab}] = \{ s \overrightarrow{cd} : t \langle \overrightarrow{ab}, \overrightarrow{x y} \rangle = s \langle \overrightarrow{cd}, \overrightarrow{x y} \rangle \forall x, y \in X \}$. For more details see [3].
**Definition 1.1.** (See [3]) A sequence \( \{x_n\} \) in a CAT(0) space \( X \) is said to \( w \)-converges to \( x \) if
\[
\lim_{n \to \infty} \langle x_n, ax \rangle = 0 \quad \text{for all } a, b \in X.
\]

It is known that \( X^* \) separates the convex sets [3, Proposition 2.3 Hossein Dehghan and Jamal Rooin], more precisely, if \( A \) and \( B \) are nonempty, closed, and disjoint convex subsets of \( X \) such that one of them is compact, then there exist \( x, y, z \in X \) and two real numbers \( \alpha < \beta \) such that
\[
\langle x_\beta, z \rangle \leq \alpha < \beta \leq \langle x_\alpha, z \rangle \quad (a \in A, b \in B).
\]

**Lemma 1.2.** Let \( C \) be a nonempty closed convex subset of a complete CAT(0) space \( X \) and \( \{x_n\} \) be a sequence in \( C \). If \( \{x_n\} \) \( w \)-converges to \( x \), then \( x \in C \).

**Proof.** It is sufficient to prove that \( C \) is closed in weak topology. To this end, let \( x_0 \in C \). Using (4), we have
\[
\langle x_\beta, z_0 \rangle < \alpha < \langle x_\alpha, z_0 \rangle
\]
for some \( x, y, z \in X \), some real number \( \alpha \) and for all \( c \in C \). Set
\[
V = \{u \in X : \langle x_\beta, z_0 \rangle < \alpha\}.
\]
Then \( x_0 \in V, V \cap C = \emptyset \) and \( V \) is open in the weak topology. \( \square \)

**Definition 1.3.** Let \( C \) be a nonempty subset of a CAT(0) space \( X \). A mapping \( T : C \to X \) is called strict pseudo-contraction if there exists a constant \( 0 \leq \kappa < 1 \) such that
\[
d^2(Tx, Ty) \leq d^2(x, y) + 4\kappa d^2(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}Ty, \frac{1}{2}x \oplus \frac{1}{2}y)
\]
for all \( x, y \in C \). If (5) holds, we also say that \( T \) is a \( \kappa \)-strict pseudo-contraction.

The definition of pseudo-contractions finds its origin in Hilbert spaces. Note that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings. That is, \( T \) is nonexpansive if and only if \( T \) is a 0-strict pseudo-contraction.

We need the following lemmas in the sequel.

**Lemma 1.4.** [4, Lemma 2.5 Hossein Dehghan and Jamal Rooin] A geodesic space \( X \) is a CAT(0) space if and only if the following inequality
\[
d^2(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d^2(x, z) + (1 - \lambda)d^2(y, z) - \lambda(1 - \lambda)d^2(x, y)
\]
is satisfied for all \( x, y, z \in X \) and \( \lambda \in [0, 1] \).

**Lemma 1.5.** [4, Lemma 2.4 Hossein Dehghan and Jamal Rooin] Let \( X \) be a CAT(0) space. Then
\[
d(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d(x, z) + (1 - \lambda)d(y, z)
\]
for all \( x, y, z \in X \) and \( \lambda \in [0, 1] \).

### 2 Main Result

We recall that, given a self-mapping \( T \) of a closed convex subset \( C \) of a CAT(0) space \( X \), Mann’s algorithm generates a sequence \( \{x_n\} \) in \( C \) by the recursive formula
\[
x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n)Tx_n, \quad n \geq 0,
\]
where the initial guess \( x_0 \) is arbitrary, and where \( \{\alpha_n\} \) is a real control sequence in the interval \((0, 1)\).

Mann’s algorithm has been extensively investigated for nonexpansive mappings in CAT(0) spaces. In this section, we establish \( w \)-convergence of Mann iterative algorithm for strict pseudo-contractive mappings. To proceed in this direction we need the following results.
Proposition 2.1. Let $C$ be a nonempty subset of a CAT(0) space $X$ and $T : C \to X$ be a mapping. If $T$ is a $\kappa$-strict pseudo-contraction, then $T$ satisfies the Lipschitz condition
\[ d(Tx, Ty) \leq \frac{1 + \kappa}{1 - \kappa} d(x, y). \] (9)

Proof. Using Cauchy-Schwarz inequality and (6), we have
\[
d^2(Tx, Ty) \leq d^2(x, y) + 4\kappa d^2\left(\frac{1}{2} x \oplus \frac{1}{2} Ty, \frac{1}{2} Tx \oplus \frac{1}{2} y\right)
\[
\leq d^2(x, y) + \kappa (d^2(x, y) + d^2(Ty, Tx) + d^2(x, Tx) + d^2(y, Ty) - d^2(x, Ty) - d^2(y, Tx))
\[
= d^2(x, y) + \kappa (d^2(x, y) + d^2(Ty, Tx)) + 2\kappa d(x, y) d(Ty, Tx).
\] (10)

It follows that
\[ (1 - \kappa)d^2(Tx, Ty) - 2\kappa d(x, y) d^2(Tx, Ty) - (1 + \kappa)d^2(x, y) \leq 0. \]
Solving this quadratic inequality, we obtain the Lipschitz condition (9). □

Theorem 2.2. (Demiclosed principle) Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : C \to X$ be a $\kappa$-strict pseudo-contraction. If the bounded sequence $\{x_n\}$ w-converges to $x^*$ and $d(x_n, Tx_n) \to 0$, then $Tx^* = x^*$.

Proof. By a simple computation one can see that
\[ d^2(a, b) = d^2(a, c) + d^2(b, c) - 2\langle \overrightarrow{ac}, \overrightarrow{bc} \rangle \] (11)
for all $a, b, c \in X$. Let $\{x_n\}$ w-converges to $x^*$ and
\[ f(x) = \limsup_{n \to \infty} d^2(x_n, x), \quad (x \in X). \]

It follows from (11) that $f(x) = f(x^*) + d^2(x, x^*)$ for all $x \in X$. In particular,
\[ f(Tx^*) = f(x^*) + d^2(Tx^*, x^*). \] (12)

On the other hand, using similar method as in (10), we have
\[
d^2(Tx_n, Tx^*) \leq d^2(x_n, x^*) + \kappa (d^2(x_n, Tx_n) + d^2(x^*, Tx^*)) + 2\kappa d(x_n, Tx_n) d(x^*, Tx^*).
\]

It follows from the assumption $d(x_n, Tx_n) \to 0$ that
\[ f(Tx^*) = \limsup_{n \to \infty} d^2(x_n, Tx^*) = \limsup_{n \to \infty} d^2(Tx_n, Tx^*) \leq f(x^*) + \kappa d^2(x^*, Tx^*). \]

This together with (12) that $Tx^* = x^*$. □

Theorem 2.3. Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$, $T : C \to C$ be a $\kappa$-strict pseudo-contraction for some $0 \leq \kappa < 1$ such that the fixed point set $F(T)$ is nonempty. Let $\{x_n\}$ be the sequence generated by Mann’s algorithm (8). If the control sequence $\{\alpha_n\}$ is chosen so that $\kappa < \alpha_n < 1$ for all $n \geq 0$ and $\sum_{n=0}^{\infty} (\alpha_n - \kappa) (1 - \alpha_n) = \infty$, then $\{x_n\}$ w-converges to a fixed point of $T$. 
Proof. Let $p \in F(T)$. It follows from (5) and (6) that
\[
d^2(x_{n+1}, p) = d^2(\alpha_n x_n \oplus (1 - \alpha_n)Tx_n, p) \\
\leq \alpha_n d^2(x_n, p) + (1 - \alpha_n)d^2(Tx_n, p) - \alpha_n(1 - \alpha_n)d^2(x_n, Tx_n) \\
\leq \alpha_n d^2(x_n, p) + (1 - \alpha_n)(d^2(x_n, p) + \kappa d^2(x_n, Tx_n)) - \alpha_n(1 - \alpha_n)d^2(x_n, Tx_n) \\
= d^2(x_n, p) - (\alpha_n - \kappa)(1 - \alpha_n)d^2(x_n, Tx_n).
\]
That is, the sequence $\{d(x_n, p)\}$ is decreasing. \hfill \Box

References


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Application of a new version of Haar wavelet for solving linear integral equations

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Abstract

In the present paper, a new version of Haar wavelet which we'd like to call M-Haar wavelet is introduced to be used for solving linear Fredholm and Volterra integral equations. To demonstrate this new method's efficacy and its advantages over Haar wavelet method, some numerical examples with exact solutions are solved.

Keywords: M-Haar wavelet, Haar wavelet, Integral equations, Collocation method

Mathematics Subject Classification: 42C15

1 Introduction

Many problems of theoretical physics and other disciplines lead to linear integral equations. Beginning from 1991 the wavelet method has been applied for solving integral equations, a short survey on these papers can be found in [4]. Haar wavelet is the most simple one of different wavelets.

The Haar wavelet method for solving linear integral equations of different type was proposed in [1] and for nonlinear Fredholm integral equations in [2]. In the present paper, the M-Haar wavelet, a new version of Haar wavelet, is applied for solving linear Fredholm and Volterra integral equations. The proposed method is based on the collocation technique. The method is tested by the help of some numerical examples, for which the exact solution is known.

In order to solve linear Fredholm equations, we define a new version of Haar wavelet which we'd like to call M-Haar wavelet. Let $t \in [0, 1]$; The M-Haar wavelet family of level resolution $J$ is defined as

$$r_i(t) = \begin{cases} 
1 & \text{for } t \in [\lambda_1, \lambda_2), \\
\frac{1}{m} & \text{for } t \in [\lambda_2, \lambda_3), \\
\frac{1}{m} & \text{for } t \in [\lambda_3, \lambda_4), \\
-1 & \text{for } t \in [\lambda_4, \lambda_5], \\
0 & \text{elsewhere}
\end{cases} \quad (1)$$

such that

$$\lambda_1 = \frac{k}{m}, \quad \lambda_2 = \frac{(k + \frac{1}{2})}{m}, \quad \lambda_3 = \frac{(k + \frac{1}{2})}{m}, \quad \lambda_4 = \frac{(k + \frac{3}{2})}{m}, \quad \lambda_5 = \frac{(k + 1)}{m}.$$ 

and $m = 2^j; j = 0, 1, ..., J$ indicates the level of the wavelet; $k = 0, 1, ..., m - 1$ is the translation parameter. The index $i$ is calculated according to the formula $i = m + k + 1$. In the case $m = 1$ and $k = 0$ we have $i = 2$; the maximal value of $i$ is $2M$, where $M = 2^J$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $r_1(t) = 1$ for $t \in [0, 1]$.
The function $r_i$ can be discretized by dividing the interval $[0,1]$ into $2M$ subintervals with equal length $\triangle(t) = \frac{1}{2M}$. We introduce the collocation points as

$$t(l) = \left( l - \left( \frac{J}{2J+1} \right) \right) \triangle(t), \quad l = 1, 2, ..., 2M.$$  \hfill (2)

The Haar coefficient matrix $R$ is defined as $R(i;l) = r_i(t(l))$.

Following is the Haar coefficient matrix for $J = 1$ and $J = 2$.

$$R = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1/2 & -1/2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1/2 & -1/2 & -1 \\
1 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1/4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1/4 \\
\end{bmatrix}.$$  \hfill (3)

2 Main result

To solve integral equations, we have to find a real function $u$ which is defined in the interval $[0,1]$. The function $u$ can be expanded with respect to the M-Haar wavelet series

$$u(t) = \sum_{i=0}^{2M} a_i r_i(t),$$  \hfill (4)

where $a_i$ are the wavelet coefficients. Therefore, the discrete form of the equation is

$$u(t_l) = \sum_{i=0}^{2M} a_i r_i(t_l) = \sum_{i=0}^{2M} a_i R(i,l).$$  \hfill (5)

Now for $2M \times 2M$ Matrix $R$ and $2M$–dimensional row vectors $u$ and $a$, the matrix representation of Equation (5) becomes $u = aR$.

Now we solve linear Fredholm integral equation by using the M-Haar wavelet method and then, this method is compared with the Haar wavelet method. A linear form of Fredholm integral equation [3] is defined as

$$u(x) - \int_0^1 K(x,t)u(t)\,dt = f(x) \quad x, t \in [0,1].$$  \hfill (6)

Replacing (4) into (6), we find

$$\sum_{i=1}^{2M} a_i r_i(x) - \sum_{i=1}^{2M} a_i G_i(x) = f(x),$$  \hfill (7)
where

\[ G_i(x) = \int_1^x K(x, t) r_i(t) \, dt. \]  

(8)

The following linear system of equations for the coefficients \( a_i \) is obtained by putting collocation points (2) in the (7),

\[ \sum_{i=1}^{2M} a_i [r_i(x_l) - G_i(x_l)] = f(x_l) \quad l = 1, 2, ..., 2M. \]  

(9)

By using the matrix form of

\[
\begin{align*}
    u &= u(x_l), \\
    F &= f(x_l), \\
    G &= G_i(x_l),
\end{align*}
\]

the equations (5) and (9) get the form

\[ u = aR, \quad a(R - G) = F. \]  

(10)

The advantages of the M-Haar method over the Haar method are shown by solving some numerical examples. We consider some equations for which the exact solution \( u_{ex} \) is known and then obtain the error estimates \( u \) and \( u_H \) corresponding to M-Haar wavelets and Haar wavelets, respectively, where accuracy of the results is estimated by the error functions

\[
\begin{align*}
    \beta_J &= \max\{|u(x_l) - u_{ex}(x_l)|\} \quad l = 1, \ldots, 2M, \\
    \alpha_J &= \max\{|u_H(x_l) - u_{ex}(x_l)|\} \quad l = 1, \ldots, 2M.
\end{align*}
\]

All calculations are made with the aid of MATLAB programs.

**Example 2.1.** If \( K(x, t) = x + t \) and \( f(x) = x^2 \), then Equation (6) has the exact solution

\[ u_{ex} = x^2 - 5x - \frac{17}{6}. \]

By evaluating the integrals (8), entries of \( 2M \times 2M \) matrix \( G \) are obtained as

\[
G_i(x_l) = \begin{cases} 
    x_l + 0.5 & \text{for } i = 1, \\
    -\frac{3m + 1}{16m^3} & \text{for } i = 2, \ldots, 2M.
\end{cases}
\]

(11)

The errors for \( J = 1, 2 \) are shown in Table 1.

**Table 1:** Error functions \( \alpha_J \) and \( \beta_J \)

<table>
<thead>
<tr>
<th>( J )</th>
<th>( 2M )</th>
<th>( \alpha_J )</th>
<th>( \beta_J )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>2.6E - 1</td>
<td>8.0E - 2</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>7.2E - 2</td>
<td>8.7E - 3</td>
</tr>
</tbody>
</table>

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As demonstrated in Table 1, the value of the error in the M-Haar method in the first step, when $J = 1$ and matrices are $4 \times 4$, is close to the value of the error in the Haar method in the second step, when $J = 2$ and matrices are $8 \times 8$. Therefore, we get more exact solutions in the early steps.

As in the Example 1, the value of the error obtained in the M-Haar method significantly decreases. In the new method, by the use of $4 \times 4$ matrices, a negligible value of error is obtained whereas the same is obtained using $8 \times 8$ matrices in the Haar method. Therefore, the number of operations in the latter is nearly 4 times of the new method.

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Ternary structure of \( l_1^\omega (S) \)

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Abstract

Let \( S \) be a ternary semigroup. In this article we introduce our notation and prove some elementary properties of a ternary weight function \( \omega \) on \( S \). Also, we make ternary weighted algebra \( l_1^\omega (S) \) and show that \( l_1^\omega (S) \) is a ternary Banach algebra.

Keywords: Ternary Banach Algebra, Ternary Semigroup, Ternary Weight Function.

Mathematics Subject Classification: 43A10, 43A07, 43A15, 43A15

1 Introduction

Ternary algebraic operations were considered in the XIX-th century by several mathematicians such as A. Cayley [1] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 (see [7]). The simplest example of such non-trivial ternary operation is given by the following composition rule:

\[
\{a,b,c\}_{ijk} = \sum_{1 \leq i,m,n \leq N} a_{nl} b_{jm} c_{mkn} \quad (i,j,k = 1,2,\cdots,N).
\]

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their possible applications in physics. Some significant physical applications are described in [9] and [10]. The notion of an n-ary group was introduced by Dörnte [4] (inspired by E. Nöther) and is a natural generalization of the notion of a group and a ternary group considered by Certain [2] and Kasner [8]. In 1940 E. L. Post [11] published an extensive study of n-groups in which the well-known Post’s Coset Theorem appeared.

Definition 1.1. A nonempty set \( G \) with one ternary operation \( [ \ ] : G \times G \times G \to G \) is called a ternary groupoid and denoted by \((G, [\ ] )\). The ternary groupoid \((G, [\ ] )\) is called a ternary semigroup if the operation \( [ \ ] \) is associative, i.e., if \([xyz]uv = [xyzu]v = [xy[zuv]]\) hold for all \(x,y,z,u,v \in G\).

Definition 1.2. A ternary semigroup \((G, [\ ] )\) is a ternary group if for all \(a,b,c \in G\), there are \(x,y,z \in G\) such that

\([xab] = [ayb] = [abz] = c\).

In a ternary group, the equation \([xzx] = x\) has a unique solution which is denoted by \(z = \bar{x}\) and called the skew element to \(x\) [4]. Other properties of skew elements are described in [5] and [6].
**Definition 1.3.** [12] Let \((G, [\cdot])\) be a ternary group, \(^{-1}\) it's inverse operation, and \(G\) be equipped with a topology \(O\). Then, we say that \((G, [\cdot], O)\) is a topological ternary group iff

(i) ternary operation \([\cdot]\) is continuous in \(O\); and

(ii) The \(2\)-operation \(^{-1}\) is continuous in \(O\).

Let \(G\) be a ternary group and \(A\) any subset of \(G\). We denote by \(\overline{A}\) denote set of all \(x\) (skew element) such that \(x \in A\) i.e. :

\[\overline{A} = \{x : x \in A\}\]

**Definition 1.4.** A ternary Banach algebra is a complex Banach space \(A\), equipped with a ternary product \((x, y, z) \rightarrow [x, y, z]\) of \(A^3\) into \(A\), which is associative in the sense that \([x, y, z], u, v] = [x, [y, z], u, v]\), and satisfy \[\|[x, y, z]\| \leq \|x\|\|y\|\|z\|\].

Let \(A\) be a ternary Banach algebra and \(A_1, A_2,\) \(A_3\) subsets of \(A\). We define

\[A_1, A_2, A_3 = \{[a_1, a_2, a_3] : a_1 \in A, a_2 \in A_2, a_3 \in A_3\}\]

Let \(A\) and \(B\) be a ternary Banach algebra. A linear mapping \(\phi : A \rightarrow B\) is called to be a ternary homomorphism if \(\phi(x, y, z) = [\phi(x), \phi(y), \phi(z)]\).

### 2 Main Result

**Definition 2.1.** A ternary weight on ternary semigroup \((S, [\cdot])\) is a positive real function \(\omega : S \rightarrow \mathbb{R}^+\) such that

\[\omega([rst]) \leq \omega(r)\omega(s)\omega(t)\]

for all \(r, s, t \in S\).

**2.2.** If \(\omega_1\) and \(\omega_2\) are two ternary weight function, then \(\omega_1\omega_2\) is a ternary weight function.

**Example 2.3.** Let \((\mathbb{Z}, [\cdot])\) with \([xyz] = x + y + z\) be a ternary group. For \(\alpha > 0\), define \(\omega_\alpha = (1 + |n|)^\alpha\). Then \(\omega_\alpha\) is a weight ternary function.

**Theorem 2.4.** Let \(K\) be a compact subset of a topological ternary group \(G\), \(\omega\) a ternary weight function on \(G\) and the interior of \(\{x : \omega(x) < n\}\) is nonempty for some \(n \in \mathbb{N}\). Then there exists \(a, b \in \mathbb{R}\) such that

\[0 < a \leq \omega(x) \leq b\]

for all \(x \in K\).

**Proposition 2.5.** Let \(\omega\) be a ternary weight function on a ternary semigroup \(S\) such that \(\{x \in S : \omega(x) < \epsilon\}\) is finite for some \(\epsilon > 0\). Then \(\omega(x) \geq 1\), for all \(x \in S\).

**Corollary 2.6.** Let \(\omega\) be a ternary weight function on a compact topological ternary group \(G\) and the interior of \(\{x : \omega(x) < n\}\) is nonempty for some \(n \in \mathbb{N}\). Then \(\omega(x) \geq 1\) for all \(x \in G\).
Lemma 2.7. Let $\omega$ be a continuous ternary weight function on a ternary semigroup $S$. Then $x,y,f \in C(S,\omega)$, for all $f \in C(S,\omega)$ and $x \in S$, if and only if $\Omega(x,y)$ is finite.

Corollary 2.8. Let $\omega$ be a continuous ternary weight function on a compact topological ternary group $G$ and the interior of $\{ x : \omega(x) < n \}$ is nonempty for some $n \in \mathbb{N}$. Then $x,y,f \in C(S,\omega)$ for all $f \in C(S,\omega)$ and $x \in G$.

Let $S$ be a ternary semigroup. In [3] introduce ternary Banach algebra $l_1(S)$. Now, we make ternary Beurling algebra $l_1^r(S)$ and show that some elementary properties.

Definition 2.9. Let $S$ be a ternary semigroup, let $\omega$ be a ternary weight on $S$ and let $l_1^r(S)$ denote the set of mappings $f$ of $S$ into $\mathbb{C}$ such that $\sum_{s \in S} |f(s)\omega(s)| < \infty$, with pointwise addition and scalar multiplication, with ternary convolution

$$[f,g,h]_*(x) = \sum_{x=rst} f(r)g(s)h(t) \quad (x \in S, f, g, h \in l_1^r(S)),$$

and with the norm

$$\|f\|_1,\omega = \|f\omega\|_1 = \sum_{s \in S} |f(s)\omega(s)|.$$

The ternary algebra $l_1^r(S)$ is called the ternary Beurling algebra on $S$ associated with the ternary weight $\omega$.

Theorem 2.10. Let $S$ be a ternary semigroup and $\omega$ be a ternary weight function on $S$. Then $(l_1^r(S),[,]_*,\|\|_1,\omega)$ is a ternary Beurling algebra.

Proposition 2.11. Let $\omega$ and $\omega'$ be ternary weights on $S$ and $\phi : l_1^r(S) \rightarrow l_1^r(S)$ is a continuous nonzero ternary homomorphism. If $[l_1^r(S),f,g]_*$ is norm dense in $l_1^r(S)$, then the norm closure of $[l_1^r(S),\phi(f),\phi(g)]_* [\phi(h)]_*$ for all $h \in l_1^r(S)$.

Proposition 2.12. Let $\omega$ and $\omega'$ be ternary weights on $S$ and $\phi : l_1^r(S) \rightarrow l_1^r(S)$ is a continuous nonzero ternary homomorphism. If $[l_1^r(S),l_1^r(S),f]_*$ is norm dense in $l_1^r(S)$, then the norm closure of $[l_1^r(S),\phi(f)]_* [\phi(g)]_*$ for all $g \in l_1^r(S)$.

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On some Hadamard type inequalities for $(\alpha, m)$-convex functions

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Abstract

In this paper, we establish new inequalities for functions whose first derivative in absolute value aroused to the $q$th ($q \geq 1$) power are $(\alpha, m)$-convex. Some estimates to the right hand side of Hadamard type inequality for those functions are also given. Applications to some special means are considered.

Keywords: Hermite-Hadamard inequality, $(\alpha, m)$-convex function, convexity, Hölder inequality

Mathematics Subject Classification: 26D10, 39B62

1 Introduction

The following inequality is well known as the Hermite-Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

where $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$.

In [2], Mihesan introduced the class of $(\alpha, m)$-convex functions as the following:

**Definition 1.1.** The function $f : [0, b] \rightarrow \mathbb{R}$ is said to be $(\alpha, m)$-convex, where $(\alpha, m) \in [0, 1]^2$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y).$$

For recent results and generalizations concerning $(\alpha, m)$-convex functions, see [1, 4].

The aim of this paper is to establish some inequalities like those given in [3], but now for functions whose first derivative in absolute value is $(\alpha, m)$-convex.

2 Main Result

In order to prove main theorems, we need the following lemma.

**Lemma 2.1.** Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $I^o$ and assume that $a, b \in I$ with $a < b$ and $m \in (0, 1]$. If $f' \in L^1[a, b]$, then the following equality holds:

$$\frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x)dx = \frac{(mb - a)}{2} \int_0^1 (1 - 2t)f'(ta + m(1-t)b)dt.$$
In the following theorems, we propose new upper bound for the right hand side of Hadamard type inequality for \((\alpha,m)\)-convex functions.

**Theorem 2.2.** Let \(f : I \subseteq [0,b^*] \rightarrow \mathbb{R}\) be a differentiable mapping on \(f^0\) such that \(f' \in L^1[a,b]\), where \(a,b \in I\) with \(a < b\) and \(b^* > 0\). If \(|f'|^q\) is \((\alpha,m)\)-convex on \([a,b]\) for \((\alpha,m) \in [0,1]^2\), \(q \geq 1\), then the following inequality holds:

\[
\frac{|f(a) + f(mb)|}{2} - \frac{1}{mb - a} \int_a^{mb} f(x)dx \leq \frac{|mb - a|}{2} \left[ \left( \alpha + \frac{1}{2} \right) \frac{|f'(a)|^q + m|f'(b)|^q}{(\alpha + 1)(\alpha + 2)} + \left( \frac{1}{2} - \frac{\alpha + \frac{1}{2}}{\alpha + 1} \right) m|f'(b)|^q \right]^{\frac{1}{2}}.
\]

**Remark 2.3.** If in Theorem 2.2, we choose \(m = \alpha = q = 1\), we obtain

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{8} \left[ |f'(a)| + |f'(b)| \right],
\]

which is as in the previous works.

**Theorem 2.4.** Let \(f : I \subseteq [0,b^*] \rightarrow \mathbb{R}\) be a differentiable mapping on \(f^0\) such that \(f' \in L^1[a,b]\), where \(a,b \in I\) with \(a < b\) and \(b^* > 0\). If \(|f'|^q\) is \((\alpha,m)\)-convex on \([a,b]\) for \((\alpha,m) \in [0,1]^2\), \(q \geq 1\), then the following inequalities hold:

\[
\frac{|f(a) + f(mb)|}{2} - \frac{1}{mb - a} \int_a^{mb} f(x)dx \leq \frac{|mb - a|}{2} \left( \frac{q - 1}{2q - 1} \right)^{\frac{q-1}{q}} \left( \frac{1}{\alpha + 1} \right)^{\frac{q-1}{q}} \left( |f'(a)|^q + m|f'(b)|^q \right)^{\frac{1}{q}}.
\]

**Corollary 2.5.** In Theorem 2.4, if we choose \(m = \alpha = 1\), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \left( \frac{b - a}{2} \right) \left( \frac{q - 1}{2q - 1} \right)^{\frac{q-1}{q}} \left( \frac{1}{2} \right)^{\frac{q-1}{q}} \left( |f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}}.
\]

In order to prove next theorems we need the following lemma.

**Lemma 2.6.** Let \(f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}\) be a differentiable mapping on \(f^0\) and assume that \(a,b \in I\) with \(a < b\). If \(f' \in L^1[a,b]\), then the following equality holds:

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx = \left( \frac{b - a}{2} \right) \left[ \int_0^1 (-t)f'(ta + (1-t)b)dt + \int_0^1 t f'(tb + (1-t)a)dt \right].
\]
In the following theorems, we give new upper bound for the right hand side of Hadamard type inequality for \((\alpha, m)\)-convex functions.

**Theorem 2.7.** With the assumptions of Theorem 2.2, we have the following inequalities:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\
\leq \left( \frac{b-a}{4} \right) \left( \frac{2}{\alpha + 2} \right) \left[ \left( |f'(a)|^q + \frac{ma}{2} \right) \left( \frac{b}{m} \right)^q \right]^{\frac{1}{q}} + \left( |f'(b)|^q + \frac{ma}{2} \right) \left( \frac{a}{m} \right)^q \left[ |f'(a)| + |f'(b)| \right].
\]

**Theorem 2.8.** With the assumptions of Theorem 2.4, we have the following inequalities:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \left( \frac{b-a}{2} \right) \left( \frac{q-1}{2q-1} \right) \left( \frac{1}{\alpha + 1} \right) \left[ |f'(a)|^q + |f'(b)|^q + \frac{ma}{2} \right] \left( \frac{a}{m} \right)^q \left[ |f'(a)| + |f'(b)| \right].
\]

**Remark 2.9.** If in Theorem 2.7, we choose \(m = \alpha = 1\), then we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \left( \frac{b-a}{4} \right) \left( \frac{2}{3} \right) \left( \frac{1}{2} \right) \left( |f'(a)| + |f'(b)| \right).
\]

and if in Theorem 2.8, we choose \(m = \alpha = 1\), then we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \left( \frac{b-a}{2} \right) \left( \frac{2q-2}{2q-1} \right) \left( \frac{2q-2}{2q-1} \right) \left( |f'(a)| + |f'(b)| \right).
\]

**Remark 2.10.** From Theorems 2.2, 2.4, we have

\[
\left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x)dx \right| \leq \min\{E_1, E_2\}
\]

where

\[
E_1 = \left[ \frac{mb-a}{2} \right] \left( \alpha + \left( \frac{1}{2} \right)^\alpha \right) \left( |f'(a)|^q + \left( \frac{1}{\alpha + 1} \right) \left( 1 \right) \right) \left( \frac{a}{m} \right)^q \left( \frac{b}{m} \right)^q \left( |f'(a)| + |f'(b)| \right).
\]

\[
E_2 = \left[ \frac{mb-a}{2} \right] \left( \frac{q-1}{2q-1} \right) \left( \frac{1}{\alpha + 1} \right) \left( |f'(a)| + |f'(b)| \right).
\]

We consider the special means for arbitrary real numbers \(\alpha, \beta \ (\alpha \neq \beta)\). We take
(1) Arithmetic mean: \( A(\alpha, \beta) = \frac{\alpha + \beta}{2} \), \( \alpha, \beta \in \mathbb{R} \).

(2) Logarithmic mean: \( L(\alpha, \beta) = \frac{\alpha - \beta}{\ln |\alpha| - \ln |\beta|} \), \( |\alpha| \neq |\beta| \), \( \alpha, \beta \neq 0 \), \( \alpha, \beta \in \mathbb{R} \).

(3) Generalized log-mean: \( L_n(\alpha, \beta) = \left[ \frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)} \right]^\frac{1}{n} \), \( n \in \mathbb{N} \), \( \alpha, \beta \in \mathbb{R} \), \( \alpha \neq \beta \).

Now, using the results, we give some applications to special means of real numbers.

**Proposition 2.11.** Let \( n \in (-\infty, 0) \cup [1, \infty) \setminus \{ -1 \} \) and \( [a, b] \subset [0, \infty) \) and \( q > 1 \). Then we have the following inequalities:

\[
|A(a^n, b^n) - L_n^n(a, b)| \leq n \left( \frac{b - a}{2} \right) \left( \frac{2q - 2}{2q - 1} \right) A^{n-1}(a^{n-1}, b^{n-1})
\]

and

\[
|A(a^n, b^n) - L_n^n(a, b)| \leq n \left( \frac{b - a}{2} \right) \left( \frac{2}{3} \right)^\frac{1}{q} \left( 1 + \left( \frac{1}{2} \right)^\frac{1}{q} \right) A^{n-1}(a^{n-1}, b^{n-1}).
\]

**Proposition 2.12.** Let \( q > 1 \) and \( [a, b] \subset (0, \infty) \). Then we have the following inequality:

\[
|A(a^{-1}, b^{-1}) - L^{-1}(a, b)| \leq \left( \frac{b - a}{2} \right) \left( \frac{2}{3} \right)^\frac{1}{q} \left( 1 + \left( \frac{1}{2} \right)^\frac{1}{q} \right) A(a^{-2}, b^{-2}).
\]

**References**


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Essential norm of finite sum of weighted composition operators on $L^p(\Sigma)$

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Abstract

In this paper, we will give some upper and lower bounds for essential norm of finite sum of weighted composition operators on $L^p(\Sigma)$.

Keywords: Weighted composition operators, bounded, compact, essential norm.

Mathematics Subject Classification: 47B20, 46B38

1 Introduction

Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space. The support of a measurable function $f$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set. We denote the linear space of all complex-valued $\Sigma$-measurable functions on $X$ by $L^0(\Sigma)$. For a sub-$\sigma$-finite algebra $A \subseteq \Sigma$, the conditional expectation operator associated with $A$ is the mapping $f \mapsto E_A f$, defined for all non-negative $f$ as well as for all $f \in L^p(\Sigma)$, $1 \leq p \leq \infty$, where $E_A f$ is the unique $A$-measurable function satisfying

$$\int_A f d\mu = \int_A E_A f d\mu, \quad A \in A.$$ 

As an operator on $L^p(\Sigma)$, $E_A$ is an idempotent and $E_A (L^p(\Sigma)) = L^p(A)$. For more details on the properties of $E_A$ see [6] and [8].

In this paper we consider finite sum of weighted composition operators defined on $L^p(\Sigma)$ of the form

$$W = \sum_{i=1}^n u_i C_{\varphi_i},$$

where for each $1 \leq i \leq n$, $u_i : X \to \mathbb{C}$ is measurable function and $\varphi_i : X \to X$ is nonsingular transformation.

The basic properties of weighted composition operators on measurable function spaces are studied by Lambert [6, 7], Singh and Manhas [10], Takagi [11], Hudzik and Krbec [3], Cui, Hudzik, Kumar and Maligranda [4], Arora [1] and some other mathematicians.

In this paper, first we give some sufficient and necessary conditions for boundedness and compactness of finite sum of weighted composition operator $W$ on $L^p(\Sigma)$. Then, by making use of these conditions we determine the lower and upper estimates for the essential norm of these type operators.
2 Main Result

First we give some necessary and sufficient conditions for boundedness of finite sum of weighted composition operators on $L^p(\Sigma)$.

**Proposition 2.1.** Let $1 \leq p < \infty$. Then the following assertions hold.

(a) If $J_i \in L^\infty(\Sigma)$, then $\|W\| \leq n^{\frac{1}{p}} \|\sum_{i=1}^n J_i\|_{\infty}$.

(b) If $u_i$’s are nonnegative, then $W$ is bounded if and only if $J_i \in L^\infty(\Sigma)$ and

\[
\|\sum_{i=1}^n J_i\|_{\infty} \leq \|W\| \leq n^{\frac{1}{p}} \|\sum_{i=1}^n J_i\|_{\infty}.
\]

Let $T$ be a linear operator on a Banach space $\mathfrak{B}$. Then $T$ is said to be compact if, for every bounded sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathfrak{B}$, the sequence $\{Tf_n\}_{n \in \mathbb{N}}$ has a convergent subsequence in $\mathfrak{B}$. When $1 \leq p < \infty$, a characterization for compact weighted composition operator $L^p(\Sigma)$ spaces was obtained by Takagi [11] and independently around the same time by Chan [2]. Chan has showed that $uC_{\varphi}$ is compact on $L^p(\Sigma)$ if and only if

\[
\text{for each } \varepsilon > 0, \{x \in X : J_{\varphi}(x) \geq \varepsilon\} \text{ consists of finitely many atoms.}
\]

Recall that an atom of the measure $\mu$ is an element $A \in \Sigma$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space $(X, \Sigma, \mu)$ with no atoms is called non-atomic measure space. It is well-known fact that every $\sigma$-finite measure space $(X, \Sigma, \mu)$ can be partitioned uniquely as $X = B \cup \{A_j : j \in \mathbb{N}\}$, where $\{A_j\}_{j \in \mathbb{N}}$ is a countable collection of pairwise disjoint atoms and $B \in \Sigma$, being disjoint from each $A_j$, is non-atomic (see [12]). Since $\Sigma$ is $\sigma$-finite, so $a_j := \mu(A_j) < \infty$, for all $j \in \mathbb{N}$.

**Theorem 2.2.** Let $1 < p < \infty$, $u_i$’s are nonnegative and the sequence $\{a_j\}_{j \in \mathbb{N}}$ either has no subsequence that converges to zero or converges to zero. If $W$ is a compact operator on $L^p(\Sigma)$, then for each $\varepsilon > 0$, $N_{e}(\sqrt[p]{J}) = \{x : \sqrt[p]{J}(x) \geq \varepsilon\}$ consists of finitely many atoms, where $J = \sum_{i=1}^n J_i$.

Let $\mathfrak{B}$ be a Banach space and $\mathcal{K}$ be the set of all compact operators on $\mathfrak{B}$. For $T \in L(\mathfrak{B})$, the Banach algebra of all bounded linear operators on $\mathfrak{B}$ into itself, the essential norm of $T$ means the distance from $T$ to $\mathcal{K}$ in the operator norm, namely $\|T\|_e = \inf\{\|T - S\| : S \in \mathcal{K}\}$. Clearly, $T$ is compact if and only if $\|T\|_e = 0$. As is seen in [9], the essential norm plays an interesting role in the compact problem of concrete operators. Many people have computed the essential norm of (weighted) composition operators on various function spaces. In [5], the essential norm of $uC_{\varphi}$ on $L^p(\Sigma)$ with $1 < p < \infty$ have been computed by one of the authors as follows:

\[
\|uC_{\varphi}\|_e = \inf\{r > 0 : G_r \text{ consists of finitely many atoms}\},
\]

where $G_r = \{x \in X : \sqrt[p]{J}(x) \geq r\}$.

**Theorem 2.3.** Let $1 < p < \infty$ and let $W = \sum_{i=1}^n u_iC_{\varphi_i}$ be a bounded operator on $L^p(\Sigma)$. Put $\alpha = \inf\{r > 0 : N_r(\sqrt[p]{J}) \text{ consists of finitely many atoms}\}$. Then the followings hold.

(a) $\|W\|_e \leq n^{\frac{1}{p}}\alpha$, where $q$ is conjugate component of $p$.

(b) If the sequence $\{a_j\}_{j \in \mathbb{N}}$ either has no subsequence that converges to zero or converges to zero and $u_i$’s are nonnegative, then $\|W\|_e \geq \alpha$.

**2.4.** If the sequence $\{a_j\}_{j \in \mathbb{N}}$ either has no subsequence that converges to zero or converges to zero, and $u_i$’s are nonnegative then

\[
\alpha \leq \|W\|_e \leq n^{\frac{1}{p}}\alpha.
\]
References


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Existence of idempotents in minimal left ideals

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Abstract
If $S$ is a semigroup with partially ordered relation with semilattice property, then every minimal element in any minimal left ideal is idempotent. Also we show that this idempotent is unique.

1 Introduction
We denote the set of idempotents of a subset $X$ of a semigroup $S$ by $E(X)$. An idempotent $e$ of a semigroup $S$ is said to be a minimal idempotent if $Se$ is a minimal left ideal. Note that for a po-semigroup $S$, a minimal idempotent need not be minimal with respect to the partial order on $S$. (Here, minimality is a property of the underlying semigroup, and refers to the fact that such idempotents are minimal with respect to the partial order $\leq$ on $E(S)$ defined by $e \leq f \iff ef = fe = e$.) We show that if $S$ is a po-semigroup with semilattice property. If $L$ is a minimal left ideal in $S$ then $L$ has minimal idempotent.

Keywords: Po-semigroup, Semilattice, Upper-complete.

Mathematics Subject Classification: 00A73

Definition 1.1. A po-semigroup $S$ is a partially ordered set which is a semigroup with increasing multiplication, (i.e. if $a, b, c, d \in S$ with $a \leq b$ and $c \leq d$ then $ac \leq bd$).

Let $A$ be a subset of a partially ordered set $S$. We define $\downarrow A$ by

$$\downarrow A = \{ s \in S : s \leq a \text{ for some } a \in A \}.$$  

For $s \in S$ we define $\downarrow s = \downarrow \{ s \}$. A is said to be decreasing if $A = \downarrow A$. Increasing and $\uparrow A$ are defined similarly.

An element $x \in X \subseteq S$ is said to be maximal with respect to $X$ if for all $y \in X$ with $x \leq y$, we have $x = y$. We denote the set of maximal elements of $X$ by $M(X)$.

Lemma 1.2. Let $S$ be a po-semigroup. If $A \subseteq S$ is a subsemigroup (respectively left ideal, right ideal, ideal) of $S$, then so is $\downarrow A$.

Proof. We prove this for $A$ a left ideal, the other proofs begin similar. Suppose $\alpha, \beta \in \downarrow A$. Then $\alpha \leq a, \beta \leq b$ for some $a, b \in A$. As multiplication is increasing, $\alpha \beta \leq ab \in A$, so $\downarrow A$ is a semigroup. Also $sa \leq sa \in A$ for all $s \in S$, and so $\downarrow A$ is a left ideal. \hfill $\Box$

Lemma 1.3. Suppose $A$ and $B$ are subsets of a partially ordered set $S$ with $\downarrow A = \downarrow B$. Then

$$M(A) = M(\downarrow A) = M(\downarrow B) = M(B).$$

In particular, $A$ and $B$ share maximal elements.
Proof. We deal only with the case $M(A) \neq \emptyset$. Suppose $\mu \in M(\downarrow A)$. As $\mu \in \downarrow A$, $\mu \leq a$, for some $a \in A$. However, since both $\mu$ and $a$ are in $\downarrow A$, the maximality of $\mu$ gives us $\mu = a$, and so $\mu \in M(A)$. Thus $M(\downarrow A) \subseteq M(A)$. The reverse inclusion is obvious.

Proof. This follows from [1] as the multiplication in po-semigroups is increasing.

Theorem 1.4. Let $S$ be a semigroup with a minimal idempotent $e$. Then

(i) the minimal ideal $K$ of $S$ exists and has paragroup structure, i.e. it is isomorphic to $T_e := E(eS) \times eSe \times E(eS)$, where $E(eS)$ and $E(eS)$ are left zero and right zero semigroups respectively, and multiplication in $T_e$ is defined by

$$(x, y, z)(\xi, \eta, \zeta) = (x, y \xi \eta, \zeta).$$

The isomorphism is given by

$$\theta_e(s) = (s(ese)^{-1}, ese, (ese)^{-1}s) \quad (s \in K)$$

with inverse

$$\theta_e^{-1}(x, y, z) = xyz \quad ((x, y, z) \in T_e).$$

(ii) the minimal left ideals of $S$ are pairwise isomorphic. If $Sd$ is a minimal left ideal (d chosen to be in $E(eS)$), then the isomorphism is given by

$$\psi_{d,e} : Sd \to Se$$

$$s \mapsto s(ese)^{-1}s.$$

(iii) the maximal subgroups of $K$ are pairwise isomorphic. If $f \in E(K)$ and if $d$ is the identity of the group $eS \cap Sf$, then the isomorphism is given by

$$\phi_{f,e} : fSf \to eSe$$

$$s \mapsto dse.$$

If $e$ and $f$ are in the same minimal left ideal of $S$, then this becomes

$$\phi_{f,e} : fSf \to eSe$$

$$s \mapsto es.$$

Proposition 1.5. Let $S$ be a po-semigroup with a minimal idempotent $e$. Then the maximal subgroups of the minimal ideal are order and semigroup isomorphic.

Proof. This follows from Theorem 1.5(iii) as the multiplication in po-semigroup is increasing.

Definition 1.6. Let $S \neq \emptyset$ be a subset of a partially order set $T$. $S$ is upper semi-bounded if, for each $t \in S$, there is $x \in M(S)$ such that $t \leq x$. Also $S$ is said to be an upper-complete if, for each chain $E$ in $S$, Sups $E$ exists.

Example 1.7. Let $S = \{x, e_i, h_i : i \in N\}$. Let $L = \{x, e_i : i \in N\} \subseteq S$. This is a po-semigroup which is semi-bounded above, has a minimal idempotent which is maximal with respect to its minimal (left) ideal, and for which the minimal (left) ideal is not semi-bounded above.

It is clear that $S$ is semi-bounded above and $L$ is the minimal (left) ideal of $S$. Also $L$ is not semi-bounded above. Moreover, $x$ is a minimal idempotent which is maximal with respect to its minimal left ideal.
Corollary 1.8. Let $S$ be a po-semigroup with semi-lattice property. If $L$ is a minimal left ideal, then $L$ has at most one minimal element.

Theorem 1.9. Let $S$ be a po-semigroup with semilattice property. Let $L$ be a minimal left ideal in $S$ which contains minimal element, say $x$. Then $x$ is idempotent.

Proposition 1.10. Let $S$ be an upper-complete po-semigroup with a minimal ideal idempotent. Then the minimal left ideals of $S$ are upper-complete.

2 Main Result

Theorem 2.1. Let $S$ be a po-semigroup with semilattice property. Let $L$ be a minimal left ideal in $S$ which contains minimal element, say $x$. Then $x$ is idempotent.

References


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Condensation rank of injective Banach spaces

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Abstract
The condensation rank associates any topological space with a unique ordinal number. In this paper we prove that the condensation rank of any infinite dimensional injective Banach space is equal to or greater than the first uncountable ordinal number. Techniques and the proofs are all elementary.

Keywords: Condensation rank; Injective Banach space; Borel derivative

Mathematics Subject Classification: 46B25; 03E10; 54A05; 28A05

1 Introduction
Topological derivative has been used as a tool for the understanding of topological spaces and classification of Banach spaces, see e.g., [1-6, 11, 12]. Topological derivative is a Borel derivative that it derives the limit points set of a set. The chain of iterative condensation derived sets of any given set would converge faster than that of the corresponding chain of iterative limit point derived sets. The limit point rank of the real line and also the infinite dimensional Banach space $l_{\infty}(\mathbb{N})$ is both the first uncountable ordinal number $\Omega$. However, the condensation rank of the real line and $l_{\infty}(\mathbb{N})$ are one and $\Omega$, respectively. This motivates the possibility of using condensation derivative, and its generalizations associated with cardinal numbers, as a tool for classification of classical Banach spaces.

A “sufficiently” large ordinal number of iterations of the condensation derivative composed on any set approaches a perfect set. The necessary number of iterations depends on the initial set and the topological property of the whole space. This leads to the notion of the condensation rank of topological spaces, see [8], and is similar to the classical rank due to Cantor-Bendixson. A second motivation for defining condensation derivative and its rank is for its application in obtaining a measure theoretical version of Aleksandrov’s Theorem, see [8, 9]. The condensation derivative is measure-preserving on closed subsets and its iteration up to the condensation rank of the space provides a sufficient tool to prove “a set with finite and positive measure in any regular non-atomic Borel measure space contains a perfect set whose measure is positive.” It is also shown that for any ordinal number, say $\alpha$, there exists an appropriate totally imperfect Hausdorff topological space whose condensation rank is $\alpha$. In the case of non-limit ordinal numbers, the space can be a totally imperfect “compact” space, see [8]. Some relevant discussions and applications in locally compact groups are also presented in [9].

2 The condensation rank of injective Banach spaces
A Borel map $D : 2^{X} \to 2^{X}$ is a Borel derivative when it is monotone on the closed subsets of the Hausdorff topological space $X$. The $\alpha$-th iterated derivative $D^\alpha : 2^{X} \to 2^{X}$ is defined inductively as follows: $D^0(K) = K$, $D^{\alpha+1}(K) = D(D^\alpha(K))$ and $D^{\alpha}(K) = \bigcap_{\beta < \alpha} D^\beta(K)$, for any limit ordinal
number \( \alpha \). A point \( p \in A \subset X \) is called a condensation point of \( A \) when all of its neighborhoods contain an uncountable number of points from \( A \). The condensation derived set (CDS) of \( A \) refers to the set of all its condensation points. Denote CD for the condensation derivative, that is a set valued function mapping any set to its CDS. The maximal perfect subset of the closure of a set is called its perfect kernel.

**Lemma 2.1.** CD is a Borel derivative and for any set \( A \), there exists an ordinal number \( \alpha_0 \) which, \( \text{CD}^\alpha(A) = \text{CD}^\alpha_0(A) \), for any \( \alpha \geq \alpha_0 \). In addition, \( \text{CD}^\alpha_0(A) \) is a perfect subset of the perfect kernel of \( A \).

Denote \( \alpha_A \) for the least ordinal number satisfying Lemma 2.1. Then, \( \{\text{CD}^\alpha(A)\}_{\alpha=1}^{\alpha_A} \) is a strictly descending chain. The condensation rank \( \text{CR}(X) \) of the topological space \( X \) is defined by \( \text{CR}(X) = \sup\{\alpha_A | A \subset X\} \). Thus, we have \( \text{CD}^\beta = \text{CD}^\beta + 1 \), where \( \beta \geq \text{CR}(X) \). Now we are ready to present our main nontrivial result.

**Lemma 2.2.** The condensation rank of \( l_\infty \) is the first uncountable ordinal number \( \Omega_1 \), i.e. \( \text{CR}(l_\infty) = \Omega_1 \). In addition,

\[
\{\text{CR}(A) | A \subseteq l_\infty, A \text{ is a totally imperfect closed set}\} = [1, \Omega_1].
\]

**Proof.** Let \( \aleph_1 \) denote the first uncountable cardinal number. It is not too difficult to see that \( \text{CR}(l_\infty) \leq \Omega_1 \), since otherwise with Cantor’s Theorem we have

\[
\text{Card}(l_\infty) \geq \aleph_1^{\aleph_1},
\]

which is a contradiction. We first apply transfinite induction on ordinal number \( \alpha \) in which \( \alpha < \Omega \). Then, we separately deal with the case \( \alpha = \Omega \).

Although for \( \alpha = 1 \) the proof is trivial, we still need to consider the case \( \alpha = 2 \). We consider a real number, \( a \in \mathbb{R} \), and the sequence of natural numbers \( \{n\}_{n=1}^{\infty} = \{1, 2, 3, \ldots\} \). Denote the set of all subsequences of \( \{n\}_{n=1}^{\infty} \) with \( \Sigma_2 = \Sigma(n)_{n=1}^{\infty} \). For a natural number \( m \) and a sequence \( \{n_k\}_{k=1}^{\infty} \in \Sigma^2 \), define

\[
a_n^2(a, m, \{n_k\}_{k=1}^{\infty}) = \begin{cases} 
a + 1/m & \text{if } n = n_r \\
a & \text{otherwise.} \end{cases}
\]

Thereby,

\[
A_2 = A_2(a) = \left\{ \{a_n^2(a, m, \{n_k\}_{k=1}^{\infty})\}_{n=1}^{\infty} | m \in N, \{n_k\}_{k=1}^{\infty} \in \Sigma^2 \right\} \bigcup \{(a, a, a, \ldots)\}
\]

is an uncountable closed subset, where \( P = \{(a, a, a, \ldots)\} = \text{CD}(A_2) \) and \( \text{CD}^2(A_2) = \text{CD}(P) = \emptyset \), i.e. \( \text{CR}(A_2) = 2 \). The set \( A_2 \) is a totally imperfect closed set.

Note that it is easy to modify \( A_2(a) \) to obtain the closed set

\[
A_2(\{a_n\}_{n=1}^{\infty}) \subset l_\infty
\]

such that

\[
\text{CD}(A_2(\{a_n\}_{n=1}^{\infty})) = \{\{a_n\}_{n=1}^{\infty}\}, \text{ where } \{a_n\}_{n=1}^{\infty} \in l_\infty.
\]

Now let \( \theta \) be a non-limit ordinal number \( \alpha \) where \( \theta + 2 = \alpha < \Omega \). For a sequence \( \{b_n\} \in A_{\theta + 1}(a) \), a natural number \( N \in \mathbb{N} \) and a sequence \( \{n_k\}_{k=1}^{\infty} \in \Sigma^2 \), we define

\[
a_n^\alpha = a_n^\alpha(\{b_n\}_{n=1}^{\infty}, N, \{n_k\}_{k=1}^{\infty}) = \begin{cases} 
b_n + 1/N & \text{if } n = n_k \\
b_n & \text{otherwise.} \end{cases}
\]
and 

\[ A_\alpha(a) = \left\{ \left\{ a_n^n(b_n), N, \{ n_k \} \right\} \mid \{ b_n \}_{n=1}^{\infty} \in A_{\theta+1}(a), N \in \mathbb{N}, \{ n_k \} \in \Sigma^2 \right\}. \]

Then, \( CD(A_\alpha(a)) = A_{\theta+1} \) and \( CD^{\theta+1}(A_\alpha(a)) = \{ \{ a, a, \cdots \} \} \), i.e. \( CD(A_\alpha(a)) = \theta + 2 \).

Now consider a non-limit ordinal number \( \alpha \) such that \( \alpha = \theta + 1 < \Omega \), where \( \theta \) is a limit ordinal number. Let the set \( \{ \theta_n \mid n \in \mathbb{N} \} \) be the set of all (possibly rearranged) non-limit predecessor ordinal numbers of \( \theta \). For

\[ A_\alpha(a) = \bigcup_{n=1}^{\infty} A_{\theta_n}(a + 1/n), \]

we have \( CD^\theta(A_\alpha(a)) = \{ \{ a, a, \cdots \} \} \) and \( CR(A_\alpha(a)) = \theta + 1 \). For a limit ordinal number \( \alpha \), we define

\[ A_\alpha(a) = \bigcup_{n=1}^{\infty} A_{\alpha_n}(a + n) \cup \{ \{ a, a, \cdots \} \}, \]

in which \( \{ \alpha_n \mid n \in \mathbb{N} \} \) is the set of (possibly rearranged) PONs of \( \alpha \). Hence,

\[ CD^{\alpha_n}(A_\alpha(a)) \neq \emptyset \text{ and } CD^\alpha(A_\alpha(a)) = \emptyset. \]

Therefore, transfinite induction is complete. Modify the above to obtain \( A_\alpha(\{ a_n \}_{n=1}^{\infty}) \) for any ordinal number \( \alpha < \Omega_1 \) and \( \{ a_n \}_{n=1}^{\infty} \in l_\infty \) such that \( CD^{\alpha_n}(A_{\alpha+1}(\{ a_n \}_{n=1}^{\infty})) = \{ a_n \}_{n=1}^{\infty} \).

Finally, for constructing \( A_{\Omega_1} \), let \( A_1 = \{ (a_n)_{n=1}^{\infty} \mid a_n \in \{ 0, 1 \} \} \) and \( \Psi : [1, \Omega_1) \rightarrow A_1 \) be a bijective function. Then, define

\[ A_{\Omega_1} = \bigcup_{\alpha < \Omega_1} \left\{ A_\alpha(\Psi(\alpha)) \right\}, \]

and thereby we have

\[ CD^\alpha(A_{\Omega_1}) \neq \emptyset, \text{ where } \alpha < \Omega_1, \text{ and } CD^{\Omega_1}(A_{\Omega_1}) = \emptyset. \]

It can be seen that \( A_{\Omega_1} \) is a totally imperfect closed set and \( CR(A_{\Omega_1}) = \Omega_1 \).

**Theorem 2.3.** [10, Theorem 2.3] Majid Gazor / Every infinite-dimensional injective Banach space has a subspace isomorphic to \( l_\infty \).

Note that Theorem 2.3 implies that such a subspace is a completed “closed” subspace isomorphic to \( l_\infty \). The following is our main result in this paper.

**Theorem 2.4.** Let \( X \) be an infinite dimensional injective Banach space. Then, \( CR(X) \geq \Omega_1 \), where \( \Omega_1 \) is the first uncountable ordinal number. In addition,

\[ \{ CR(A) \mid A \subseteq X, A \text{ is a totally imperfect closed set} \} \supseteq [1, \Omega_1]. \]

**Proof.** This is straightforward based on Theorem 2.3 and Lemma 2.2.  \( \square \)
References


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On near set-valued derivation-like equation

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Abstract
In this paper for a set-valued mapping $F_0 : [0, \infty) \to CC(Y)$ we consider the following equation

$$F_0(x + y + zw) \subseteq F_0(x) + F_0(y) + zF_0(w) + wF_0(z).$$

It is proved that the equation admits a subadditive and $Q_+-$ homogeneous set-valued function $F : [0, \infty) \to CC(Y)$ which satisfies this equation and $F \subset F_0$.

Keywords: Set-valued maps; Subadditive equation

Mathematics Subject Classification: 39B72, 54C60

1 Introduction

Set valued functions in Banach spaces have received a lot of attention in the literature (see[2], [5], for example). The pioneering papers by Aumann, [2], and Debreu, [5], were inspired by problems in control theory and Mathematical Economics. We can refer to the papers by [3], [4], [10] and the survey [8]. The stability problem of functional equations originated from a question of Ulam , [13], concerning the stability of group homomorphisms.

Smajdor, [12], and R. Ger, Z. Gajda, [6], observed that if $B(0, \epsilon)$ is the closed ball of center 0 and radius $\epsilon$ in $Y$ and $CCZ(Y)$ is the family of all convex and compact member of $2^Y$ and $f : X \to Y$ satisfied $\|f(x + y) - f(x) - f(y)\| < \epsilon$ then the set-valued map $F : X \to CC(Y)$ defined by the relation

$$F(x) = f(x) + B(0, \epsilon), \quad x \in X$$

is sub-additive, i.e.

$$F(x + y) \subseteq F(x) + F(y), x, y \in X.$$  (2)

By [3] we know that there exists an additive function $g : X \to Y$ such that $\|f(x) - g(x)\| < \epsilon$, and it is proved that $f$ satisfies the relation $f(x) \in F(x)$, i.e. is a selection of $F$.

Such a mapping $F : X \to 2^Y$ is called sub-additive if it satisfies (2), for all $x, y \in X$. It is proved that (see ([7])) if $F : \mathbb{R} \to \rho_0(\mathbb{R}) := 2^\mathbb{R} \setminus \emptyset$ is additive and $F(0) = \{0\}$ , then $F$ is single-valued.

In this note we consider the set-valued equation

$$F_0(x + y + zw) \subseteq F_0(x) + F_0(y) + zF_0(w) + wF_0(z)$$

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If $f : X \to Y$ satisfied $\|f(x + y + w) - f(x) - f(y) - w f(z) - z f(w)\| < \epsilon$ for some $\epsilon > 0$ then it is easy to see that the set-valued map $F : X \to CCZ(Y)$ defined by the relation

$$F(x) = f(x) + B(0, \epsilon), \quad x \in X$$

is a solution of (1).

Let $Y$ be a real vector space. The family of all closed and convex subsets, containing $0$, of $Y$ will be denoted by $CCZ(Y)$. Let $A, B$ be nonempty subsets of a real vector space $X$ and $\lambda$ be a real number. We define

$$A + B = \{x \in X : x = a + b, a \in A, b \in B\},$$

$$A = \{x \in X : x = \lambda a, a \in A\}.$$ It is known (see [11]) that if $\lambda$ and $\mu$ are real numbers and $A$ and $B$ are nonempty subset of a real vector space $X$, then

$$\lambda(A + B) = \lambda A + \lambda B,$$

$$(\lambda + \mu)A \subseteq \lambda A + \mu B.$$ Moreover, if $A$ is a convex set and $\lambda \mu \geq 0$, then we have

$$(\lambda + \mu)A = \lambda A + \mu A.$$ A subset $A \subseteq X$ is said to be a cone if $A + A \subseteq A$ and $\lambda A \subseteq A$ for all $\lambda > 0$.

2 Main results

**Theorem 2.1.** Let $F_0 : [0, \infty) \to CCZ(Y)$ be a s.v.f satisfy

$$F_0(x + y + zw) \subseteq F_0(x) + F_0(y) + zF_0(w) + wF_0(z), \quad (5)$$

then there exists a subadditive and $Q_+$- homogeneous (i.e. $F(sx) = sF(x)$ for $s \in \mathbb{Q}, s > 0, x \in X$) s.v.f. $F : [0, \infty) \to CCZ(Y)$ such that $F \subseteq F_0$, $F(xy) \subseteq xF(y) + yF(x)$ and $F(x) = \bigcap_{n=0}^{\infty} F_n(x)$, for $x, y \in [0, \infty)$ where

$$F_n(x) = \bigcap_{s=0}^{\infty} (n+1)^{-s}F_{n-1}((n+1)^s x), \quad n = 1, 2, 3, ...$$

Our next result is a more general case of Theorem5.1.

**Theorem 2.2.** Let $A$ be a positive cone in real Banach algebra $B$, $(A \subseteq B)$. If $F_0 : A \to CCZ(B)$ is a set-valued functional satisfying

$$F_0(a + b + cd) \subseteq F_0(a) + F_0(b) + cF_0(d) + F_0(c)d, \quad \forall a, b, c, d \in A, \quad (6)$$

then there exists a subadditive and $Q_+$-homogeneous s.v.f. $F : A \to CCZ(B)$ such that $F \subseteq F_0$, $F(ab) \subseteq F(a)b + aF(b)$ which is given by formulas

$$F(x) = F_n(x), \quad x \in [0, \infty)$$

where

$$F_n(x) = \bigcap_{s=0}^{\infty} (n+1)^{-s}F_{n-1}([n+1]^s x), n = 1, 2, 3, ...$$
Now letting \( z = x, w = y \) in (1) we get the following equation

\[
F(x + y + xy) \subseteq F(x) + F(y) + xF(y) + yF(x), \quad x, y \in [0, \infty).
\]  

(7)

On can easily see that the mapping \( F : [0, \infty) \rightarrow \rho_0(\mathbb{R}) \) by \( F(x) = [0, x], \quad x \in [0, \infty) \) satisfies

\[
F(xy) \subseteq xF(y) + yF(x),
\]

\( F \) is additive and \( F(0) = \{0\} \), but it is not single-valued. This shows that (7) has a non-single point solution.

Also if \( f : [0, \infty) \rightarrow [0, \infty) \) satisfies

\[
f(x + y + xy) = f(x) + f(y) + xf(y) + yf(x)
\]

then the set-valued mapping \( F : [0, \infty) \rightarrow \rho_0(\mathbb{R}) \), defined by \( F(x) = [0, f(x)] \) satisfies

\[
F(x + y + xy) = F(x) + F(y) + xF(y) + yF(x), \quad x, y \in [0, \infty)
\]

(9)

Theorem 2.3. Suppose a set-valued function \( D : [0, \infty) \rightarrow C(\mathbb{R}) \) satisfies \( D(xy) = yD(x) + xD(y), \) for all \( x, y \geq 0 \) and let \( F(x) = D(x + 1) \) then \( F \) is a solution of (9). Also the convexity of \( D(x) \) for \( x \in [0, \infty) \) is essential.

We know that the set-valued mapping \( F : [0, \infty) \rightarrow I(\mathbb{R}) \) given by \( F(x) = [0, x] \) satisfies (7). Now define \( F : [1, \infty) \rightarrow I(\mathbb{R}) \) by \( D(x) := F(x - 1) \). Then the fact that \( xy > 1 \), for any \( x, y \in [1, \infty) \), implies that

\[
D(xy) = F(xy - 1) = [0, xy - 1] \subseteq [0, 2xy] = x[0, y] + y[0, x] = xD(x) + yD(x).
\]

(10)

References


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Spectrum preserving linear map on liminal $C^*$-algebras

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Abstract
Let $A$ and $B$ be unital semi-simple Banach algebras. If $B$ is liminal $C^*$-algebra, and $\phi$ is a surjective spectrum preserving linear mapping from $A$ to $B$, then $\phi$ is a Jordan homomorphism.

Keywords: Semi-simple Banach algebras, liminal $C^*$-algebra, spectrum preserving, Jordan homomorphism

Mathematics Subject Classification: 46H05, 47B48, 47C15

1 Introduction

In 1970[7] Kaplansky asked the following question:

Let $\phi: A \to B$ be a unital, invertibility preserving linear map between unital Banach algebras $A$ and $B$. Is $\phi$ a Jordan homomorphism? Let us quickly state that the above question of Kaplansky is too general and the answer to it is negative in this generality. Sourour[2] shows that if $\phi$ is not surjective, then it may not be a Jordan homomorphism. Aupetit [5] showed that Kaplansky’s question may not have a positive answer if the Banach algebras $A$ and $B$ are not semisimple. In view of the above discussions, it seems quite natural that researchers working on this question made the following conjecture:

Suppose $A$ and $B$ are unital semi-simple Banach algebras and $\phi: A \to B$ is a unital surjective linear map preserving invertibility. Is $\phi$ a Jordan homomorphism?

In this generality, the problem remains unsolved. A number of partial positive results have been found, especially in the case where the map is spectrum preserving. Jafrayan and Sourour proved in [1] that every surjective, spectrum preserving linear map from $B(X)$ onto $B(Y)$ is an isomorphism or an anti-isomorphism. Aupetit [3] proved that every surjective, spectrum preserving linear map from a von Neumann algebra onto another von Neumann algebra is a Jordan isomorphism. For more information one can see[4]. In this paper we will prove that every surjective spectrum preserving linear map between semi-simple unital Banach algebras $A$ and $B$ is a Jordan homomorphism provided that one of $A$ or $B$ is liminal.

2 Main Result

We recall that if $A$ and $B$ are Banach algebras a linear map $\phi: A \to B$ is said to be spectrum preserving if for all $x \in A, \sigma(\phi(x)) = \sigma(x)$, and a Jordan homomorphism if $\phi(x^2) = \phi(x)^2$ for every $x \in A$.

A $C^*$-algebra $A$ is said to be liminal if for every non-zero irreducible representation $(H, \phi)$ of $A$ we have $\phi(A) = K(H)$.

Lemma 2.1. [3] Let $A$ and $B$ be Banach algebras, with $B$ semisimple. If $\phi: A \to B$ is a linear surjective, spectrum preserving map, then $\phi(1) = 1$. 

Lemma 2.2. [3] Let A and B be two semi-simple Banach algebras. If $\varphi$ is a spectrum-preserving linear mapping from A into B. Then $\varphi$ is injective.

Lemma 2.3. Let A and B be Banach algebras, with B semi-simple. If $\varphi : A \rightarrow B$ is a linear surjective, spectrum preserving map, then $\varphi^{-1}$ is spectrum preserving.

Proof. since $\varphi$ is linear surjective, spectrum preserving, by Lemma 2.2 $\varphi$ is injective, and hence invertible. If $b \in B$, then there exists $a \in A$ such that $\varphi(a) = b$ so, $\sigma(\varphi^{-1}(b)) = \sigma(a) = \sigma(\varphi(a)) = \sigma(b)$.

Theorem 2.4. Every unital liminal $C^*$-algebra A has only finite-dimensional irreducible representations.

Proof. If $(H, \varphi)$ is a non-zero irreducible representation of A, then it is non-degenerate and therefore $\varphi(1) = id_H$. Hence $id_H$ is compact, and therefore $\text{dim}(H) < \infty$.

Corollary 2.5. Every unital liminal $C^*$-algebra A has a separating family of finite dimensional irreducible representations.

Proof. Since every unital liminal $C^*$-algebra is semi-simple, and the radical of A is the intersection of the kernels of all the irreducible representations of A, so for every $a \neq 0$ in A there exists an irreducible representations $\varphi$ of A such that $\varphi(a) \neq 0$. By Theorem 2.4 $\varphi$ has finite-dimensional.

Theorem 2.6. Let A be semi-simple Banach and B a liminal $C^*$-algebras both with identity. If $\varphi$ is a surjective spectrum preserving linear mapping from A to B , then $\varphi$ is a Jordan homomorphism.

Proof. Since B is unital liminal $C^*$-algebra, by Corollary 2.5 has a family of finite dimensional irreducible representations. Since $\varphi$ is linear surjective and invertibility preserving then there exists a Jordan homomorphism $S$ such that $\varphi(x) = \varphi(1)S(x)$ ( [6] theorem 2 ). On the other hand Lemma 2.1 implies that $\varphi(1) = 1$. So we have, $\varphi(x) = S(x)$ for all $x \in A$ and $\varphi$ a Jordan homomorphism.

Corollary 2.7. Let A and B be unital semi-simple Banach algebras and A is liminal $C^*$-algebra. If $\varphi$ is a surjective spectrum preserving linear mapping from A to B, then $\varphi$ is a Jordan homomorphism.

Proof. Since $\varphi$ linear surjective, spectrum preserving, by Lemma 2.2, $\varphi$ is injective, so is invertible and by Lemma 2.3 $\varphi^{-1}$ preserves invertibility. Then, by Theorem 2.6, $\varphi^{-1}$ is a Jordan homomorphism and therefore, $\varphi$ is Jordan homomorphism.

References


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Be careful on partial metric fixed point results

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Abstract
In this paper, we show that there is a disorder in partial metric fixed point results. In fact, we show that under which condition most fixed point generalizations to partial metric spaces are not real generalizations.

Keywords: 0-complete, Fixed point, Partial metric space.

Mathematics Subject Classification: 47H09, 47H10.

1 Introduction
In 1992, Matthews introduced partial metric spaces ([16]). In a partial metric space, the distance of a point in the self may not be zero. Recently, many authors have been focused on obtaining new and more fixed point results in partial metric spaces. But, there is a big gap on partial metric fixed point results and researchers should be careful about the disorder. Based on [13], we are going to show that under which condition most fixed point generalizations to partial metric spaces are not real generalizations. A partial metric is a function $p : X \times X \to [0, \infty)$ satisfying the following conditions (a) $p(x, y) = p(y, x)$, (b) If $p(x, x) = p(x, y) = p(y, y)$, then $x = y$, (c) $p(x, x) \leq p(x, y)$ and (d) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$. Then $(X, p)$ is called a partial metric space. If $p$ is a partial metric $p$ on $X$, then $p^* : X \times X \to [0, \infty)$ defined by $p^*(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on $X$ ([16]). The notions of completeness and 0-completeness have been defined in partial metric spaces.

2 Main Result
Now, we are ready to state and prove our main results. First, we give the following key result.

Proposition 2.1. Let $(X, p)$ be a complete partial metric space. Define the mapping $d : X \times X \to [0, +\infty)$ by $d(x, y) = 0$ whenever $x = y$ and $d(x, y) = p(x, y)$ whenever $x \neq y$. Then $(X, d)$ is a complete metric space.

Proposition 2.2. A partial metric space $(X, p)$ is a 0-complete if and only if $(X, d)$ is a complete metric space, where $d$ is the metric in Proposition 2.1.

You know there are many fixed point generalizations which are not real generalizations (see [13]). The aim of this paper is to show that there are a lot of partial metric results which obtain from the correspondence results in metric spaces. We present only some of the results which have been published after 2010 in different journals. Therefore, fixed point researchers should be careful on obtaining partial metric fixed point results. Let $(X, p)$ a partial metric space, $T$ a selfmap on $X, d$ the obtained metric in Proposition 2.1 and $x, y \in X$. We define

$$M_d(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$$
and
\[ M_p(x,y) = \max\{p(x,y), p(x,Tx), p(y,Ty)\}, \quad \frac{1}{2}[p(x,Ty) + p(y,Tx)]. \]

**Lemma 2.3.** \( M_p(x,y) = M_p(x,y) \) for all \( x, y \in X \) with \( x \neq y \).

The following result shows that Theorem 1 in [4] is a consequence of Theorem 2 in [7].

**Theorem 2.4.** Let \( (X,p) \) be a complete partial metric space, \( \phi : [0, +\infty) \rightarrow [0, +\infty) \) a continuous and nondecreasing function such that \( \phi(t) < t \) for all \( t > 0 \) and \( T \) a selfmap on \( X \) satisfying \( p(Tx,Ty) \leq \phi(M_p(x,y)) \) for all \( x, y \in X \). Then \( T \) has a unique fixed point.

The following result shows that Theorem 6 in [2] is a consequence of the main result of [12].

**Theorem 2.5.** Let \( \psi, \varphi : [0, +\infty) \rightarrow [0, +\infty) \) continuous and nondecreasing functions such that \( \phi(t) = \varphi(t) = 0 \) if and only if \( t = 0 \). \( (X,p) \) be a complete partial metric space and \( T \) a selfmap on \( X \) satisfying \( \psi(p(Tx,Ty)) \leq \psi(p(x,y)) - \varphi(p(x,y)) \) for all \( x, y \in X \) with \( x \neq y \). Then \( T \) has a unique fixed point.

The following result shows that Theorem 5 in [3] is a consequence of Theorem 1.3 of [6].

**Theorem 2.6.** Let \( (X,p) \) be a complete partial metric space, \( \phi : [0, +\infty) \rightarrow [0, +\infty) \) a continuous and nondecreasing function such that \( \phi(t) > 0 \) for all \( t > 0 \) and \( \phi(0) = 0 \). Suppose that \( T \) and \( S \) are selfmaps on \( X \) such that
\[ p(Tx,Ty) \leq M_p(x,y,S,T) - \phi(M_p(x,y,S,T)) \]
for all \( x, y \in X \), where
\[ M_p(x,y,S,T) = \max\{p(x,y), p(Sx,x), p(y,Ty), \frac{1}{2}[p(Sx,y) + p(x,Ty)]\}. \]
Then \( T \) and \( S \) have a unique common fixed point.

The following result shows that Theorem 2 in [4] is a consequence of the main result of [10].

**Theorem 2.7.** Let \( (X,p) \) be a complete partial metric space, \( a, b, c, d, e \) nonnegative numbers such that \( a + b + c + d + e < 1 \) when \( d \geq e \) and \( a + b + c + d + 2e < 1 \) when \( d < e \) and \( T \) a selfmap on \( X \) satisfying
\[ p(Tx,Ty) \leq ap(x,y) + bp(x,Tx) + cp(y,Ty) + dp(x,Ty) + ep(y,Tx) \]
for all \( x, y \in X \). Then \( T \) has a unique fixed point.

Note that, cone version of the main result of [10] has been proved by the authors in [17]. The following result shows that Theorem 8 in [1] is a consequence of Theorem 3.1 in [9].

**Theorem 2.8.** Let \( \psi, \varphi : [0, +\infty) \rightarrow [0, +\infty) \) continuous and nondecreasing functions such that \( \phi(t) = \varphi(t) = 0 \) if and only if \( t = 0 \). \( (X,p) \) be a complete partial metric space and \( T \) a selfmap on \( X \) satisfying
\[ \psi(p(Tx,Ty)) \leq \psi(M_p(x,y)) - \phi(\max\{p(x,y), p(y,Ty)\}) \]
for all \( x, y \in X \) with \( x \neq y \). Then \( T \) has a unique fixed point.

The following result shows that Theorem 2.1 in [11] is a consequence of Theorem 2.4 of [6].

**Theorem 2.9.** Let \( (X,p) \) be a complete partial metric space, \( \phi : [0, +\infty) \rightarrow [0, +\infty) \) a continuous and nondecreasing function such that \( \phi(t) < t \) for all \( t > 0 \), \( \phi(t) = 0 \) if and only if \( t = 0 \) and \( A, B, S \) and \( T \) four selfmaps on \( X \) such that \( AX \subseteq TX, BX \subseteq SX \) and
\[ p(Ax,By) \leq \phi(\max\{p(Sx,Ty), p(Ax,Sx), p(By,Ty), \frac{1}{2}[p(Sx,By) + p(Ax,Ty)]\}) \]
for all \( x, y \in X \). If one of the ranges \( AX, BX, TX \) and \( SX \) is a closed subset of \( X \), then \( A \) and \( S \) have a coincidence point and \( A \) and \( S \) have a coincidence point. Moreover, if the pairs \( \{A,S\} \) and \( \{B,T\} \) are weakly compatible, then \( A, B, T \) and \( S \) have a unique common fixed point.
Let \((X, d)\) be a complete metric space, \(\phi : X \to [0, +\infty)\) a lower semi-continuous function and \(T\) a selfmap on \(X\) such that \(d(x, Tx) \leq \phi(x) - \phi(Tx)\) for all \(x \in X\). In 1976, Caristi proved that the selfmap \(T\) has a fixed point ([8]). Immediately, Kirk proved that a metric space \((X, d)\) is complete if and only if each Caristi selfmap on \(X\) has a fixed point ([15]). Recently, Romaguera generalized the result of Kirk to partial metric spaces. In fact, he proved that a partial metric space \((X, p)\) is 0-complete if and only if every \(p^s\)-Caristi selfmap \(T\) on \(X\) has a fixed point ([18]; Theorem 2.3).

**Theorem 2.10.** Theorem 2.3 in [18] is not a real generalization, that is, it could be obtained from the metric version result.

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Some new fixed point results of contractive multifunctions

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Abstract
Recently Samet, Vetro and Vetro introduced the notion of \( \alpha \)-\( \psi \)-contractive type mappings. They have been establish some fixed point theorems for the mappings in complete metric spaces. In this paper, we introduce the notion of \( \alpha \)-\( \psi \)-contractive multifunctions and give a fixed point result about fixed points of the multifunctions. Also, we give a result about fixed point of some selfmaps on complete metric satisfy a contractive condition.

Keywords: \( \alpha \)-\( \psi \)-contractive multifunction, Fixed point, Partial metric.

Mathematics Subject Classification: 47H04, 54H10

1 Introduction
You know, Fixed point theory has many applications and was extended by several authors from different views (see for example [1]-[23]). Recently Samet, Vetro and Vetro introduced the notion of \( \alpha \)-\( \psi \)-contractive type mappings ([20]). Denote with \( \Psi \) the family of nondecreasing functions \( \psi : [0, \infty) \to [0, \infty) \) such that \( \sum_{n=1}^{\infty} \psi^n(t) < \infty \) for all \( t > 0 \), where \( \psi^n \) is the \( n \)th iterate of \( \psi \). It is known that \( \psi(t) < t \) for all \( t > 0 \) and \( \psi \in \Psi ([20]) \). Let \( (X, d) \) be a metric space, \( T \) a selfmap on \( X \), \( \psi \in \Psi \) and \( \alpha : X \times X \to [0, \infty) \) a function. Then \( T \) is called a \( \alpha \)-\( \psi \)-contraction mapping whenever \( \alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \) for all \( x, y \in X \). Also, we say that \( T \) is \( \alpha \)-admissible whenever \( \alpha(x, y) \geq 1 \) implies \( \alpha(Tx, Ty) \geq 1 \) ([20]). Also, we say that \( T \) has the property (B) if \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \geq 1 \) and \( x_n \to x \), then \( \alpha(x_n, x) \geq 1 \) for all \( n \geq 1 \). Let \( (X, d) \) be a complete metric space and \( T \) a \( \alpha \)-admissible \( \alpha \)-\( \psi \)-contractive mapping on \( X \). Suppose that there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \). If \( T \) is continuous or \( T \) has the property (B), then \( T \) has a fixed point (see [20]; Theorems 2.1 and 2.2). Finally, we say that \( X \) has the property (H) whenever for each \( x, y \in X \) there exists \( z \in X \) such that \( \alpha(x, z) \geq 1 \) and \( \alpha(y, z) \geq 1 \). If \( X \) has the property (H) in the Theorems 2.1 and 2.2, then \( X \) has a unique fixed point ([20]; Theorem 2.3). It is considerable that the results of Samet et al. generalize similar ordered results in the literature (see the results of third section in [20]). The aim of this paper is to introduce the notion of \( \alpha \)-\( \psi \)-contractive multifunctions and give a fixed point result about the multifunctions. Let \( (X, d) \) be a metric space, \( T : X \to 2^X \) a multifunction, \( \psi \in \Psi \) and \( \alpha : X \times X \to [0, \infty) \) a function. In this case, we say that \( T \) is a \( \alpha \)-\( \psi \)-contractive multifunction whenever

\[
\alpha_{\psi}(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y))
\]

for \( x, y \in X \), where \( H \) is the Hausdorff metric and \( \alpha_{\psi}(A, B) = \inf \{\alpha(a, b) \colon a \in A, b \in B\} \). Also, we say that \( T \) is \( \alpha \)-admissible whenever \( \alpha(x, y) \geq 1 \) implies \( \alpha_{\psi}(Tx, Ty) \geq 1 \).

Example 1.1. Let \( X = [0, \infty) \) and \( d(x, y) = |x - y| \). Define \( T : X \to 2^X \) by \( Tx = [0, \sqrt{x}] \) for all \( x \in X \) and \( \alpha : X \times X \to [0, \infty) \) by \( \alpha(x, y) = 1 \) whenever \( x, y \in [0, 1] \) and \( \alpha(x, y) = 0 \) whenever \( x \notin [0, 1] \) or \( y \notin [0, 1] \). Then it is easy to check that \( T \) is a \( \alpha \)-admissible and \( \alpha \)-\( \psi \)-contractive multifunction, where \( \psi \) is the identity map.
Let \((X, \leq, d)\) be an ordered metric space and \(A, B \subseteq X\). We say that \(A \leq B\) whenever for each \(a \in A\) there exists \(b \in B\) such that \(a \leq b\). Also, we say that \(A \lesssim B\) whenever for each \(a \in A\) and \(b \in B\) we have \(a \lesssim b\). Finally, we should emphasize that throughout this paper we suppose that all multifunctions on a metric space \((X, d)\) have closed values.

## 2 Main Result

Now, we are ready to state and prove our main results.

**Theorem 2.1.** Let \((X, d)\) be a complete metric space and \(T\) a \(\alpha_\ast\)-admissible and \(\alpha_\ast\)-\(\psi\)-contractive multifunction on \(X\). Suppose that there exists \(x_0 \in X\) such that \(\alpha_\ast\{x_0\}, Tx_0 \geq 1\) and if \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n\) and \(x_n \to x\), then \(\alpha(x_n, x) \geq 1\) for all \(n\). Then \(T\) has a fixed point.

**Example 2.2.** Let \(X = \{0, \infty\}\) and \(d(x, y) = \|x - y\|\). Define \(T : X \to 2^X\) by \(Tx = \{2x - \frac{3}{2}, \infty\}\) for all \(x > 1\), \(Tx = [0, \frac{2}{3}]\) for all \(0 \leq x \leq 1\) and \(\alpha : X \times X \to [0, \infty]\) by \(\alpha(x, y) = 1\) whenever \(x, y \in [0, 1]\) and \(\alpha(x, y) = 0\) whenever \(x \notin [0, 1]\) or \(y \notin [0, 1]\). Then it is easy to check that \(T\) is a \(\alpha_\ast\)-admissible and \(\alpha_\ast\)-\(\psi\)-contractive multifunction, where \(\psi(t) = \frac{1}{t}\) for all \(t \geq 0\). Also, \(\alpha_\ast\{1\}, T1 = 1\) and if \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n\) and \(x_n \to x\), then \(\alpha(x_n, x) \geq 1\) for all \(n\). Note that, \(T\) has infinitely many fixed points.

**Corollary 2.3.** Let \((X, \leq, d)\) be a complete ordered metric space, \(\psi \in \Psi\) and \(T\) a multifunction on \(X\) such that \(H(Tx, Ty) \leq \psi(d(x, y))\) for all \(x, y \in X\) with \(x \leq y\). Suppose that there exists \(x_0 \in X\) such that \(\{x_0\} \subseteq Tx_0\) and if \(\{x_n\}\) is a sequence in \(X\) such that \(x_n \leq x_{n+1}\) for all \(n\) and \(x_n \to x\), then \(x_n \leq x\) for all \(n\). If \(x \leq y\) implies \(Tx \leq Ty\), then \(T\) has a fixed point.

Now, we prove the following result for selfmaps.

**Theorem 2.4.** Let \((X, d)\) be a complete metric space, \(\alpha : X \times X \to [0, \infty)\) a function, \(\psi \in \Psi\) and \(T\) a selfmap on \(X\) such that \(\alpha(x, y)d(Tx, Ty) \leq \psi(m(x, y))\) for all \(x, y \in X\), where \(m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{3}d(x, Ty) + d(y, Tx)\}\). Suppose that \(T\) is an \(\alpha\)-admissible and there exists \(x_0 \in X\) such that \(\alpha(x_0, Tx_0) \geq 1\). Assume that if \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n\) and \(x_n \to x\), then \(\alpha(x_n, x) \geq 1\) for all \(n\). Then \(T\) has a fixed point.

**Example 2.5.** Let \(X = \{0, \infty\}\) and \(d(x, y) = \|x - y\|\). Define the selfmap \(T\) on \(X\) by \(Tx = 2x - \frac{5}{4}\) for \(x > 1\), \(Tx = \frac{2}{3}\) for \(0 \leq x \leq 1\) and \(\alpha : X \times X \to [0, \infty)\) by \(\alpha(x, y) = 1\) whenever \(x, y \in [0, 1]\) and \(\alpha(x, y) = 0\) whenever \(x \notin [0, 1]\) or \(y \notin [0, 1]\). Then it is easy to check that \(T\) is an \(\alpha\)-admissible and \(\alpha(x, y)d(Tx, Ty) \leq \psi(m(x, y))\) for all \(x, y \in X\), where \(\psi(t) = \frac{1}{t}\) for all \(t \geq 0\). Also, \(\alpha\{1\}, T1 = 1\) and if \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n\) and \(x_n \to x\), then \(\alpha(x_n, x) \geq 1\) for all \(n\). Note that, \(T\) has two fixed points.

**Corollary 2.6.** Let \((X, \lesssim, d)\) be a complete ordered metric space, \(\psi \in \Psi\) and \(T\) a multifunction on \(X\) such that \(H(Tx, Ty) \leq \psi(m(x, y))\) for all \(x, y \in X\) with \(x \lesssim y\). Suppose that there exists \(x_0 \in X\) such that \(x_0 \lesssim Tx_0\) and if \(\{x_n\}\) is a sequence in \(X\) such that \(x_n \lesssim x_{n+1}\) for all \(n\) and \(x_n \to x\), then \(x_n \lesssim x\) for all \(n\). If \(x \lesssim y\) implies \(Tx \lesssim Ty\), then \(T\) has a fixed point.

If we substitute a partial metric \(\rho\) instead the metric \(d\) in Theorem 1.7, it is easy to check that a similar result holds for the partial metric case as following.

**Theorem 2.7.** Let \((X, \rho)\) be a complete partial metric space, \(\alpha : X \times X \to [0, \infty)\) a function, \(\psi \in \Psi\) and \(T\) a selfmap on \(X\) such that \(\alpha(x, y)\rho(Tx, Ty) \leq \psi(m(x, y))\) for all \(x, y \in X\), where \(m(x, y) = \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{3}(\rho(x, Ty) + \rho(y, Tx))\}\). Suppose that \(T\) is an \(\alpha\)-admissible and there exists \(x_0 \in X\) such that \(\alpha(x_0, Tx_0) \geq 1\). Assume that if \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n\) and \(x_n \to x\), then \(\alpha(x_n, x) \geq 1\) for all \(n\). Then \(T\) has a fixed point.
Some new fixed point results of contractive multifunctions

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Some results of weighted composition operators on weighted Bergman spaces and weighted Bloch spaces

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Abstract
In this paper we present some conditions for boundedness and compactness of the weighted composition operators \( \psi C_\varphi f = \psi(f \circ \varphi) \) on weighted Bergman spaces and weighted Bloch spaces.

Keywords: weighted composition operators, weighted Bergman spaces, weighted Bloch spaces.

Mathematics Subject Classification: 47B33, 47B38.

1 Introduction and Preliminaries
Let \( \mathbb{D} \) be the open unit ball in \( \mathbb{C} \) and \( H(\mathbb{D}) \) be the class of all analytic functions on \( \mathbb{D} \). For two analytic functions \( \psi \) and \( \varphi \) with \( \varphi(\mathbb{D}) \subset \mathbb{D} \), the well-known weighted composition operator \( \psi C_\varphi \) is defined by \( \psi C_\varphi(f) = \psi f \circ \varphi \), for any \( f \in H(\mathbb{D}) \). This operator is a generalization of composition operator and multiplication operator. For more information about composition operators we can refer to [1]. These operators are studied on various spaces of analytic functions. For instance, see [2] and [3], for weighted composition operators acting from logarithmic and standard weighted Bergman spaces into Bloch spaces.

Let \( \omega \) and \( v \) be weights (strictly positive bounded continuous functions on \( \mathbb{D} \)). For \( p > 0 \) and a weight \( w \) the weighted Bergman space \( A_{w,p} \) consists of all \( f \in H(\mathbb{D}) \) for which
\[
||f||_{w,p}^p = \int_{\mathbb{D}} w(z)|f(z)|^p \, dA(z) < \infty,
\]
where \( dA(z) \) is the normalized area measure on \( \mathbb{D} \). Also the weighted Bloch space is defined by
\[
B_v = \{ f \in H(\mathbb{D}) : ||f||_{B_v} = \sup_{z \in \mathbb{D}} v(z)|f'(z)| < \infty \}.
\]
By letting \( w_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \) with \( \alpha > -1 \) and \( v_\alpha(z) = (1 - |z|^2)^\alpha \) with \( \alpha > 0 \), we get to the standard weighted Bergman space \( A_{w_\alpha,p} = A_\alpha^p \) (if \( \alpha = 0 \), then \( A^p_\alpha = A^p \) and \( ||.|||_{w_\alpha,p} = ||.|||_p \)) and \( \alpha \)-Bloch space \( B^{\alpha} \). Moreover the little weighted Bloch space defined as follows
\[
B^{\alpha}_v = \{ f \in B_v : \lim_{|z| \to 1} v(z)|f'(z)| = 0 \}.
\]

We are going to study weighted composition operators on weighted Bergman spaces and weighted Bloch spaces. In the second section we compute the norm of a function in Bloch space and give a sufficient condition for the boundedness of weighted composition operators on the weighted Bloch spaces. In the third section these operators will be considered between (weighted) Bergman spaces and weighted Bloch spaces.
2 Between weighted Bloch spaces

In this section we give a sufficient condition for the boundedness of weighted composition operators on the weighted Bloch spaces.

A weight $v$ is called radial if $v(z) = v(|z|)$. We say, a radial weight $v$ satisfies (L1) if

$$\inf_k \frac{v(1 - 2^{-k-1})}{v(1 - 2^{-k})} > 0.$$  

This is equivalent to the following condition:

There are $0 < r < 1$ and $0 < C < \infty$ with $\frac{v(z)}{v(a)} \leq C$ for every $a, z \in \mathbb{D}$ with $|\varphi_a(z)| \leq r$, where $\varphi_a$ is the Mobius transformation which is defined by $\varphi_a(z) = \frac{z - a}{1 - \overline{a}z}$. Also the pseudo-hyperbolic metric is defined by

$$\rho(z, a) = |\varphi_a(z)|,$$

for every $z, a \in \mathbb{D}$ and $E(a, r)$ denotes the open ball in this metric.

**Lemma 2.1.** If $f \in H(\mathbb{D})$, then

$$\|f\|_{B_v} \approx \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} v(z)|f'(z)| |\varphi_a'(z)|^2 dA(z). \tag{1}$$

**Theorem 2.2.** Let $\psi C_\varphi : B_v^0 \to B_v^0$ be a bounded operator. Then $\psi C_\varphi : B_v \to B_v$ is a bounded operator.

3 Between (weighted) Bergman spaces and weighted Bloch spaces

We first consider the standard weighted Bergman spaces.

**Theorem 3.1.** Let $p \geq 1$. Then $\psi C_\varphi$ is bounded from $A_{w_{\alpha,p}} = A_{\alpha,p}^w$ into $B_v$ if and only if

1. $\sup_{z \in \mathbb{D}} \frac{v(z)|\varphi'(z)|}{\varphi(z)|\varphi'(z)|} < \infty$
2. $\sup_{z \in \mathbb{D}} \frac{v(z)|\varphi'(z)|}{\varphi(z)|\varphi'(z)|} < \infty$.

For the compactness, we have the following theorem.

**Theorem 3.2.** Let $p \geq 1$ and $\psi C_\varphi$ is bounded from $A_{w_{\alpha,p}} = A_{\alpha,p}^w$ into $B_v$. Then $\psi C_\varphi : A_{w_{\alpha,p}} \to B_v$ is a compact operator if and only if

1. $\lim_{|\varphi(z)| \to 1} \frac{v(z)|\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1+(2+\alpha)/p}} = 0$
2. $\lim_{|\varphi(z)| \to 1} \frac{v(z)|\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1+(2+\alpha)/p}} = 0$.

If $\alpha = 0$ and $v(z) = 1 - |z|^2$, then the results of Sharma and Kumari in [3] are derived. If $\alpha = 0$ and $v(z) = (1 - |z|^2) \log \frac{2}{1 - |z|^2}$, the results of Li in [2] are derived. Note that in the previous theorem, we don’t need to the boundedness of $v(z)$, so we can omit it.

**Theorem 3.3.** Suppose that $w$ be a weight of the form $w = |W|$ such that $W$ is an analytic function on $\mathbb{D}$. If
then the weighted composition operator $\psi C_{\varphi} : A_{w,p} \rightarrow B_v$ is bounded.

In the following theorem the weights for the Bergman spaces are as follow: Let $W$ be an analytic function on $\mathbb{D}$ such that $W$ is non-vanishing, strictly positive and decreasing on $[0,1)$. Then the corresponding weight is defined by $w(z) = W(|z|^2)$, for every $z \in \mathbb{D}$. Here are some examples:

- If $W(z) = (1 - z)^\alpha$, $\alpha \geq 1$ then $w(z) = (1 - |z|^2)^\alpha$ (standard weight).
- If $W(z) = e^{-\frac{1}{1-|z|^2}}$, $\alpha \geq 1$ then $w(z) = e^{-\frac{1}{1-|z|^2}}$.

Also we assume that there exists a positive constant $C$ such that

$$\sup_{a \in \mathbb{D}} \sup_{z \in \mathbb{D}} \frac{w(z)}{W(\varphi(a)z)} \leq C.$$  

**Theorem 3.4.** Suppose that $w$ is a weight which we defined above. If $\psi C_{\varphi} : A_{w,p} \rightarrow B_v$ be bounded and

$$\sup_{z \in \mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{2/p}w(\varphi(z))^{1/p}} < \infty,$$

then

$$\sup_{z \in \mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1+2/p}w(\varphi(z))^{1/p}} < \infty,$$

if and only if

$$\sup_{z \in \mathbb{D}} \frac{v(z)|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1+2/p}w(\varphi(z))^{1/p}} < \infty.$$  

**Theorem 3.5.** Suppose that $w$ be a weight of the form $w = |W|$ where $W$ is an analytic function on $\mathbb{D}$ and $\psi C_{\varphi} : A_{w,p} \rightarrow B_v$ be bounded. If

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{v(z)|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^2/pw(\varphi(z))^{1/p}} = 0,$$

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{v(z)|\psi(z)\varphi'(z)|}{w(\varphi(z))^{1/p}} = 0,$$

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{v(z)|\psi(z)\varphi'(z)|}{w(\varphi(z))^{1/p}} = 0,$$

then the weighted composition operator $\psi C_{\varphi} : A_{w,p} \rightarrow B_v$ is compact.

**Theorem 3.6.** Suppose that $w$ is a weight which we defined above. If $\psi C_{\varphi} : A_{w,p} \rightarrow B_v$ be compact and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{v(z)|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{2/p}w(\varphi(z))^{1/p}} = 0,$$

then

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{v(z)|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1+2/p}w(\varphi(z))^{1/p}} = 0,$$

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{v(z)|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1+2/p}w(\varphi(z))^{1/p}} = 0.$$  

(4)
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Positively full Hilbert $C^*$-modules

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Abstract
We introduce the concept of positive fullness of a Hilbert $C^*$-module and we show that every positively full Hilbert $C^*$-module is full. If $A$ is a unital $C^*$-algebra and $M$ is a positively full Hilbert $A$-module, then there exists a Hilbert $A$-module $N$ such that $M \cong A \oplus N$.

Keywords: Hilbert $C^*$-Modules, $\sigma$-unital $C^*$-algebra, Positively Full Hilbert $C^*$-Modules.

Mathematics Subject Classification: 46L08

1 Introduction
Suppose $A$ is a $C^*$-algebra and $M$ is a right $A$-module, then $M$ is called a pre-Hilbert $A$-module if there exists a sesquilinear form $\langle \cdot , \cdot \rangle : M \times M \rightarrow A$ with the following properties:

i) $\langle x,x \rangle \geq 0$ and $\langle x,x \rangle = 0$ if and only if $x = 0$;

ii) $\langle y,x \rangle = \langle x,y \rangle^*$;

iii) $\langle x,ya \rangle = \langle x,y \rangle a$;

for each $x,y \in M$ and $a \in A$. Let $M$ be a pre-Hilbert $A$-module and $x \in M$. Then $M$ is called a Hilbert $C^*$-module or a Hilbert $A$-module if it is complete with respect to the norm $\|x\|_M = \|\langle x,x \rangle\|^{\frac{1}{2}}$.

Definition 1.1. Let $A$ be a $C^*$-algebra. An increasing net $\{u_\alpha\}_{\alpha \in I}$ in $A^+$ with $\|u_\alpha\| \leq 1$ for all $\alpha \in I$ is called an approximate unit for $A$ if $\lim_{\alpha} ||a(1-u_\alpha)|| = 0$ for each $a$ in $A$. A $C^*$-algebra is called $\sigma$-unital if it has a countable approximate unit. If $M$ is a Hilbert $A$-module then we denote by $\langle M,M \rangle$ the closure of the linear span of all $\langle x,x \rangle$, where $x \in M$. In fact,

$\langle M,M \rangle = \text{span}\{\langle x,x \rangle; \ x \in M\}$.

A Hilbert $A$-module $M$ is called full if $\langle M,M \rangle = A$.

If $M$ is a Hilbert $A$-module then for every $x$ in $M$, $\langle x,x \rangle$ is positive. Now this question arise in mind whether for every positive element $a$ in $A$, there exists an element $x$ in $M$ such that $\langle x,x \rangle = a$?

Definition 1.2. Suppose $A$ is a $C^*$-algebra and $M$ is a Hilbert $A$-module. The Hilbert $A$-module $M$ is called positively full if for every positive element $a$ in $A$, there exists an element $x$ in $M$ such that $\langle x,x \rangle = a$. 

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Example 1.3. Suppose $A$ is a $C^*$-algebra. The Hilbert $A$-module $A$ is positively full since the $A$-valued inner product in Hilbert $A$-module $A$ is given by $\langle a, b \rangle = a^* b$ and for every positive element $a$ in $A$, there exists $b$ in $A$ such that $a = b^* b = \langle b, b \rangle$.

Definition 1.4. Let $A$ be a $C^*$-algebra and $M, N$ be Hilbert $A$-modules. A bounded $A$-linear map $T : M \rightarrow N$ is called an operator. The set of all operators from $M$ to $N$ is denoted by $\text{Hom}_A(M, N)$. We say that $T \in \text{Hom}_A(M, N)$ is adjointable if there exists an operator $T^* \in \text{Hom}_A(N, M)$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x \in M$ and $y \in N$. The set of all adjointable operators from $M$ to $N$ is denoted by $\text{Hom}_A^*(M, N)$.

Lemma 1.5. Let $M, N$ be Hilbert $A$-modules and $T \in \text{Hom}_A^*(M, N)$ an operator with closed range, then

i) $\text{Ker} T$ is an orthogonally complementable submodule in $M$.

ii) $\text{Im} T$ is an orthogonally complementable submodule in $N$.

Proof. See [3].

2 Main Result

Theorem 2.1. Every positively full Hilbert $C^*$-module is full.

Proof. Since

$$\langle M, M \rangle = \text{span}\{\langle x, x \rangle; \ x \in M\}$$

for every $t$ in $\langle M, M \rangle$ there exists a sequence $\{t_n\}$ in $\text{span}\{\langle x, x \rangle; \ x \in M\}$ such that $t_n \rightarrow t$. But for every $n \in \mathbb{N}$, $t_n \in A$ and $A$ is complete. So $\langle M, M \rangle \subseteq A$. On the other hand for every $a$ in $A$, $a = a_1 + i a_2$ where $a_1$ and $a_2$ are self adjoint elements of $A$. So $a = a_1 - a_1 + i(a_2^+ - a_2^-)$. Since $M$ is positively full, there exists $x_1, x_2, x_3$ and $x_4$ in $M$ such that

$$a_1^+ = \langle x_1, x_1 \rangle, \quad a_1^- = \langle x_2, x_2 \rangle, \quad a_2^+ = \langle x_3, x_3 \rangle, \quad a_2^- = \langle x_4, x_4 \rangle.$$ 

So

$$a = \langle x_1, x_1 \rangle - \langle x_2, x_2 \rangle + i(\langle x_3, x_3 \rangle - \langle x_4, x_4 \rangle).$$

Therefore $A \subseteq \langle M, M \rangle$. 

Theorem 2.2. If a Hilbert $A$-module $M$ is positively full then there exists a net $\{x_\alpha\}_{\alpha \in I}$ in $M$ such that $\{(x_\alpha, x_\alpha)\}_{\alpha \in I}$ is an approximate unit for $A$.

Proof. Since each $C^*$-algebra contains an approximate unit, say $\{u_\alpha\}_{\alpha \in I}$, and $M$ is positively full, for every $\alpha$ in $I$ there exists $x_\alpha$ in $M$ such that $u_\alpha = \langle x_\alpha, x_\alpha \rangle$.

Theorem 2.3. If $A$ is a $\sigma$-unital $C^*$-algebra and $M$ is a Hilbert $A$-module then there exists a sequence $\{x_n\}$ in $M$ such that $\left\{\sum_{i=1}^{k} (x_i, x_i)\right\}_{k=1}^{\infty}$ is a countable approximate unit for $A$.

Proof. Since $A$ is $\sigma$-unital, $A$ has a countable approximate unit $\{u_n\}$. So

$$0 \leq u_1 \leq u_2 \leq u_3 \leq \cdots \leq u_n \leq u_{n+1} \leq \cdots.$$ 

Put $e_1 = u_1$ and $e_n = u_n - u_{n-1}$ ($n = 2, 3, \ldots$). Since every $e_i$ is positive and $M$ is positively full, for each $i$ in $\mathbb{N}$ there exists $x_i$ in $M$ such that $e_i = \langle x_i, x_i \rangle$. Since $u_k = \sum_{i=1}^{k} e_i$, the sequence $\left\{\sum_{i=1}^{k} (x_i, x_i)\right\}_{k=1}^{\infty}$ is a countable approximate unit for $A$. 

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**Theorem 2.4.** If $A$ is a unital $C^*$-algebra and $M$ is a positively full Hilbert $A$-module, then there exists a Hilbert $A$-module $N$ such that $M \cong A \oplus N$.

**Proof.** Suppose $1_A$ is the unit element of $A$. Since $M$ is positively full there exists $e$ in $M$ such that $1_A = \langle e, e \rangle$. Define $T : A \to M$ by $Ta = ea$ for each $a$ in $A$. Then

$$||Ta||^2 = ||\langle Ta, Ta \rangle|| = ||\langle ea, ea \rangle|| = ||a^* \langle e, e \rangle a|| = ||a^* a|| = ||a||^2.$$  

So $||Ta|| = ||a||$. Moreover

$$\langle Ta, m \rangle = \langle ea, m \rangle = a^* \langle e, m \rangle = \langle a, \langle e, m \rangle \rangle = \langle a, T^* m \rangle.$$ 

So $T^* : M \to A$ exists and is defined by $T^* m = \langle e, m \rangle$. But $Im T \cong A$, then $Im T$ is closed. By Lemma 12, we have $M = Im T \oplus Im T^\perp \cong A \oplus N$, where $N = Im T^\perp$. \hfill $\square$

**References**


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Weighted Frobenius-Perron operators on $L^p$ spaces

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Abstract
In this note weighted Frobenius-Perron operator $P$ and their finite sums on $L^1(\Sigma)$ is considered and then, boundedness, compactness and essential norm of these type operators are investigated.

Keywords: Frobenius-perron operator, conditional expectation, essential norm.

Mathematics Subject Classification: 47B20, 46B38.

1 Introduction
Let $(X, \Sigma, \mu)$ be a complete $\sigma$-finite measure space and let $\varphi : X \to X$ be a non-singular transformation, i.e. $\varphi$ is $\Sigma$-measurable and $\mu(\varphi_j^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. This assumption about $\varphi$ just says that the measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to the measure $\mu$ (we write $\mu \circ \varphi^{-1} \ll \mu$, as usual), where $\mu \circ \varphi^{-1}(A) = \mu(\varphi_j^{-1}(A))$ for $A \in \Sigma$.

We shall assume that the restriction of $\mu$ to $\sigma$-subalgebra $\varphi_j^{-1}(\Sigma)$ of $\Sigma$, is $\sigma$-finite, and we denote by $(X, \varphi_j^{-1}(\Sigma), \mu)$ the completion of $(X, \varphi_j^{-1}(\Sigma), \mu_{\varphi_j^{-1}(\Sigma)})$. We denote by $h$ the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1}/d\mu$. We will write $L^1(\varphi_j^{-1}(\Sigma))$ for $L^1(X, \varphi_j^{-1}(\Sigma), \mu_{\varphi_j^{-1}(\Sigma)})$. $L^1(\varphi_j^{-1}(\Sigma))$ may then be viewed as a subspace of $L^1(\Sigma)$ and denote its norm by $\| \cdot \|_1$. Support of a measurable function $f$ will be denoted by $\text{supp}(f) = \{x \in X; f(x) \neq 0\}$. Relationships between functions $f$ and between sets are interpreted in the almost every where sense. For any non-negative $\Sigma$-measurable functions $f$ as well as for any $f \in L^p(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\varphi_j^{-1}(\Sigma)$-measurable function $E(f)$ such that

$$\int_A Ef d\mu = \int_A f d\mu, \quad \text{for all } A \in \varphi_j^{-1}(\Sigma).$$

Hence we obtain an operator $E$ from $L^1(\Sigma)$ onto $L^1(\varphi_j^{-1}(\Sigma))$ which is called conditional expectation operator associated with the $\sigma$-algebra $\varphi_j^{-1}(\Sigma)$. It is easy to show that for each $f \in L^1(\Sigma)$, there exists a $\Sigma$-measurable function $g$ such that $E(f) = g \circ \varphi$. To obtain a unique $g$ with this property we may assume and do that $\text{supp}(g) \subseteq \text{supp}(h)$. We therefore write $g = E(f) \circ \varphi^{-1}$, though we make no assumptions regarding the invertibility of $\varphi$ (see [3]). It is easy to check that $E(f) \circ \varphi^{-1} - E(g) \circ \varphi^{-1} = E(f - g) \circ \varphi^{-1}$ and $|E(f) \circ \varphi^{-1}| = |E(f)| \circ \varphi^{-1}$ for all $f, g \in L^1(\Sigma)$.

We list here some of its useful properties:

- $E(fg) = Ef g$ whenever $g$ is $\varphi_j^{-1}(\Sigma)$-measurable and both conditional expectations are defined.
- $|E(f)|^p \leq E(|f|^p)$, for each $p \geq 1$.
- If $f \geq 0$ then $E(f) \geq 0$; if $E(|f|) = 0$ then $f = 0$.

Let $f$ be a real-valued measurable function. Consider the set $B_f = \{x \in X : E(f^+)(x) = E(f^-)(x) = \infty\}$. The function $f$ is said to be conditionable with respect to $\varphi_j^{-1}(\Sigma)$, if $\mu(B_f) = 0$. 

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If \( f \) is complex-valued, then \( f \) is conditionable if the real and imaginary parts of \( f \) are conditionable and their respective expectations are not both infinite on the same set of positive measure. For more details on the properties of \( E \) see [3, 5].

In the next section we consider finite sum of weighted Frobenius-Perron operators on \( L^1(\Sigma) \) of the form \( P = \sum_{i=1}^{n} P_{\varphi_i}^{u_i} \), where for each \( 1 \leq i \leq n, u_i : X \to \mathbb{C} \) is measurable function and \( \varphi_i : X \to X \) is nonsingular transformation. Also we assume that for each \( 1 \leq i \leq n, \varphi_i^{-1}(\Sigma) \) is a sub-\( \sigma \)-finite algebra of \( \Sigma \). Put \( h_i = d\mu \circ \varphi_i^{-1}/d\mu \) and \( E_i = E^{\varphi_i^{-1}(\Sigma)} \). It follows that \( P^* = W \) and for all \( f \in L^1(\Sigma), P(f) = \sum_{i=1}^{n} h_iE_i(u_if) \circ \varphi_i^{-1} \). Note that the set of all these kind bounded operators is an operator algebra.

## 2 Main Result

**Theorem 2.1.** Let \( P \) be a finite sum of weighted Frobenius-Perron operators on \( L^1(\Sigma) \). If \( P \) is bounded, then \( \sum_{i=1}^{n} u_i \in L^\infty(\Sigma) \) and \( \| \sum_{i=1}^{n} u_i \|_\infty \leq \| P \| \). Moreover, if \( u_i \)'s are nonnegative, then \( P \) is a bounded operator if and only if each \( 1 \leq i \leq n, P_{\varphi_i}^{u_i} \) is bounded and in this case its norm is given by \( \| P \| = \| \sum_{i=1}^{n} u_i \|_\infty \).

**Remark 2.2.** Let \( ca(X, \Sigma, \mu) \) be the set of all complex measures absolutely continuous with respect to \( \sigma \)-finite measure \( \mu \). Define a mapping \( \Psi : L^1(X, \Sigma\mu) \to ca(X, \Sigma, \mu) \) by \( \Psi(f) = \mu_f \) with inverse \( \Psi^{-1}(\nu) = \frac{d\mu}{d\nu} \), where \( \mu_f(A) := \int_A f \, d\mu \) for all \( A \in \Sigma \). Then \( \Psi \) is bounded and \( W^* \Psi = \Sigma_{i=1}^{n} \Psi^{-1}(u_iC_{\varphi_i})^* \Psi = \sum_{i=1}^{n} P_{\varphi_i}^{u_i} = P \) (see [1, 2]). Thus, \( P \) is bounded (compact) on \( L^1(\Sigma) \) if and only if \( W \) is bounded (compact) on \( L^\infty(\Sigma) \).

**Theorem 2.3.** Let \( u = \sum_{i=1}^{n} |a_i|, a_j = \mu(A_j) < \infty \) and let \( N_\varepsilon(u) = \{ x \in X : |u(x)| \geq \varepsilon \} \). If for each \( \varepsilon > 0, N_\varepsilon(u) \) consists of finitely many atoms, then \( P \) is compact.

**Theorem 2.4.** (a) Let \( P = \sum_{i=1}^{n} P_{\varphi_i}^{u_i} \) be a bounded operator on \( L^1(\Sigma) \). Put \( u = \sum_{i=1}^{n} |u_i| \) and \( \beta = \inf \{ r > 0 : N_r(u) \mbox{ consists of finitely many atoms} \} \). Then \( \| P \| \leq \beta \).

(b) Let \( (X, \Sigma, \mu) \) be a purely atomic measure space. If the sequence \( \{ a_j \}_{j \in \mathbb{N}} \) has no subsequence that converges to zero and \( u_i \)'s are nonnegative, then \( \| P \| \geq \beta \).

## References


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Numerical range of operators acting on Banach spaces

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Abstract
The aim of this paper is to propose a definition of numerical range of an operator on reflexive Banach spaces. In order to this definition the numerical range will satisfy the basic properties of a canonical numerical range. We will determine necessary and sufficient conditions from which the numerical range of a composition operator on a weighted Hardy space is closed.

Keywords: Numerical range, weighted Hardy space, compact operator, composition operator.

Mathematics Subject Classification: 47B37, 47A12.

1 Introduction
The concept of numerical range of an operator on Hilbert spaces was presented by O. Toeplitz in
1918 ([9]). The numerical range of a bounded operator $T$ on a Hilbert space $H$ is the set of complex
numbers $V(T) = \{ < Th, h : h \in H, ||h|| = 1 \}$.

The Toeplitz-Hausdorff Theorem ([6, 9]) establishing the convexity of the numerical range for any
operator on a Hilbert space. Some properties and further developments of the numerical range of a bounded linear operator on a Hilbert space can be found in [4, 5]. The concept of numerical range on a Banach space $X$, have extended by Baure and Lumer in [1, 8], that is not necessarily convex, see [2, Example 21.6].

Let $X$ be a Banach space. Every $x \in X$, defines an element $\hat{x} \in X^{**}$ (second dual of $X$) by $\hat{x}(x^*) = x^*(x)$, where $x^* \in X^*$ and $||x|| = ||\hat{x}||$. Recall that $X$ is a reflexive Banach space if $X^{**} = \{ \hat{x} : x \in X \}$.

In this paper by giving a definition of numerical range for operators on Banach spaces, we want to investigate some properties that are consistent for the Hilbert case.

A holomorphic function $\varphi$ that takes the open unit disc $\mathbb{D}$ into itself induces a linear composition operator $C_{\varphi}$ on the space $H(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$ as follows:

$$C_{\varphi}(f) = f \circ \varphi, \quad (f \in H(\mathbb{D})).$$

A lot of works has been done to studying composition operators on Hardy spaces and weighted Hardy spaces (see for example [3, 11]).
Writing $\beta(n) = ||z^n||$, the orthogonality implies that the norm of the formal power series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ in $H^2(\beta)$ is given by

$$||f||_\beta^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \beta(n)^2.$$
We can extend the definitions into Banach spaces $H^p(\beta)$ as follows: Let $\{\beta(n)\}$ be a sequence of nonzero complex numbers with $\beta(0) = 1$ and $1 \leq p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_p^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$ 

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ will be used weather or not the series converges for any value of $z \in \mathbb{D}$. These are called formal power series. Let $H^p(\beta)$ denote the space of such formal power series. It is called a weighted Hardy space.

Let $\lambda$ be a complex number, the functional of evaluation at $\lambda$, $e_{\lambda}$, is defined by $e_{\lambda}(p) = p(\lambda)$, for all polynomials $p$. Also, $\lambda$ is said to be a bounded point evaluation on $H^p(\beta)$ if the function $e_{\lambda}$ extends to be a bounded linear functional on $H^p(\beta)$. In this case we have $e_{\lambda}(f) = f(\lambda)$, $f \in H^p(\beta)$.

**Theorem 1.1.** ([11]) A complex number $\lambda$ is bounded point evaluation on $H^p(\beta)$ if and only if $\{\frac{\lambda^n}{p!}\}_{n=0}^{\infty}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

The functional of evaluation of the $j$-th derivative at $\lambda$ is denoted by $e^{(j)}_{\lambda}$. Also, we note that $e_{\lambda}(z) = \sum_{n=0}^{\infty} \frac{\lambda^n}{p!} z^n$ and $e^{(j)}_{\lambda} = \frac{d^j}{dx^j}(e_{\lambda})$. Thus

$$e^{(j)}_{\lambda}(z) = \sum_{n \geq j} n(n-1)...(n-j+1) \frac{(\lambda)^{n-j}}{\beta(n)^p} z^n.$$ 

## 2 Main Result

In this section we give a definition of numerical range for Banach spaces which extends the earlier definition in the case of Hilbert spaces. Then, under this definition we will investigate some well-known properties that are consistent for Hilbert spaces.

**Definition 2.1.** Let $X$ be a reflexive Banach space and $T \in B(X)$. The numerical range of $T$ is defined by

$$W(T) = \text{co}(V(T)),$$

where $\text{co}(V(T))$ is the convex hull of $V(T)$ and

$$V(T) = \{x^*(T(x)) : x \in X, \ x^* \in X^* ; \ |x| = \|x^*\| = x^*(x) = 1\}.$$ 

Obviously, $W(T)$ is convex and contains $V(T)$. We use the usual notations $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{sp}(T)$ respectively for the spectrum, eigenspace, and approximate point spectrum of $T$. Note that $\sigma_p(T) \subseteq \sigma_{sp}(T)$.

**Proposition 2.2.** Let $X$ be a reflexive Banach space and $T \in B(X)$. Then

(i) $W(T) = W(T^*)$.

(ii) $W(T)$ is a convex subset of the complex plane that lies in the closed disc with radius of $\|T\|$ at origin.

(iii) $\sigma_p(T) \subseteq W(T)$.

(iv) $\sigma(T) \subseteq W(T)$.
Note that in [11], we used a definition of a numerical range of an operator acting on $H^p(\beta)$ that is not necessarily convex. Here by general definition 2.1. for Banach spaces, the numerical range is convex and we can extend some earlier results. In this section, we prove some results about the numerical range of a compact operator acting on the Banach space $H^p(\beta)$.

**Theorem 2.3.** If $T$ is a compact operator on a reflexive Banach space $X$, then

$$V(T) \subseteq \{c \alpha : 0 \leq c \leq 1, \ \alpha \in V(T)\}.$$ 

**Corollary 2.4.** If $T$ is a compact operator on a reflexive Banach space, then

$$W(T) \subseteq \{c \alpha : 0 \leq c \leq 1, \ \alpha \in W(T)\}.$$ 

**Theorem 2.5.** Let $\frac{1}{p} + \frac{1}{q} = 1$ and $\sum_{n \geq 0} \frac{n^{q_j}}{\beta(n)^q} = \infty$ for some $j$. If $C_\phi$ is bounded on $H^p(\beta)$, then $0 \in W(C_\phi)$.

**Corollary 2.6.** Under the conditions of Theorem 2.5., if $C_\phi$ is compact, then $0 \in W(C_\phi)$ if and only if $W(C_\phi)$ is closed.

**Remark 2.7.** As we saw we have denoted the numerical range of an operator $T$ acting on a Banach space by $W(T)$ and defined it by $W(T) = \text{co}(V(T))$, and we note that the set $V(T)$ is not necessarily convex ([2, Example 21.6]). Our idea of this definition is to obtain the same properties holds from the earlier definition of numerical range of an operator acting on a Hilbert space such as the properties stated in Proposition 2.2. Specially, the property of convexity of $W(T)$ is useful and it is used in the proof of Theorem 2.5. In the proof of Theorem 2.5, we should show that $W(C_\phi)$ is normed closed, but we could only prove that $W(C_\phi)$ is weakly closed. Then it’s convexity and boundedness implies that it should be also normed closed.

Recall that $H^p(\beta)$ is small weighted Hardy space, if $H^p(\beta)$ is contained in the disc algebra $A(D)$.

**Theorem 2.8.** Suppose that $\{\beta(n)\}_n$ is an increasing sequence such that $\sum_{n=0}^{\infty} \frac{\beta(n)}{\beta(n)^q} \leq \infty$, then the weighted Hardy space $H^p(\beta)$ is small.

$$W(C_\phi) = \{ (1 - r) + r z : 0 \leq r \leq 1 \},$$

then $0 \in W(C_\phi)$ for all $0 < r < \alpha$.

**References**


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Abstract

In this paper we introduce a set of basis functions known as weighted partition on a bounded subset of \( \mathbb{R} \) and a weighted transform in discrete case. We investigate the properties of the weighted transform and its discrete version in the context of approximation theory.

Keywords: Basic Function, Weighted Partition, Fuzzy Transform.

Mathematics Subject Classification: 41A30, 41A45.

1 Introduction

In [3]Perfilieva has developed a general method known as fuzzy transform as well as approximation methods based on IF-THEN rules studied in fuzzy modeling. By defining fuzzy transform she has established a correspondence between a set of continuous functions on an interval of real numbers and the set of n-dimensional vectors. Many authors have been worked on fuzzy transform and its application such as approximation of discrete data [2], image analysis and compression [1], ordinary and partial differential equation [4,6] and image fusion [5].

In [2], Pantane has investigated the relation between the fuzzy transform and approximation of a set of scattered data \( \{(x_i, f(x_i))\}_{i=1}^{n} \).

In this article, by using a continuous function \( \varphi \) from \( E \) into \((0, 1]\), as a weight, we introduce the weighted partition and follow the idea of [2].

2 Main Result

In the sequel, let \( E \) be a bounded subset of \( \mathbb{R} \) which contains its infimum and supremum so that \( E = [\inf E, \sup E] \). Suppose that \( \varphi \) is a continuous function from \( E \) into \((0, 1]\).

Definition 2.1. Let \( x_0 = x_1 < \ldots < x_n = x_{n+1} \) be such that \( x_1 = \inf E, x_n = \sup E \) and \( x_k \in E \) for all \( 1 \leq k \leq n \). We say that \( B = \{B_1, ..., B_n\} \) is a weighted partition of \( E \) if the following statements hold:

(i) \( B_k \) is a continuous function from \( E \) into \([0, 1] \) for all \( k = 1, ..., n \);

(ii) \( B_k(x_k) = \varphi(x_k) \), for all \( k = 1, ..., n \);

(iii) \( B_k(x) = 0 \) whenever \( x \in E \setminus (x_{k-1}, x_{k+1}) \) for all \( k = 1, ..., n \);

(iv) \( \sum_{k=1}^{n} B_k(x) = \varphi(x) \) for all \( x \in E \).
Definition 2.1 is in fact a slight generalization of basic function [3, Definition 2.1].
In this way, we have not limited ourselves to a special partition by normal fuzzy subsets.

**Definition 2.2.** Let \( \varphi \) be a continuous function from \( E \) into \((0, 1]\) and a real valued function \( f \) be given at points \( \{p_1, ..., p_s\} \) of \( E \). Suppose that \( \mathcal{B} = \{B_1, ..., B_n\} \) is a weighted partition of \( E \), \( s \leq n \).

The weighted transform \( f \) is the vector \( F = (F_1, ..., F_n) \) with respect to \( \mathcal{B} \) where

\[
F_k = \frac{\sum_{i=1}^{s} \varphi(p_i)B_k(p_i)}{\sum_{i=1}^{s} B_k(p_i)}, \quad k = 1, ..., n.
\]

Let \( K = (B_k(p_i)), 1 \leq k \leq n, 1 \leq i \leq s, n \leq s, \) be the coefficient matrix and \( D = \text{diag}(d_1, ..., d_n) \) with \( d_k = \sum_{i=1}^{s} B_k(p_i) \). Then with respect to \( \{p_1, ..., p_s\} \) the weighted transform \( F \) can be represented in the matrix form as \( F = (f(p_1), ..., f(p_s)) \).

**Theorem 2.3.** Let function \( f \) be given at points \( \{p_1, ..., p_s\} \) of \( E \) and \( F = (F_1, ..., F_n) \), \( n \leq s \), be the weighted transform of \( f \) with respect to \( \mathcal{B} = \{B_1, ..., B_n\} \). Then \( \|F\|_{2} \leq \sqrt{n} \|A\|_{\infty} \) and \( \|F\|_{\infty} \leq \|A\|_{\infty} \) where \( A^T = (f(p_1), ..., f(p_s)) \).

In the following we introduce the inverse weighted transform at points \( \{p_1, ..., p_s\} \).

**Definition 2.4.** Let \( f \) be a real valued function which known only at the given points \( \{p_1, ..., p_s\} \) of \( E \) and \( \varphi \) be a continuous function from \( E \) into \((0, 1]\). Suppose that \( F = (F_1, ..., F_n) \) is the weighted transform of \( f \) with respect to \( \mathcal{B} = \{B_1, ..., B_n\} \). The function \( T_{\mathcal{B}, f}(p) = \sum_{k=1}^{n} \frac{F_k}{\varphi(p)} \) is called the inverse weighted transform.

Since \( E \) is a bounded subset of \( \mathbb{R} \) containing its infimum and supremum and \( \overline{E} = [\inf E, \sup E] \) then this guarantees the existence of at least one point in each subset \( E \cap (x_{k-1}, x_{k+1}), \quad k = 1, ..., n \).

So we can prove that inverse weighted transform can approximate the original function \( f \) at \( \{p_1, ..., p_s\} \).

**Theorem 2.5.** Let \( \varphi \) be a continuous function from \( E \) into \((0, 1]\). Suppose that \( f \) is known at only \( \{p_1, ..., p_s\} \subseteq E \). Then for any positive \( \varepsilon \), there exists a weighted partition \( \mathcal{B}_\varepsilon = \{B_1, ..., B_{n_\varepsilon}\} \) such that

\[
\|f(p_i) - T_{\mathcal{B}_\varepsilon, f}(p_i)\|_{\infty} < \varepsilon.
\]

Following proposition gives an estimation of \( \ell^2 - \) norm of weighted transform.

**Proposition 2.6.** Let \( \varphi \) be a continuous function from \( E \) into \((0, 1]\). Suppose that function \( f \) is known only at \( \{p_1, ..., p_s\} \subseteq E \) and \( F = (F_1, ..., F_n) \) is the weighted transform of \( f \) with respect to \( \mathcal{B} = \{B_1, ..., B_n\} \).

(a) The \( \ell^2 - \) norm of \( F \) is proportional to norm of \( F \) and the maximum singular value of the coefficient matrix \( (B_k(p_i)), 1 \leq k \leq n, 1 \leq i \leq s \).

(b) Weighted transform is stable with respect to input data.

**References**


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Some properties of Lambert multipliers

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Abstract
In this paper, we investigate some the algebraic structure of $T_u(f) = u \ast f$, so-called Lambert operator on $L^2(\Sigma)$ spaces, that $u$ will be any measurable and conditional function.

Keywords: conditional expectation, multipliers, commutative operator algebra.

Mathematics Subject Classification: Primary 47B20; Secondary 47B38.

1 Introduction
Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space and let $\mathcal{A}$ be a $\sigma$-subalgebra of $\Sigma$ such that $(X, \mathcal{A}, \mu)$ is also $\sigma$-finite. For any complete $\sigma$-finite sub-algebra $\mathcal{A} \subseteq \Sigma$ with $1 \leq p \leq \infty$, the $L^p$-space $L^p(X, \mathcal{A}, \mu; \mathcal{A})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\|\cdot\|_p$. We understand $L^p(\mathcal{A})$ as a Banach sub-space of $L^p(\Sigma)$. We shall adopt the convention that all function and set-theoretic relations are assumed to hold $\mu - a.e.$, and such disclaimers will be omitted. To examine the weighted composition operators efficiently, Alan Lambert in [3] associated with each transformation $T$, the so-called conditional expectation operator $E(\cdot|\mathcal{A}) = E(\cdot)$ is defined for each non-negative measurable function $f$ or for each $f \in L^p(\Sigma)$, and is uniquely determined by the conditions: (i) $E(f)$ is $\mathcal{A}$-measurable and (ii) If $A$ is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ converges we have

$$\int_A f d\mu = \int_A E(f) d\mu.$$ 

This operator will play major role in our work. Next, we list basic properties of $E$, all of which can be found in texts such as [2]; • If $g$ is $\mathcal{A}$-measurable then $E(fg) = E(f)g$. • $|E(f)|^p \leq E(|f|^p)$. • $\|E(f)\|_p \leq \|f\|_p$. • If $f \geq 0$ then $E(f) \geq 0$; if $f > 0$ then $E(f) > 0$. • $E(|f|^2) = |E(f)|^2$ if and only if $f \in L^2(\mathcal{A})$. The mapping $E$ is a linear operator and, in particular, it is a contraction. In the case when $p = 2$, it is the orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathcal{A})$. The real-valued $\Sigma$-measurable function $f$ is said to be conditionable with respect to $\mathcal{A}$ if $\mu(\{x \in X : E(f^+)(x) = E(f^-)(x) = \infty\}) = 0$. In this case $E(f) := E(f^+) - E(f^-)$.(see [2]).

We denote the linear space of all conditionable $\mathcal{A}$-measurable functions on $X$ by $L^0(\mathcal{A})$. For $f$ and $g$ in $L^0(\Sigma)$, we define $f \ast g = fE(g) + gE(f) - E(f)E(g)$. Let $1 \leq p, q \leq \infty$. A measurable function $u \in L^0(\Sigma)$ for which $u \ast f \in L^q(\Sigma)$ for each $f \in L^p(\Sigma)$, is called Lambert multiplier. In other words, $u \in L^0(\Sigma)$ is Lambert multiplier if and only if the corresponding $\ast$-multiplication operator $T_u : L^p(\Sigma) \to L^q(\Sigma)$ defined as $T_u f = u \ast f$ is bounded.

2 Main Result
In this section we bring some facts and definitions, which will be used later. In this note we will assume $\mu(X) < \infty$. 

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Definition 2.1. For $f$ and $g$ in $L^0(\Sigma)$, we define

$$f \ast g = fE(g) + gE(f) - E(f)E(g).$$

Definition 2.2. A measurable function $u \in L^0(\Sigma)$ for which $T_u(f) = u \ast f$ for each $f \in L^2(\Sigma)$, is called Lambert operator.

Definition 2.3. A measurable function $u \in L^0(\Sigma)$ for which $u \ast f \in L^2(\Sigma)$ for each $f \in L^2(\Sigma)$, is called Lambert multiplier.

In other words, $u \in L^0(\Sigma)$ is Lambert multiplier if and only if the corresponding $\ast$-multiplication operator $T_\lambda : L^2(\Sigma) \to L^2(\Sigma)$ defined as $T_\lambda f = u \ast f$ is bounded.

Definition 2.4. Define $K_2^*$, the set of all Lambert multipliers as follows:

$$K_2^* = \{ u \in L^0(\Sigma) : u \ast L^2(\Sigma) \subseteq L^2(\Sigma) \}.$$

Theorem 2.5. Suppose $u \in L^0(\Sigma)$. Then $u \in K_2^*$ if and only if $E(|u|^2) \in L^\infty(A)$.

Proof. See [1].

Note $K_2^*$ is a vector subspace of $L^0(\Sigma)$.

Definition 2.6. Suppose $u \in L^0(\Sigma)$. Then we define

$$\mathcal{R} = \{ \lambda I + T_u : u \in K_2^* \text{ and } \lambda \in \mathbb{C} \}.$$  

Hereafter we shall denote $\lambda I + T_u$ ($I$ is identity operator) simply as $\lambda + T_u$.

Proposition 2.7. $\mathcal{R}$ is closed under addition and $T_{u+v} = T_u + T_v$, for every $u, v \in K_2^*$.

Proof. Let $\lambda + T_u$ and $\gamma + T_v$ be in $\mathcal{R}$. Then, for any $f \in L^2(\Sigma),

$$[(\lambda + T_u) + (\gamma + T_v)]f = (\lambda + T_u)f + (\gamma + T_v)f = (\lambda f + T_u f) + (\gamma f + T_v f) = \lambda f + \left( uE(f) + fE(u) - E(u)E(f) \right) + \gamma f + \left( vE(f) + fE(v) - E(v)E(f) \right) = \lambda f + \gamma f + (u + v)E(f) + fE(u + v) - E((u + v)E(f)).$$

Then $(\lambda + \gamma)f + T_{u+v}f = (\delta + T_y)f$ where $\delta = \lambda + \gamma$ and $y = u + v$. Since $|u + v|^2 \leq 2 \left( |u|^2 + |v|^2 \right)$ and $E$ is a linear operator thus $E \left( |u + v|^2 \right) \leq 2 \left( E(|u|^2) + E(|v|^2) \right)$. It follows that $y \in K_2^*$. Therefore, $\mathcal{R}$ is closed under addition and $T_{u+v} = T_u + T_v$.  

Proposition 2.8. a. $\mathcal{R}$ is closed under multiplication operators and $T_{\lambda u} = \lambda T_u$, for any $u \in K_2^*$ and $\lambda \in \mathbb{C}$.

b. $\mathcal{R}$ is commutative under composition operators and $T_u T_v = T_v T_u$ for any $u, v \in K_2^*$.

Proof. a. Suppose $f \in L^2(\Sigma)$ and $\lambda \in \mathbb{C}$, thus we have

$$(\lambda T_u)f = \lambda T_u f = \lambda u E(u) + fE(u) - E(u)E(f) = (\lambda u)E(f) + fE(\lambda u) - E(\lambda u)E(f) = T_{\lambda u} f,$$

b. Suppose $f, g \in L^2(\Sigma)$ and $u \in K_2^*$, then we have

$$(u \ast f) \ast g = u \ast (f \ast g) = u \ast (fE(g) + gE(f) - E(f)E(g)) = u \ast (u \ast (f \ast g)) = (u \ast (f \ast g)) \ast u.$$
it follows $\lambda T_u = T_{\lambda u}$. Otherwise since
\[
E\left(|\lambda u|^2\right) = E\left(|\lambda|^2 |u|^2\right) = |\lambda|^2 E(|u|^2),
\]
this equality show that if $E(|u|^2) \in L^\infty(A)$, then $E\left(|\lambda u|^2\right) \in L^\infty(A)$. Thus if $T_u \in R$, then $T_{\lambda u} = \lambda T_u$ for any $\lambda \in \mathbb{C}$.

b. First we show $R$ is commutative. Suppose $T_u$ and $T_v$ be in $R$. Thus
\[
T_u T_v f = T_u \left(v E(f) + f E(v) - E(v) E(f)\right)
= v E(u) E(f) + f E\left(v E(u)\right) - E\left(v E(u)\right) E(f) + u E(v) E(f) - E\left(u E(v)\right) E(f).
\]
Similarly we have
\[
T_v T_u f = T_v \left(u E(f) + f E(u) - E(u) E(f)\right)
= v E(u) E(f) + f E\left(v E(u)\right) - E\left(v E(u)\right) E(f) + u E(v) E(f) - E\left(u E(v)\right) E(f).
\]
The above equality shows $T_u T_v = T_v T_u$. Therefore, $R$ is commutative.

Now, we show $T_u T_v \in R$. Note that
\[
T_u T_v f = T_{(u Ev + v Eu)} f - T_{E Ev u} f = T_{(u Ev + v Eu)} f - T_{E (u Ev)} f.
\]
also
\[
E\left(|u Ev + v Eu|^2\right) \leq 2 \left(E|u Ev|^2 + E|v Eu|^2\right) \\
\leq 2 \left(E|u|^2 E|v|^2 + E|v|^2 E|u|^2\right) \\
= 4E|v|^2 E|u|^2,
\]
and
\[
E|Ev Eu|^2 = E\left(|Ev|^2 |Eu|^2\right) \leq E \left(E|v|^2 E|u|^2\right) = E|v|^2 E|u|^2,
\]
which implies $T_v T_u$ be in $R$, since by $E|u|^2$ and $E|v|^2$ are bounded operators. In general, for every $\gamma, \lambda \in \mathbb{C}$ we have
\[
(\lambda + T_v)(\gamma + T_u) = \lambda \gamma + \lambda T_v + \gamma T_u + T_v T_u.
\]
Thus by the above equalities (b) hold.

**Theorem 2.9.** $R$ is a commutative operator algebra.

**Proof.** By Propositions [2.7] and [2.8], it is trivial.

**References**


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Characterization of unitary and self-adjoint operators using elementary operators

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Abstract
In this paper, we want to study the elementary operators of the form $SXS^{-1} + S^{-1}XS$. Using some inequalities on this operator, we can characterize some of the important subspaces of $\mathcal{B}(H)$.

Keywords: elementary operators, self-adjoint operators, normal operators

Mathematics Subject Classification: 47B47, 47B15

1 Introduction
Let $H$ be a Hilbert space and $B(H)$ be the set of all bounded linear operators on $H$. An operator $R : B(H) \to B(H)$ is called an elementary operator if it can be written of the form

$$R(X) = \sum_{i=1}^{n} A_i X B_i$$

for some $A_i, B_i \in B(H)$. One can introduce some useful classes of elementary operators as follows:

- inner derivation $\delta_A : X \mapsto AX -XA$,
- generalized derivation $\delta_{A,B} : X \mapsto AX -XB$,
- two-sided multiplication $M_{A,B} : X \mapsto AXB$,
- symmetric two-sided multiplication $U_{A,B} : X \mapsto AXB + BXA$.

There is an important relation between the norm of elementary operators and the injective norm defined on tensor product of operators.

Definition 1.1. Let $X$ and $Y$ be two normed space with dual space $X'$ and $Y'$. For $x \in X$ and $y \in Y$, we can define $x \otimes y \in \mathbb{B}(X', Y'; \mathbb{C})$ by $x \otimes y(f, g) = f(x)g(y)$ and the injective norm is defined as

$$\|\sum_{i=1}^{n} x_i \otimes y_i\|_{\lambda} = \sup_{f \in X'_i, g \in Y'_i} |\sum_{i=1}^{n} f(x_i)g(y_i)|,$$

where $X'_i$ and $Y'_i$ are the unit ball of the dual space of $X$ and $Y$, respectively.

Theorem 1.2. For operators $A_1, \ldots, A_n, B_1, \ldots, B_n \in B(H)$ we have

$$\|\sum_{i=1}^{n} x_i \otimes y_i\|_{\lambda} = \sup\{\|R_{A,B}(X)\| : \|X\| = 1 = \text{rank}(X)\}.$$
2 Main Result

In 1990, Corach-Porta and Recht [1] proved that if $S$ is an invertible self-adjoint operator, then

$$\forall X \in \mathcal{B}(\mathcal{H}) \quad ||SXS^{-1} + S^{-1}XS|| \geq 2||X||.$$  

This inequality is a key factor in their study on differential geometry. Note that in this theorem, the self-adjointness of $S$ is not the necessary condition. In what follows, we state the necessary and sufficient conditions for which this inequality and some other similar inequalities hold true. These results can be found in [2,3,4].

**Theorem 2.1.** Let $S$ be an invertible operator on a Hilbert space $\mathcal{H}$. Then $S$ is a nonzero complex coefficient of a self-adjoint operator, if and only if one of the following relations holds.

$$\forall X \in \mathcal{B}(\mathcal{H}), \quad ||SXS^{-1} + S^{-1}XS|| \geq 2||X||$$

**Theorem 2.2.** For an invertible operator $S$, the following conditions are equivalent.

- $S$ is normal,
- $\forall X \in \mathcal{B}(\mathcal{H}), \quad ||SXS^{-1}|| + ||S^{-1}XS|| \geq 2||X||$,
- $\forall X \in \mathcal{B}(\mathcal{H}), \quad ||SXS^{-1}|| + ||S^{-1}XS|| = ||S^{-1}XS^*|| + ||S^*S^{-1}||$.

$I(\mathcal{H})$ is set of all invertible operators.

**Theorem 2.3.** Let $S$ be an invertible operator. The following equalities and inequality are true only when $S$ is a nonzero real coefficient of a unitary operator.

$$\forall X \in \mathcal{B}(\mathcal{H}), \quad ||S^*XS^{-1} + S^{-1}XS^*|| = 2||X||$$

$$\forall X \in \mathcal{B}(\mathcal{H}), \quad ||S^*XS^{-1}|| + ||S^{-1}XS^*|| = 2||X||$$

$$\forall X \in \mathcal{B}(\mathcal{H}), \quad ||S^* \otimes S^{-1} + S^{-1} \otimes S^*|| = 2||X||$$

$$\forall X \in \mathcal{B}(\mathcal{H}), \quad ||SXS^{-1}|| + ||S^{-1}XS|| \leq 2||X||$$

If $S$ is an invertible operator with $||S|| = 1$ then the above relations be equivalent to the fact that $S$ is unitary.

The authors could proved a modification of the above theorems for closed range operators instead of invertible operators.

**References**


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An operator extension of Csizár’s result

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Abstract
We introduce the non-commutative $f$-divergence functional and give its properties. More precisely, an operator extension of Csizár’s result regarding $f$-divergence functional is presented. As some applications, we establish a refinement of the Choi–Davis–Jensen operator inequality.

Keywords: continuous field of operators, operator Jensen inequality, $f$-divergence functional, perspective function, (jointly) operator convex.

Mathematics Subject Classification: 47A63, 46L05, 26D15

1 Introduction
Let $B(ℋ)$ be the algebra of all bounded linear operators on a complex Hilbert space $ℋ$ and $I$ denote the identity operator. An operator $A$ is said to be positive (denoted by $A \geq 0$) if $⟨Ax,x⟩ \geq 0$ for all vectors $x ∈ ℋ$. If, in addition, $A$ is invertible, then it is called strictly positive (denoted by $A > 0$). By $A \geq B$ we mean that $A - B$ is positive, while $A > B$ means that $A - B$ is strictly positive. A map $Φ$ on $B(ℋ)$ is called positive if $Φ(A) ≥ 0$ for each $A ≥ 0$.

A continuous real valued function $f$ defined on an interval $J$ is said to be operator convex if

$$f(λA + (1 − λ)B) ≤ λf(A) + (1 − λ)f(B),$$

for all self-adjoint operators $A, B$ with spectra contained in $J$ and any $λ ∈ [0, 1]$. If $−f$ is operator convex, then $f$ is said to be operator concave. Let $J_1$ and $J_2$ be two real intervals. A jointly operator convex function is a function $f$ defined on $J_1 × J_2$ such that

$$f(λ(A, B) + (1 − λ)(C, D)) ≤ λf(A, B) + (1 − λ)f(C, D),$$

for all self-adjoint operators $A, C$ with spectra contained in $J_1$, all self-adjoint operators $B, D$ with spectra contained in $J_2$ and all $λ ∈ [0, 1]$. The Choi–Davis–Jensen inequality states that if $f$ is operator convex, then

$$f(Φ(A)) ≤ Φ(f(A)),$$

for any unital positive linear map $Φ$ and any self-adjoint operator $A$, whose spectrum is contained in the domain of $f$. See e.g. [4].

Let $f$ be a convex function on a convex set $K ⊆ ℝ$. The perspective function $g$ associated to $f$ is defined on the set $\{(x, y) : y > 0 \text{ and } \frac{x}{y} \in K\}$ by $g(x, y) := yf\left(\frac{x}{y}\right)$. As an operator extension of the perspective function, Effros [3] introduced the perspective function of an operator convex function $f$ by $g(L, R) := Rf\left(\frac{L}{R}\right)$ for commuting strictly positive operators $L$ and $R$ and proved that $f$ is operator convex if and only if $g$ is jointly operator convex. The authors of [2] extended Effros results by removing the restriction to commuting operators.
For a convex function \( f : [0, \infty) \to \mathbb{R} \), Csiszár [1] introduced the \( f \)-divergence functional by

\[
I_f(\tilde{p}, \tilde{q}) := \sum_{i=1}^{n} q_i f \left( \frac{p_i}{q_i} \right),
\]

for positive \( n \)-tuples \( \tilde{p} = (p_1, \cdots, p_n) \) and \( \tilde{q} = (q_1, \cdots, q_n) \). Also Csiszár and Körner [1] showed that if \( f : [0, \infty) \to \mathbb{R} \) is convex, then \( I_f(\tilde{p}, \tilde{q}) \) is jointly convex in \( \tilde{p} \) and \( \tilde{q} \)

\[
\sum_{i=1}^{n} q_i f \left( \frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} q_i} \right) \leq I_f(\tilde{p}, \tilde{q})
\]

(1)

for all positive \( n \)-tuples \( \tilde{p} = (p_1, \cdots, p_n), \tilde{q} = (q_1, \cdots, q_n) \). A series of results and related inequalities can be found in [1, 5].

In this note, we introduce an operator extension of \( f \)-divergence functional and give some of its properties. We establish its relationship to the perspective of \( f \). In particular, an operator extension of (1) is presented. As an application, we give a refinement of the Choi–Davis–Jensen operator inequality.

## 2 Main Result

Throughout this section, assume that \( T \) is a locally compact Hausdorff space with a bounded Radon measure \( \mu \) and \( \mathcal{A} \) and \( \mathcal{B} \) are \( C^* \)-algebras of Hilbert space operators. Assume that \( \tilde{A} = (A_t)_{t \in T} \) and \( \tilde{B} = (B_t)_{t \in T} \) are continuous fields of self-adjoint and strictly positive operators in \( \mathcal{A} \), respectively, and \( (\Phi_t)_{t \in T} : \mathcal{A} \to \mathcal{B} \) is a unital field of positive linear maps. Let \( f \) be an operator convex function. Let us define the non-commutative \( f \)-divergence functional \( \Theta \) by

\[
\Theta(\tilde{A}, \tilde{B}) := \int_{T} B_t \frac{2}{f} J \left( B_t^{-\frac{2}{f}} A_t B_t^{-\frac{1}{f}} \right) B_t^\frac{1}{f} d\mu(t).
\]

It is easy to see that \( \Theta \) is jointly operator convex if and only if \( f \) is operator convex. The following result is an extension of (1).

**Theorem 2.1.** Let \( f \) be an operator convex function, and \( g \) be the corresponding perspective function. Then

\[
g(A, B) \leq \Theta(\tilde{A}, \tilde{B}),
\]

(2)

where \( A = \int_{T} A_t d\mu(t) \) and \( B = \int_{T} B_t d\mu(t) \).

**Corollary 2.2.** Let \( f \) be an operator convex function and \( g \) be the perspective function of \( f \). Then

(i) The perspective function \( g \) of an operator convex function \( f \) is sub-additive. More general,

\[
g \left( \sum_{i=1}^{n} L_i, \sum_{i=1}^{n} R_i \right) \leq \sum_{i=1}^{n} g(L_i, R_i).
\]

(3)

(ii) \( f \left( \sum_{i=1}^{n} L_i \right) \leq \sum_{i=1}^{n} g(L_i, R_i) \), whenever \( \sum_{i=1}^{n} R_i = I \).

For continuous functions \( f \) and \( h \) and commuting matrices \( L \) and \( R \), Effros [3] defined the function \( (L, R) \mapsto (f \Delta h)(L, R) \) by

\[
(f \Delta h)(L, R) := f \left( \frac{L}{h(R)} \right) h(R).
\]

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He also proved that if \( f \) is operator convex with \( f(0) \leq 0 \) and \( h \) is operator concave with \( h > 0 \), then \( f \Delta h \) is jointly operator convex.

Let \( f \) and \( h \) be continuous real valued functions defined on an interval \( J \) and \( \mu \) be a probability measure on \( T \). As a generalization of \( f \Delta h \), we define \( f \nabla h \) by

\[
(f \nabla h)(\bar{A}, \bar{B}) := \int_T h(B_t)^{\frac{1}{2}} f \left( h(B_t)^{-\frac{1}{2}} A_t h(B_t)^{-\frac{1}{2}} \right) h(B_t)^{\frac{1}{2}} d\mu(t).
\]

It is not hard to see that \( f \) is operator convex with \( f(0) \leq 0 \) and \( h \) is operator concave with \( h > 0 \) if and only if \( f \nabla h \) is jointly operator convex. The next result, is a Choi–Davis–Jensen type inequality for \( f \Delta h \).

**Theorem 2.3.** Let \( f \) be an operator convex function with \( f(0) \leq 0 \) and \( h \) be an operator concave function with \( h > 0 \). If \( \int_T \Phi_t(I) d\mu(t) \leq I \), then

\[
(f \Delta h) \left( \int_T \Phi_t(A_t) d\mu(t), \int_T \Phi_t(B_t) d\mu(t) \right) \leq \int_T \Phi_t((f \Delta h)(A_t, B_t)) d\mu(t),
\]

where \( B_t \) is strictly positive for any \( t \in T \).

**Theorem 2.4.** Let \( f_1 \) and \( f_2 \) be operator convex functions with \( f_1(0) \leq 0 \) and \( f_2(0) \leq 0 \) and let \( h \) be an operator concave function with \( h > 0 \). The following assertions are equivalent:

1. \( f_1 \leq f_2 \);

2. \( (f_1 \Delta h) \left( \int_T \Phi_t(A_t) d\mu(t), \int_T \Phi_t(B_t) d\mu(t) \right) \leq \int_T \Phi_t((f_2 \Delta h)(A_t, B_t)) d\mu(t) \) for all unital fields \( (\Phi_t)_{t \in T} \) and all continuous fields of operators \( (A_t)_{t \in T} \) and \( (B_t)_{t \in T} \);

3. \( f_1 \left( \int_T \Phi_t(A_t) d\mu(t) \right) \leq \int_T \Phi_t(f_2(A_t)) d\mu(t) \) for all continuous fields of operators \( (A_t)_{t \in T} \).

The next result, gives the relation between two functions \( f \Delta h \) and \( f \nabla h \).

**Theorem 2.5.** Let \( f \) be an operator convex function with \( f(0) \leq 0 \) and \( h \) be an operator concave function with \( h > 0 \). If \( \mu \) is a probability measure on \( T \), then

\[
(f \Delta h)(A, B) \leq (f \nabla h)(\bar{A}, \bar{B}),
\]

where \( A = \int_T A_t d\mu(t) \) and \( B = \int_T B_t d\mu(t) \).

As an application, we give the following refinement of the Choi–Davis–Jensen inequality.

**Theorem 2.6.** Let \( T_1 \) and \( T_2 \) be disjoint locally compact Hausdorff spaces, \( T_1 \cup T_2 = T \) and \( \mu \) be a bounded Radon measure on \( T \). Let \( f \) be an operator convex function, \( (A_t)_{t \in T} \) be a continuous field of self-adjoint operators in \( \mathcal{A} \), \( (\Phi_t)_{t \in T} : \mathcal{A} \to \mathcal{B} \) be a unital field of positive linear maps, \( D_{T_1} = \int_{T_1} \Phi_t(I) d\mu(t) \) and \( D_{T_2} = \int_{T_2} \Phi_t(I) d\mu(t) \). Then

\[
f \left( \int_T \Phi_t(A_t) d\mu(t) \right)
\leq D_{T_1}^{\frac{1}{2}} f \left( D_{T_1}^{-\frac{1}{2}} \int_{T_1} \Phi_t(A_t) d\mu(t) D_{T_1}^{-\frac{1}{2}} \right) D_{T_1}^{\frac{1}{2}} + D_{T_2}^{\frac{1}{2}} f \left( D_{T_2}^{-\frac{1}{2}} \int_{T_2} \Phi_t(A_t) d\mu(t) D_{T_2}^{-\frac{1}{2}} \right) D_{T_2}^{\frac{1}{2}}
\leq \int_T \Phi_t(I)^{\frac{1}{2}} f \left( \Phi_t(I)^{-\frac{1}{2}} \Phi_t(A_t) \Phi_t(I)^{-\frac{1}{2}} \right) \Phi_t(I)^{\frac{1}{2}} d\mu(t)
\leq \int_T \Phi_t(f(A_t)) d\mu(t).
\]
References


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The existence of non-zero weakly compact left multipliers on ideals of \( L^\infty(G)^* \)

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Abstract

Let \( I \) be an ideal in \( L^\infty(G)^* \). In this paper, we find conditions under which the existence of a non-zero weakly compact left multiplier on \( I \) is equivalent to compactness of \( G \).

Keywords: Locally compact group, multiplier, weakly compact operator, ideal.

Mathematics Subject Classification: 43A15, 43A20, 47B07, 47B48, 47B65.

1 Introduction

Throughout this paper, \( G \) denotes a locally compact group with a fixed left Haar measure \( \lambda \). Let \( L^1(G) \) be the group algebra of \( G \) defined as in [5] equipped with the convolution product \( \ast \) and norm \( \| \cdot \|_1 \). Let \( L^\infty(G) \) be the usual Lebesgue space as defined in [5] equipped with the essential supremum norm \( \| \cdot \|_\infty \). Then \( L^\infty(G) \) is the dual of \( L^1(G) \). We recall that the first dual \( L^\infty(G)^* \) is a Banach algebra with the first Arens product \( " \cdot " \) defined by

\[
(F \cdot H, f) = (F, Hf) \quad \text{where} \quad \langle Hf, \phi \rangle = \langle H, f\phi \rangle \quad \text{and} \quad \langle f\phi, \psi \rangle = \langle f, \phi^* \psi \rangle
\]

for all \( F, H \in L^\infty(G)^*, f \in L^\infty(G) \) and \( \phi, \psi \in L^1(G) \).

Let \( A \) be a Banach algebra; a bounded operator \( T : A \rightarrow A \) is called a left multiplier if \( T(ab) = T(a)b \) for all \( a, b \in A \). For any \( a \in A \), the left multiplier \( b \mapsto ab \) on \( A \) is denoted by \( \lambda_a \); also \( a \) is said to be a left weakly completely continuous element of \( A \) if \( \lambda_a \) is a weakly compact operator on \( A \). We denote by \( L_{wcc}(A) \) the set of all left weakly completely continuous elements of \( A \).

Akemann [1] and Sakai [8] have studied weakly compact left multipliers on the Banach algebra \( L^1(G) \). Akemann [1] has shown that if \( G \) is compact, then any \( \phi \in L^1(G) \) is a left weakly completely continuous element of \( L^1(G) \). He also has characterized weakly compact left multiplier on \( L^1(G) \). In fact, he has shown that any weakly compact left multiplier on \( L^1(G) \) is of the form \( \lambda_\phi \) for some \( \phi \in L^1(G) \). Sakai [8] has shown that if \( G \) is a locally compact non-compact group, then zero is the only left weakly completely continuous element of \( L^1(G) \). We apply left multipliers on the Banach algebra \( L^\infty(G)^* \) of a locally compact group \( G \) have been studied by Ghahramani and Lau in [3, 4]. In the same papers, they have obtained some results on the question of existence of non-zero weakly compact left multipliers on \( L^\infty(G)^* \). Losert [6], among other things, has proved that if \( G \) is non-compact, then there is no non-zero weakly compact left multipliers on \( L^\infty(G)^* \); see also [7].

In this paper, we find conditions under which the existence of a non-zero weakly compact left multiplier on ideal \( I \) of \( L^\infty(G)^* \) is equivalent to compactness of \( G \).
2 Main Results

We commence this section with the following result which is needed in the sequel.

Proposition 2.1. Let $G$ be a locally compact group and let $X$ be a closed subalgebra of $L^\infty(G)^\ast$. If $I$ is a closed right ideal in $L^\infty(G)^\ast$ with a bounded approximate identity, then

$$L_{wec}(I) \cap X \subseteq L_{wec}(X) \cap I.$$ 

Corollary 2.2. Let $G$ be a locally compact group and $I, J$ be closed right ideals in $L^\infty(G)^\ast$. If $I$ and $J$ have bounded approximate identities, then the following statements hold.

(i) $L_{wec}(I) \cap J = L_{wec}(J) \cap I$.
(ii) $L_{wec}(I) = L_{wec}(L^\infty(G)^\ast) \cap I$.

Let us recall that an element $u \in L^\infty(G)^\ast$ is called a mixed identity if $\phi \cdot u = u \cdot \phi = \phi$ for all $\phi \in L^1(G)$. Denote by $\Lambda(G)$ the set of all mixed identities $u$ with norm one in $L^\infty(G)^\ast$ and note that $u \in L^\infty(G)^\ast$ is a mixed identity if and only if it is a weak$^\ast$-cluster point of an approximate identity in $L^1(G)$ bounded by one; or equivalently, a right identity for $L^\infty(G)^\ast$ with norm one; see [2].

In the following, let $M_{wcl}(I)$ be the set of all weakly compact left multipliers on a right ideal $I$ of $L^\infty(G)^\ast$. The next result can be considered as a more general statement of Theorem 1 of Sakai [8] and Theorem 4 of Akeman [1].

Theorem 2.3. Let $G$ be a locally compact group and $I$ be a closed right ideal in $L^\infty(G)^\ast$ such that $I$ has a bounded approximate identity or $I$ is a subspace of $u \cdot L^\infty(G)^\ast$ for some $u \in \Lambda(L^\infty(G)^\ast)$. Then $G$ is compact if and only if $M_{wcl}(I) \neq \{0\}$.

Let $I$ be a subset of $L^\infty(G)^\ast$. The left annihilator of $I$ is denoted by $\text{lan}(I)$ and is defined by $\text{lan}(I) = \{ l \in L^\infty(G)^\ast : l \cdot I = \{0\} \}$.

Theorem 2.4. Let $G$ be a locally compact group and $I$ be a closed ideal in $L^\infty(G)^\ast$ such that $\text{lan}(I) = \{0\}$. Then $G$ is compact if and only if $M_{wcl}(I) \neq \{0\}$.

$\text{ran}(L^\infty(G)^\ast)$ will be understood the set of all $r \in L^\infty(G)^\ast$ such that

$$L^\infty(G)^\ast \cdot r = \{0\}.$$ 

We finish the paper with the following result.

Proposition 2.5. Let $G$ be a locally compact group and let $I$ be a closed right ideal in $L^\infty(G)^\ast$ such that $I \cap \text{ran}(L^\infty(G)^\ast) = \{0\}$. If $G$ is compact, then $M_{wcl}(I) \neq \{0\}$.

References


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(\phi, \psi)-Derivations mapping into the primitive ideals

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Abstract
The question of a (\phi, \psi)-derivation on a Banach algebra has quasinilpotent value, and we also establish a nice generalized Leibniz formula and extend the Kleineck-Sirokov theorem for (\phi, \psi)-derivations under certain conditions.

Keywords: (\phi, \psi)-derivation

Mathematics Subject Classification: 47B47.

1 Introduction
Throughout the paper \( A \) denotes a Banach algebra, \( D \) denotes a subalgebra of \( A \).

Definition 1.1. \( d \) is called a (\phi, \psi)-derivation if \( d(ab) = d(a)\phi(b) + \psi(a)d(b) \) for all \( a, b \in D \).

Example 1.2. Let \( \phi \) and \( \psi \) be an arbitrary homomorphisms on \( D \) and suppose that \( u \) is an element of \( A \). Then the mapping \( d : D \to A \) defined by \( d(a) = \psi(a)u - u\phi(a) \) is a (\phi, \psi)-derivation.

The above (\phi, \psi)-derivation is called inner (\phi, \psi)-derivation and we denote by \( \psi \phi d u \).

In this section we gather together a few useful elementary observations on (\phi, \psi)-derivations. We omit the proof the first result it is just a straightforward verification.

Theorem 1.3. Let \( A \) be an algebra, let \( \phi \) and \( \psi \) be an automorphism of \( A \), and \( d \) be a (\phi, \psi)-derivation on \( A \). Set

\[
\begin{align*}
\Delta_{n,0} &= \phi^n, \\
\Lambda_{n,0} &= \psi^n, \\
\Delta_{n,n} &= d^n, \quad \text{and} \quad \Delta_{n+1,k} = \Delta_{n,k}\phi + \Delta_{n,k-1}d \quad \text{if} \quad k = 1, \ldots, n; \\
\Lambda_{n,n} &= d^n, \quad \text{and} \quad \Lambda_{k,n+1} = \Lambda_{k,n}\psi + \Lambda_{k-1,n}d \quad \text{if} \quad k = 1, \ldots, n,
\end{align*}
\]

and if \( a, b \in D \)

\[
\begin{align*}
\Lambda_{k,n+1}(a) \ast \Delta_{n+1,n+1-k}(b) &= (\Lambda_{k,n}\psi + \Lambda_{k-1,n}d)(a) \ast (\Delta_{n,n+1-k}\phi + \Delta_{n,n-k}d)(b) \\
&= \Lambda_{k,n}\psi(a) \ast \Delta_{n,n-k}d(b) + \Lambda_{k-1,n}d(a) \ast \Delta_{n,n+1-k}\phi(b)
\end{align*}
\]

for each \( n \in \mathbb{N} \) and \( k = 1, \ldots, n \). If \( a, b \in D \) and \( n \in \mathbb{N} \), then

\[
d^n(ab) = \sum_{j=0}^{n} \Lambda_{j,n}(a) \ast \Delta_{n,n-j}(b). \quad \text{(Leibnitz rule extension)} \quad (1)
\]

Accordingly,

\[
\begin{align*}
d^n(ab) &= \sum_{j=0}^{n} \binom{n}{j} d^{n-j}(\psi^j(a)) d^j(\phi^{n-j}(b)) \\
&= \sum_{j=0}^{n} \binom{n}{j} \psi^j(d^{n-j}(a)) \phi^{n-j}(d^j(b)) \quad (2)
\end{align*}
\]

provided that \([d, \phi] = [d, \psi] = 0\).
**Theorem 1.4.** Let \( A \) be an algebra, let \( \phi \) be an automorphism of \( A \), and \( d \) be a \( \phi \)-derivation on \( A \) such that \([d, \phi] = [d, \psi] = [\phi, \psi] = 0\). Suppose that \( a \in D \) is such that \( d^2(a) = 0 \). Then

\[
d^n \left( \prod_{j=1}^n \psi^{-(n-j)}(a) (a) \right) = n! (d(a))^n
\]

**Proof.** To shorten notation, let \( c_n \) stand for

\[
\prod_{j=1}^n \psi^{-n-j}(a) = \psi^{-n-1}(a) \psi^{-n-2}(a) \cdots \psi^{-n-1}(a)
\]

and note that \( c_{n+1} = \psi^{-n}(a) \phi^{-1}(c_n) \). If \( d^n(c_n) = n! d(a)^n \), then it easy seen that \( d^{n+1}(c_n) = n! d(d(a))^n \) and so, using the Theorem 1.3, we get

\[
d^{n+1}(c_{n+1}) = d^{n+1}(\psi^{-n}(a) \phi^{-1}(c_n))
\]

\[
= \sum_{j=0}^{n+1} \binom{n+1}{j} d^{n+1-j}(\psi^j(\psi^{-n}(a))) d^j(\phi^{n+1-j}(\phi^{-1}(c_n)))
\]

\[
= (n + 1) d(\psi^n(\psi^{-n}(a))) d^n(\phi(\phi^{-1}(c_n))) + \psi^{n+1}(\psi^{-n}(a)) d^{n+1}(\phi^{-1}(c_n))
\]

\[
= (n + 1) d(a) n! d(a) ^n
\]

\[\square\]

**Theorem 1.5.** Let \( A \) be an algebra, let \( \phi \) and \( \psi \) be an automorphism of \( A \), and \( d \) be a \( \phi, \psi \)-derivation on \( A \). Let \( I \) be two-side ideal of \( A \) and suppose \( \phi, \psi, [d, \phi] \) and \([d, \psi] \) leave \( I \) invariant then

\[
d^n(\psi^{-n+1}(a_1) \psi^{-n+2}(a_2) \cdots \psi^{-n}(a_{n-1}) a_n) \in n! d(a_1) d(\phi(a_2)) \cdots d(\phi^{n-1}(a_n)) + I
\]

\[= n! d(a_1) \phi(a_2) \cdots \phi^{n-1}(a_n) + I \quad \text{(3)}
\]

for all \( n \in \mathbb{N} \) and \( a_1, a_2, \ldots, a_n \in I \).

**Remark 1.6.** Note that, in the case when \( \phi \) and \( \psi \) are an inner automorphism, \([d, \phi] \) and \([d, \psi] \) leaves each ideal \( I \) invariant. Trivially the same holds true for \( \phi \) and \( \psi \), and so the requirements of Theorem 1.5 are automatically satisfied.

2. continuous \((\phi, \psi)\)-derivations

**Theorem 2.1.** Let \( A \) be a Banach algebra, let \( \phi \) be a continuous surjective homomorphism of \( A \), and \( d \) be a continuous \((\phi, \psi)\)-derivation on \( A \) such that \([d, \phi] = [d, \psi] = [\phi, \psi] = 0\), suppose that \( a \in A \) is such that \( d^2(a) = 0 \), then \( d(a) \) is quasinilpotent.

**Proof.** Write \( c_1 = a \) and \( c_{n+1} = \psi^{-n}(a) \phi^{-1}(c_n) = \psi^{-1}(c_n) \phi^{-n}(a) \), then

\[
c_n = \prod_{j=1}^n \psi^{-n-j}(a) \phi^{-j-1}(a)
\]

for each \( n \in \mathbb{N} \). By induction it is easily seen that \( \|e_{2^n}\| \leq \|\psi^{-1}\|^{2^n-1} \|\phi^{-1}\|^{2^n-1} \|a\|^{2^n} \). This fact together with Theorem 1.3 given

\[
\|d(a)^{2^n}\| \leq \|(2^n!)^{-1} d^2(e_{2^n})\| \leq \|(2^n!)^{-1} d\| \|e_{2^n}\| \|a\|^{2^n}
\]

\[
\leq (2^n!)^{-1} \|d\| \|\psi^{-1}\|^{1-1/2^n} \|\phi^{-1}\|^{1-1/2^n} \|a\|
\]

for each \( n \in \mathbb{N} \) and this latter term tends to 0 as \( n \to \infty \). \[\square\]
Remark 2.2. It should be noted that in the case when $[d, \phi] \neq 0$ or $[d, \psi] \neq 0$, Theorem 2.1 fails to hold true. Indeed, set $A = M_2(\mathbb{C})$,

$$a = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and let $d$ be the $(\phi, \psi)$-derivation on $A$ defined by $d(x) = c^{-1}[x, e]b$ for each $x \in A$. Then $d(a) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not quasinilpotent; however, $d^2(a) = 0$.

For an ideal $I$ of an algebra $A$, we shall write $\pi_I$ for the quotient map from $A$ onto $A/I$.

Theorem 2.3. Let $A$ be a unital Banach algebra, let $\phi$ and $\psi$ be a continuous automorphism of $A$, and $d$ be a continuous $(\phi, \psi)$-derivation on $A$. Let $I$ be a closed two-sided ideal of $A$, and suppose that $\phi$ and $\psi$ leave $I$ invariant. Then $\pi_I(d(a))$ is quasinilpotent for each $a \in I$.

Proof. Define $e_n$ as in the proof of Theorem 2.1. Due to the Theorem 1.5, we have

$$\pi_I(d^n(e_n)) = n!(\pi_I(d(a)))^n$$

and so

$$\|\pi_I(d(a))\|^{1/2} \leq (2n!)^{-1/2} \|d\| \|c_{2^n}\|^{1/2} \leq (2n!)^{-1/2} \|d\| \|\phi^{-1}\|^{1/2} \|\psi^{-1}\|^{1/2} \|a\|$$

for each $n \in \mathbb{N}$. Since the latter term tends to 0 as $n \to \infty$, this yields the result.

Corollary 2.4. Let $A$ be a unital Banach algebra, let $\phi$ and $\psi$ be an inner automorphism of $A$, and let $d$ be a continuous $(\phi, \psi)$-derivation on $A$. Then $d$ leaves each primitive ideal of $A$ invariant.

Proof. Let $P$ be primitive ideal of $A$. According to the Theorem 2.3 together with Remark 1.6, $\pi_P(d(P))$ consists of quasinilpotent elements. On the other hand, it easily seen that $\pi_P(d(P))$ is two-sided ideal of $A/P$ and thus $\pi_P(d(P)) \subset \text{Rad}(A/P) = 0$. That is, $d(P) \subset P$.

Corollary 2.5. Let $A$ be a unital Banach algebra, let $\phi$ and $\psi$ be a continuous automorphisms of $A$, and let $d$ be a continuous $(\phi, \psi)$-derivation on $A$. Let $I$ be a two-sided ideal of $A$ such that $\phi$, $\psi$, $[d, \phi]$ and $[d, \psi]$ leave $I$ invariant. If $d(a) = 1$ for some $a \in I$, then $I = A$.

Theorem 2.6. Let $A$ be a unital Banach algebra, let $\phi$ and $\psi$ be a continuous automorphism of $A$, and $d$ be a continuous $(\phi, \psi)$-derivation on $A$. Let $P$ be a primitive ideal of $A$, if $\phi$ and $\psi$ leave $P$ invariant then $d$ is.

Proof. By the Thorem 2.3 $d(a)P + P \subset \Omega(A/P)$ for each $a \in P$. Consequently, the ideal $(d(P) + P)/P$ is contained in $\Omega(A/P)$, hence $\text{rad}(A/P)$, and $dP \subset P$ follows.

References


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Some results on $\langle \cdot \rangle_h$-approximate fixed points in $\langle \cdot \rangle$ for two maps

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Abstract

In this paper, we consider pre Hilbert non-expansive maps. Also approximate fixed point theorems for a class of non-expansive maps defined on a pre-Hilbert spaces.

Keywords: Pre-Hilbert space, non-expansive mapping, $\langle \cdot \rangle_h$-approximate fixed point.

Mathematics Subject Classification: 37C25, 47H10

1 Introduction

The maps on pre-Hilbert spaces theory plays a major role in mathematics (geometry, topology, analysis, ... ) and applied sciences, such that optimization, economic theories [5]. There exists papers on approximate fixed point for various contractive conditions whose comprehensive survey can be found in Berinde [1]. Recently, Nigam et al. [3] proved fixed point theorem in Hilbert space using the parallelogram identity. we know that pre-Hilbert spaces was introduced by Cuenca Mira and Jos Antonio [2].

Definition 1.1. Let $V$ be a complex vector space. A complex-valued function

$\langle \cdot \rangle : V \times V \to \mathbb{C}$

of two variables on $V$ is an inner product if satisfying the following properties:

$(P_{h1}) \quad \langle x, y \rangle = \langle y, x \rangle$

$(P_{h2}) \quad \langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle,$

$(P_{h3}) \quad \langle x, y + y' \rangle = \langle x, y \rangle + \langle x, y' \rangle$

$(P_{h4}) \quad \langle x, x \rangle \geq 0$

$(P_{h5}) \quad \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

$(P_{h6}) \quad \langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$

Then $V$ equipped with such a $\langle \cdot \rangle$ is an inner space.

Remark 1.2. The associated norm $|\cdot|$ on inner space $V$ is defined by

$|x| = \langle x, x \rangle^{\frac{1}{2}}$
and the associated metric is
\[ d(x, y) = (x - y, x - y)^{\frac{1}{2}}. \]

**Remark 1.3.** The reflexivity, symmetry, and positivity of this alleged distance function are clear from the definitional properties of \( \langle \cdot, \cdot \rangle \).

**Remark 1.4.** [2] The triangle inequality \( |x - z| \leq |x - y| + |y - z| \) is equivalent to the triangle inequality for the alleged metric
\[ d(x, y) = (x - y, x - y)^{\frac{1}{2}}. \]

## 2 \( \langle \cdot, \cdot \rangle_h \)-approximate fixed points in \( \langle \cdot, \cdot \rangle \) for two maps

In this section, we will consider the existence of \( \langle \cdot, \cdot \rangle_h \)-approximate fixed points in \( \langle \cdot, \cdot \rangle \) for two maps \( \varphi : E \to E, \psi : E \to E \), and diameter it.

**Remark 2.1.** pre-Hilbert spaces \( (V, \langle \cdot, \cdot \rangle_h) \) has a smallest norm, that is \( x \in V \), s.t.
\[ k_h = |x| = \inf_{y \in V} |y|. \]

**Definition 2.2.** Let \( E \) and \( F \) be two nonempty sets in a pre-Hilbert space \( (V, \langle \cdot, \cdot \rangle_h) \) and \( \varphi : E \cup F \to E \cup F, \psi : E \cup F \to E \cup F \) be two linear functionals such that \( \lambda(E) \subseteq F \) and \( \lambda(F) \subseteq E \).

A point \( (\nu, \omega) \) in \( E \times F \) is said to be an \( \langle \cdot, \cdot \rangle_h \)-approximate-pair fixed point for \( (\varphi, \psi) \) in \( V \), if for every \( \epsilon > 0 \)
\[ \langle \varphi \nu - \psi \omega, \varphi \nu - \psi \omega \rangle \leq k_h + \epsilon. \]

We say that the pair \( (\varphi, \psi) \) has the \( \langle \cdot, \cdot \rangle_h \)-approximate-pair fixed property in \( V \) if
\[ P^\epsilon_{(\varphi, \psi)}(E, F) = \{ (\nu, \omega) \in E \times E : \langle \varphi \nu - \psi \omega, \varphi \nu - \psi \omega \rangle \leq k_h + \epsilon \}, \]
is nonempty for some \( \epsilon > 0 \).

**Theorem 2.3.** Let \( E \) and \( F \) be two nonempty sets in a pre-Hilbert space \( (V, \langle \cdot, \cdot \rangle_h) \) and \( \varphi : E \cup F \to E \cup F, \psi : E \cup F \to E \cup F \) be two linear functionals. If, for every \( (\nu, \omega) \in E \times F \),
\[ \langle \varphi^n(\nu) - \psi^n(\omega), \varphi^n(\nu) - \psi^n(\omega) \rangle_h \to k_h \]
then \( (\varphi, \psi) \) has an \( \langle \cdot, \cdot \rangle_h \)-approximate-pair fixed in \( E \cup F \).

**Theorem 2.4.** Let \( E \) and \( F \) be two nonempty sets in a pre-Hilbert space \( (V, \langle \cdot, \cdot \rangle_h) \) and \( \varphi : E \cup F \to E \cup F, \psi : E \cup F \to E \cup F \) be two linear functionals and, for every \( (\nu, \omega) \in E \times F \),
\[ \langle \varphi \nu - \psi \omega, \varphi \nu - \psi \omega \rangle_h \leq \alpha(\nu - \omega, \nu - \omega) + \beta(\nu - \psi \nu, \nu - \psi \nu) + (\omega - \psi \omega, \omega - \psi \omega) + \gamma k_h \]
where \( \alpha, \beta, \gamma \geq 0 \) and \( \alpha + 2 \beta + 2 < 1 \). Then if \( \nu \) is an \( \langle \cdot, \cdot \rangle_h \)-approximate fixed point for \( \varphi \), or \( \omega \) is an \( \langle \cdot, \cdot \rangle_h \)-approximate fixed point for \( \psi \), then \( (\varphi, \psi) \) has an \( \langle \cdot, \cdot \rangle_h \)-approximate-pair fixed in \( E \cup F \).

**Theorem 2.5.** Let \( E \) and \( F \) be two nonempty sets in a pre-Hilbert space \( (V, \langle \cdot, \cdot \rangle_h) \) and \( \varphi : E \cup F \to E \cup F, \psi : E \cup F \to E \cup F \) be two linear functionals and \( \varphi \) is continuous and, for every \( (\nu, \omega) \in E \times F \),
\[ \langle \varphi \nu - \psi \omega, \varphi \nu - \psi \omega \rangle_h \leq \alpha(\nu - \omega, \nu - \omega) + \gamma k_h \]
where \( \alpha, \gamma \geq 0 \) and \( \alpha + \gamma = 1 \), also let \( \{\nu_n\} \) and \( \{\omega_n\} \) as flowing:
\[ \nu_{n+1} = \psi \omega_n, \quad \omega_{n+1} = \varphi \nu_n \quad \text{for some} \quad (\nu_0, \omega_0) \in E \times F, \quad n \in \mathbb{N}. \]
If \{\nu_n\} has a convergent subsequence in E then there exists a \nu_1 \in E \cup F such that \langle \nu_1 - \varphi \nu_1, \nu_1 - \varphi \nu_1 \rangle_h = k_h

**Definition 2.6.** Let \varphi : E \cup F \to E \cup F, \psi : E \cup F \to E \cup F be two linear functionals. For \epsilon > 0 define diameter \text{diam}(P_{(\varphi, \psi)}^\epsilon (E, F)) by,

\text{diam}(P_{(\varphi, \psi)}^\epsilon (E, F)) = \sup\{\langle \nu - \omega, \nu - \omega \rangle_h : \langle \varphi \nu - \varphi \omega, \varphi \nu - \varphi \omega \rangle_h \leq k_h + \epsilon\}.

**Example 2.7.** Suppose \(E = \{(\nu, 0) : 0 \leq \nu \leq 1\}, \ F = \{(\nu, 1) : 0 \leq \nu \leq 1\}, \ \varphi(\nu, 0) = \varphi(\nu, 1) = (1, 1) \) and \(\psi(\nu, 1) = \psi(\nu, 0) = (1, 0)\). Then \(\langle (\varphi(\nu, 0) - \varphi(\nu, 0), \psi(\omega - 1, \omega - 1) \rangle_h = 1\) and \(\text{diam}(P_{(\varphi, \psi)}(E, F)) = \text{diam}(E \times F) = \sqrt{2}\).

**Theorem 2.8.** Let \varphi : E \cup F \to E \cup F, \psi : E \cup F \to E \cup F be two linear functionals such that \(\lambda(E) \subseteq F\) and \(\lambda(F) \subseteq E\). If, there exists a \(p \in [0, 1]\),

\(\langle \nu - \varphi \nu, \nu - \varphi \nu \rangle_h + \langle \psi \omega - \omega, \psi \omega - \omega \rangle_h \leq p(\nu - \omega, \nu - \omega)\).

Then

\text{diam}(P_{(\varphi, \psi)}(E, F)) \leq \frac{\epsilon}{1 - p} + \frac{k_h}{1 - p} \text{ for some } \epsilon > 0.

**References**


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σ-Approximately contractible Banach algebras

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Abstract
The notion of σ-amenability for Banach algebras and its related notions were introduced and extensively studied by M.S. Moslehian and A.N. Motlagh in [1]. We develop these notions parallel to the approximately contractible and amenable Banach algebras introduced by F. Ghahramani and R.J. Loy in [2]. Briefly, we investigate σ-approximate contractibility and σ-approximate amenability of Banach algebras, which are extensions of usual notions of contractiblity and amenability, respectively, where σ is a dense range or is an idempotent bounded endomorphism of the corresponding Banach algebra.

Keywords: Banach algebras, σ-approximately contractible, σ-approximately inner, σ-derivation.

Mathematics Subject Classification: Primary: 46H20; Secondary: 46H35, 46H25

1 Introduction
For a Banach algebra $A$, an $A$-bimodule will always refer to a Banach $A$-bimodule $X$. A derivation $D : A \rightarrow X$ is a linear map, always taken to be continuous, satisfying
\[ D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in A). \]

A Banach algebra $A$ is amenable if for any $A$-bimodule $X$, any derivation $D : A \rightarrow X^*$ is inner, that is, there is $x^* \in X^*$, with
\[ D(a) = a \cdot x^* - x^* \cdot a = \delta_{x^*}(a) \quad (a \in A). \]

Let $A$ be a Banach algebra and $\sigma$ be a bounded endomorphism of $A$. A $\sigma$-derivation from $A$ into a Banach $A$-bimodule $X$ is a bounded linear map $D : A \rightarrow X$ satisfying
\[ D(ab) = \sigma(a) \cdot D(b) + D(a) \cdot \sigma(b) \quad (a, b \in A). \]

For each $x \in X$, the mapping
\[ \delta_x^\sigma : A \rightarrow X \]
defined by $\delta_x^\sigma(a) = \sigma(a) \cdot x - x \cdot \sigma(a)$, for all $a \in A$, is a $\sigma$-derivation called an inner $\sigma$-derivation.

Remark. Throughout this paper, we shall assume that $A$ is a Banach algebra and $\sigma$ is a bounded endomorphism of $A$ unless otherwise specified. Also, we write $(\sigma$-a.i) for $\sigma$-approximately inner, $(\sigma$-a.a) for $\sigma$-approximately amenable and $(\sigma$-a.c) for $\sigma$-approximately contractible.

The basic definition for the present paper is the following.
Definition 1.1. A $\sigma$-derivation $D : A \rightarrow X$ is $\sigma$-a.i. if there exists a net $(x_\alpha) \subseteq X$ such that for every $a \in A$, $D(a) = \lim_\alpha \sigma(a) \cdot x_\alpha - x_\alpha \cdot \sigma(a)$, the limit being in norm and we write $D = \lim_\delta \sigma x_\alpha$.

Note we don’t suppose that $(x_\alpha)$ be bounded.

Definition 1.2. A Banach algebra $A$ is called $\sigma$-a.c ($\sigma$-a.a) if for any $A$-bimodule $X$, every $\sigma$-derivation $D : A \rightarrow X$ ($D : A \rightarrow X^*$) is $\sigma$-a.i.

Definition 1.3. Let $A$ be a Banach algebra and let $X$ and $Y$ be Banach $A$-bimodules. The linear map $T : X \rightarrow Y$ is called a $\sigma$-$A$-bimodule homomorphism if

$$T(a \cdot x) = \sigma(a) \cdot T(x), \quad T(x \cdot a) = T(x) \cdot \sigma(a) \quad (a \in A, \ x \in X).$$

2 Basic properties

Proposition 2.1. Let $A$ be a $\sigma$-a.c Banach algebra. Then $\sigma(A)$ has a left and right approximate identity.

Corollary 2.2. Let $A$ be a $\sigma$-a.c Banach algebra and $\sigma$ is a continuous epimorphism of $A$. Then $A$ has a left and right approximate identity.

Proposition 2.3. Let $\varphi$ be a bounded endomorphism of Banach algebra $A$. If $A$ is $\sigma$-a.c, then $A$ is $(\varphi \sigma)$-a.c too.

Corollary 2.4. Let $A$ be an $a.c$ Banach algebra. Then $A$ is $\sigma$-a.c for each bounded endomorphism $\sigma$ of $A$.

Proposition 2.5. Let $A$ be a $\sigma$-a.c Banach algebra where $\sigma$ is a bounded epimorphism of $A$. Then $A$ is $a.c$.

Corollary 2.6. Let $A$ be a $\sigma$-a.a Banach algebra where $\sigma$ is a bounded epimorphism of $A$. Then $A$ is $a.a$.

Proposition 2.7. Suppose that $B$ is a Banach algebra and $\varphi : A \rightarrow B$ is a continuous epimorphism. If $A$ is $a.c$ then $B$ is $\sigma$-a.c for each bounded endomorphism $\sigma$ of $B$.

Proposition 2.8. Suppose that $A$ and $B$ be Banach algebras and let $\sigma$ and $\tau$ be bounded endomorphism of $A$ and $B$, respectively. Let $\varphi : A \rightarrow B$ be a bounded epimorphism such that $\varphi \sigma = \tau \sigma$. If $A$ is $\sigma$-a.c. Then $B$ is $\tau$-a.c.

3 $\sigma$-Approximate Amenability when $A$ has a B.A.I.

Lemma 3.1. Let $A$ be a Banach algebra with bounded approximate identity and $X$ be a Banach $A$-bimodule with trivial left or right action, then every $\sigma$-derivation $D : A \rightarrow X^*$ is $\sigma$-inner.

The following definitions extends the definition of the neo-unital and essential Banach $A$-bimodule in the classical sense.

Definition 3.2. Let $X$ be a Banach $A$-bimodule. $X$ is called $\sigma$-neo-unital ($\sigma$-pseudo-unital), if $X = \sigma(A) \cdot X \cdot \sigma(A)$. Similarly, one defines $\sigma$-neo-unital left and right Banach modules.

Definition 3.3. Let $X$ be a Banach $A$-bimodule. $X$ is called $\sigma$-essential if $X = \sigma(A)X \sigma(A) = \varpropto \sigma(A) \cdot X \cdot \sigma(A)$. Similarly, one defines $\sigma$-essential left and right Banach modules.

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Proposition 3.4. Assume that $A$ has a left bounded approximate identity, $\sigma$ be a bounded idempotent endomorphism of $A$, and $X$ be a left Banach $A$-module. $X$ is $\sigma$-neu-unital if and only if $X$ is $\sigma$-essential.

Corollary 3.5. Every $\sigma$-neo-unital left Banach $A$-module is essential.

Proposition 3.6. Let $A$ be a Banach algebra with a left bounded approximate identity, $\sigma$ be a bounded idempotent endomorphism of $A$, and $X$ be a left Banach $A$-module. Then $\sigma(A) \cdot X$ is closed weakly complemented submodule of $X$.

Corollary 3.7. Let $A$ has a bounded approximate identity, $X$ be a Banach $A$-bimodule and $\sigma$ be a bounded idempotent endomorphism of $A$. Then
  
i) $\sigma(A) \cdot X \cdot \sigma(A)$ is closed weakly complemented submodule of $X$.
  
ii) $A$ is $\sigma$-a.a if and only if for every $\sigma$-neo-unital Banach $A$-bimodule $X$, every $\sigma$-derivation $D : A \rightarrow X^*$ is $\sigma$-approximately inner.

Corollary 3.8. Let $A$ has a bounded approximate identity, $X$ be a Banach $A$-bimodule and $\sigma$ be a bounded idempotent endomorphism of $A$. Then $A$ is $\sigma$-a.a if and only if for every $\sigma$-essential Banach $A$-bimodule $X$, every $\sigma$-derivation $D : A \rightarrow X^*$ is $\sigma$-approximately inner.

Corollary 3.9. Suppose that Banach algebra $A$ is $\sigma$-a.a, then $\sigma(A)$ has left and right approximate identities.

Corollary 3.10. Suppose that Banach algebra $A$ is $\sigma$-a.a and $\sigma$ is a bounded epimorphism of $A$, then $A$ has left and right approximate identities.

Proposition 3.11. Let $I$ be a closed ideal of $A$ such that $\sigma(I) \subseteq I$. If $A$ is $\sigma$-a.a, then $\frac{I}{A}$ is $\hat{\sigma}$-a.c, where $\hat{\sigma}$ is endomorphism of $\frac{I}{A}$ induced by $\sigma$ (i.e. $\hat{\sigma}(a + I) = \sigma(a) + I$ for $a \in A$).

Proof. Let $X$ be a Banach $\frac{I}{A}$-bimodule and $D : \frac{I}{A} \rightarrow X$ be a $\hat{\sigma}$-derivation. Then $X$ becomes an $A$-bimodule with the following module actions

$$a \cdot x = \pi(a) \cdot x, \quad x \cdot a = x \cdot \pi(a) \quad (a \in A, \ x \in X),$$

where $\pi$ is the canonical homomorphism $\pi : A \rightarrow \frac{I}{A}$. It is easy to see that $D\sigma : A \rightarrow X$ becomes a $\sigma$-derivation. Since $A$ is $\sigma$-a.c, there exists a net $\{x_\alpha\} \subseteq X$ such that $D\sigma(a) = \lim_\alpha \sigma(a) \cdot x_\alpha - x_\alpha \cdot \sigma(a)$ $(a \in A)$. Therefore, for each $(a \in A)$,

$$D(a + I) = D\sigma(a) = \lim_\alpha \sigma(a) \cdot x_\alpha - x_\alpha \cdot \sigma(a)$$

$$= \lim_\alpha \pi(\sigma(a)) \cdot x_\alpha - x_\alpha \cdot \pi(\sigma(a))$$

$$= \lim_\alpha (\sigma(a) + I) \cdot x_\alpha - x_\alpha \cdot (\sigma(a) + I)$$

$$= \lim_\alpha \hat{\sigma}(a + I) x_\alpha - x_\alpha \hat{\sigma}(a + I)$$

Thus $\frac{I}{A}$ is $\hat{\sigma}$-a.c. \qed

Proposition 3.12. Suppose that $I$ is a closed ideal in $A$. If $I$ is $\sigma$-amenable and $\frac{I}{A}$ is a.a, then $A$ is $\sigma$-a.a.
**Example.** Let $A$ be a Banach algebra and let $0 \neq \varphi \in \text{Ball}(A^*)$. Then $A$ with the product $a \cdot a' = \varphi(a)a'$, becomes a Banach algebra. We denote this algebra with $A_\varphi$. It is easy to see that $A_\varphi$ has a left identity $e$, and it has not right approximate identity, so $A_\varphi$ is not contractible and is not approximately contractible. Also $A_\varphi$ is biprojective. Now suppose that $\sigma : A_\varphi \rightarrow A_\varphi$ be defined by $\sigma(a) = \varphi(a)e$. We have
\[
\sigma^2(a) = \sigma(\varphi(a)e) = \varphi(a)\sigma(e) = \varphi(a)\varphi(e)e = \varphi(a)e = \sigma(a)
\]
Thus $\sigma$ is idempotent. It is easy to see that $e$ is identity for $\sigma(A_\varphi)$ and since $A$ is biprojective by [2, Corollary 5.3], $A_\varphi$ is $\sigma$-biprojective. Thus by [2, Theorem 4.3] $A_\varphi$ is $\sigma$-contractible and so $A_\varphi$ is $\sigma$-a.c.

**References**


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Positive answer to Olaleru’s open problem

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Abstract
Let $(X, d)$ be a complete metric space and $T : X \rightarrow X$ a mapping that satisfies
\[ d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Tx) + ed(y, Ty) + fd(x, Ty) \]
for all $x, y \in X$, where $0 < a < 1$, $b, c, e, f \geq 0$, $a + b + c + e + f = 1$ and $b + c > 0$.
If $e + f \geq 0$ and $X$ is a closed convex subset of a complete metrizable topological vector space $(Y, d)$, then $T$ has a unique fixed point. Above results extend some results of several authors. Also these results shall give an answer the Olaleru’s Open Problem.

Keywords: Fixed point, metrizable, topological vector space

Mathematics Subject Classification: 46J10, 46J15, 47H10

1 Introduction
Let $X$ be a Banach space and let $C$ be a closed convex subset of $X$. In 1980 Gregus [1] proved the following results.

Theorem 1.1. Let $T : C \rightarrow C$ be a mapping satisfying the inequality
\[ \|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|, \]
for all $x, y \in C$, where $0 < a < 1$, $b, c \geq 0$ and $a + b + c = 1$. Then $T$ has a unique fixed point.

Note. In above theorem obviously $b + c > 0$.

In 2006 Olaleru [2] extended the Gregus fixed point theorem. At the end of [2], he proposed an open problem as flaws:

Theorem 1.2. Let $C$ be a closed convex subset of a complete metrizable space $X$ and $T : C \rightarrow C$ a mapping that satisfies
\[ F(Tx - Ty) \leq aF(x - y) + bF(x - Tx) + cF(y - Tx) + eF(y - Ty) + fF(x - Ty) \]
for all $x, y \in X$, where $0 < a < 1$, $b, c, e, f \geq 0$ and $a + b + c + e + f = 1$. Then $T$ has a unique fixed point.

In 2007 Olaleru and Akewe [3] proved this open problem. But their proof has some flaws. In 2011 Moradi [4] gave a counterexample for Theorem 1.2 and showed that the Olaleru-Akewe’s answer is not correct. In fact he gave a counterexample for the condition $b + c = 0$. 

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2 Main Result

At first we present the following lemma that use for extension of Gregus fixed point theorem.

**Lemma 2.1.** Let \((X, d)\) be a complete metric space and let \(T : X \rightarrow X\) be a mapping such that for all \(x, y \in X\),

\[
d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + bd(y, Ty) + ed(y, Tx) + ed(x, Ty)
\]

where \(0 < a < 1, b > 0, e > 0\) and \(a + 2b + 2e = 1\). Then for each \(x \in X\) where \(x \neq Tx\), there exists \(z \in X\) such that \(d(z, Tz) < d(x, Tx)\).

**Corollary 2.2.** Let \((X, d)\) be a complete metric space and \(T : X \rightarrow X\) be a mapping such that for all \(x, y \in X\),

\[
d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(y, Tx) + fd(x, Ty)
\]

where \(0 < a < 1, b, c, e, f \geq 0, b + c > 0, e + f > 0\) and \(a + b + c + e + f = 1\). Then for each \(x \in X\) where \(x \neq Tx\), there exists \(z \in X\) such that \(d(z, Tz) < d(x, Tx)\).

The following corollary is a direct result of Corollary 2.2 in a compact metric spaces.

**Corollary 2.3.** Let \((X, d)\) be a compact metric space and \(T : X \rightarrow X\) be a mapping such that satisfying the inequality \((4)\). Then \(T\) has a unique fixed point.

**Proof.** By taking \(\alpha = \inf \{d(x, Tx) : x \in X\}\), there exists a sequence \(\{x_n\}\) such that \(\lim_{n \to \infty} d(x_n, Tx_n) = \alpha\). Since \(X\) is compact there exists \(x \in X\) and a subsequence \(\{x_{n(k)}\}\) such that \(\lim_{k \to \infty} x_{n(k)} = x\).

Also there exists \(y \in X\) and a subsequence \(\{x_{k(l)}\}\) from \(\{x_{n(k)}\}\) such that \(\lim_{l \to \infty} Tx_{k(l)} = y\). We may assume that \(\lim_{n \to \infty} x_n = x\) and \(\lim_{n \to \infty} Tx_n = y\). We prove that \(Ty = y\). By a similar method in Corollary 2.2 we can consider that \(b = c\) and \(e = f\). Obviously \(d(x, y) = \alpha\) and

\[
d(Tx_n, Ty) \leq ad(x_n, y) + bd(x_n, Tx_n) + bd(y, Ty) + ed(x_n, Ty) + ed(y, Tx_n).
\]

Taking the limit in above inequality we get

\[
d(y, Ty) \leq aa + ba + ca + cd(y, Ty),
\]

and hence

\[
(1 - b - c)d(y, Ty) \leq (a + b + c)\alpha.
\]

Therefore \(d(y, Ty) \leq \alpha\) and hence \(d(y, Ty) = \alpha\). Using Lemma 2.1 we conclude that \(\alpha = 0\) and hence \(Ty = y\).

Uniqueness of the common fixed point follows from \((4)\) and this complete the proof.

The following fixed point theorem extends Gregus fixed point theorem in complete metric spaces.

**Theorem 2.4.** Let \((X, d)\) be a complete metric space and \(T : X \rightarrow X\) be a mapping such that for all \(x, y \in X\),

\[
d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(y, Tx) + fd(x, Ty)
\]

where \(0 < a < 1, b, c, e, f \geq 0, b + c > 0\) and \(a + b + c + e + f = 1\). If \(e + f > 0\) then \(T\) has a unique fixed point.
Proof. By the same method in Corollary 2.2 we can assume that \( b = c \) and \( e = f \). So the inequality (5) is change to the following inequality.

\[
d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + bd(y, Ty) + cd(y, Tx) + ed(x, Ty),
\]

where \( b, e > 0 \) and \( a + 2b + 2e = 1 \).

We break the argument into three steps.

**Step 1.** There exists a \( 0 < \lambda < 1 \) that is for all \( x \in X \) there exists \( z \in X \) such that \( d(z, Tz) \leq \lambda d(x, Tx) \).

**Step 2.** \( T \) has a fixed point.

**Step 3.** The fixed point of \( T \) is unique.

Now we are ready to extend the Gregus theorem on closed convex subset of complete metrizable space.

**Theorem 2.5.** Let \( C \) be a closed convex subset of a complete metrizable space \( X \) and \( T : C \rightarrow C \) a mapping that satisfies

\[
F(Tx - Ty) \leq aF(x - y) + bF(x - Tx) + cF(y - Ty) + eF(y - Tx) + fF(x - Ty),  
\]

for all \( x, y \in X \), where \( 0 < a < 1, b, c, e, f \geq 0, b + c > 0 \) and \( a + b + c + e + f = 1 \). Then \( T \) has a unique fixed point.

Proof. Alike of Theorem 5 we can consider \( b = c \) and \( e = f \). So the inequality (7) is change to the following inequality.

\[
F(Tx - Ty) \leq aF(x - y) + bF(x - Tx) + bF(y - Ty) + eF(y - Tx) + eF(x - Ty),
\]

where \( a + 2b + 2e = 1 \) and \( b > 0 \).

If \( e > 0 \) then by using the proof of Theorem 5, there exists \( 0 < \lambda < 1 \) that is for every \( x \in C \) there exists \( z \in C \) such that \( F(z - Tz) \leq \lambda F(x - Tx) \).

If \( e = 0 \) then the inequality (8) is change to the following inequality.

\[
F(Tx - Ty) \leq aF(x - y) + bF(x - Tx) + bF(y - Ty),
\]

where \( a + 2b = 1 \) and \( b > 0 \). In this case we find \( 0 < \lambda < 1 \) that is for every \( x \in C \) there exists \( z \in C \) such that \( F(z - Tz) \leq \lambda F(x - Tx) \).

By the similar method in Step 2 and Step 3 in Theorem 5, we conclude that \( T \) has a unique fixed point.

**Remark 2.6.** Corollary 1.4 and Theorem 1.7 are Olaleru and Akewe [3] main results with the additional assumption that \( b + c > 0 \).

**References**


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On Edelstein-Suzuki-type fixed point theorem

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Abstract
In this paper we prove a fixed point theorem in compact cone metric spaces by using implicit relation technique.

Keywords: Cone metric spaces, Common fixed point, Edelstein’s theorem, Suzuki’s theorem

Mathematics Subject Classification: Primary 47H10, 54H25, 55M20

1 Introduction

In 1962, M. Edelstein [6] proved an other version of Banach contraction Principle. He assumed a compact metric space $(X, d)$ and a self-mapping $T$ on $X$ such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, and he proved $T$ has a unique fixed point. In 2009, T. Suzuki [12] improved the results of Banach and Edelstein. Suzuki replaced the condition “$d(Tx, Ty) < d(x, y)$” by “$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow d(Tx, Ty) < d(x, y)$” for all $x, y \in X$. By this assumption he established $T$ in compact space has a unique fixed point. Recently D. Dorić et al. in [5] proved the following theorem and extended the results of Edelstein and Suzuki on compact cone metric spaces:

**Theorem 1.1.** Let $(X, d)$ be a compact cone metric space over a normal and solid cone $P$ and let $T : X \to X$. If for all $x, y \in X$ with $x \neq y$,

\[
\frac{1}{2}d(x, Tx) - d(x, y) \notin \text{int}P \implies d(Tx, Ty) \ll d(x, y) + A d(x, Tx) + B d(y, Ty) + C d(x, y) + D d(x, Ty) + E d(y, Tx)
\]

holds true, where $A, B, C, D, E \geq 0$, $A + B + C + 2D = 1$ and $C \neq 1$, then $T$ has a fixed point in $X$. If $E \leq B + C + D$ then the fixed point of $T$ is unique.

In 2007, Huang and Zhang [7] introduced cone metric spaces and defined some properties of convergence of sequences and completeness in cone metric spaces, also they proved a fixed point theorem of cone metric spaces. During the recent years, cone metric spaces and fixed point theorems in these spaces have been studied by a number of authors.(see [10], [13]). Furthermore, many authors considered implicit relation technique to investigation of fixed point theorems in metric spaces (see [2], [3], [8]).

In this paper, we introduce an implicit relation. This helps us to extend result of D. Dorić et al. (Theorem3.8 of [5]).

Let $E$ be a real Banach space with norm $\|\cdot\|$ and $P$ be a subset of $E$. $P$ is called a cone if and only if the following conditions are satisfied: (i) $P$ is closed, nonempty and (ii) $a, b \geq 0$ implies $ax + by \in P$, (iii) $x \in P$ and $-x \in P$ implies $x = \theta$

Let $P \subset E$ be a cone, we define a partial ordering $\preceq$ on $E$ with respect to $P$ by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$(interior of $P$). The cone $P \subset E$ is called normal if there is a positive real number $K$
such that for all \(x, y \in E\), \(\theta \leq x \leq y \Rightarrow \|x\| \leq K\|y\|\). The least positive number satisfying the above inequality is called the normal constant of \(P\). If \(K = 1\) then the cone \(P\) is called monotone.

A cone metric space is an ordered pair \((X, d)\), where \(X\) is any set and \(d: X \times X \to \mathbb{R}\) is a mapping satisfying:

1. \(d(x, y) = 0\) if and only if \(x = y\).
2. \(d(x, y) = d(y, x)\) for all \(x, y \in X\).
3. \(d(x, y) \leq d(x, z) + d(z, y)\) for all \(x, y, z \in X\).

Let \((X, d)\) be a cone metric space, \(P\) be a normal cone in \(X\) with normal constant \(K\), \(x \in X\) and \(\{x_n\}\) a sequence in \(X\). The sequence \(\{x_n\}\) converges to \(x\) if and only if \(d(x_n, x) \to 0\).

We now define a metric on \(X\) by \(D(x, y) = \|d(x, y)\|\). Furthermore, it is proved that \(D\) and \(d\) give same topology on \(X\) (see [9]).

Let \(F: P^6 \to \mathbb{R}\) be a function satisfies the following conditions:

1. \(u \leq v\) implies \(F(u, v, v, v, u, u) \leq 0\).
2. \(F(u, v, v, u + v, u, \theta) < 0\) implies \(\|u\| < \|v\|\).
3. \(F(u, v, \theta, v, v) < 0\) implies \(\|u\| < \|v\|\), where \(u \geq \theta\) and \(v \gg \theta\).

2 Main results

**Theorem 2.1.** Let \((X, d)\) be a compact cone metric space and \(T\) be a self-mapping on \(X\). Suppose that \(F: P^6 \to \mathbb{R}\) is a continuous function such that (M1) – (M3) are satisfied. Assume that

\[
\frac{1}{2}D(x, Tx) < D(x, y) \implies F\left(d(Tx, Ty), d(x, y), d(x, Tx), d(x, Ty), d(y, Ty), d(y, Tx)\right) < 0
\]

for all \(x, y \in X\). Then \(T\) has at least one fixed point. Moreover, if \(F\) satisfies in the (M4), then \(T\) has a unique fixed point.

**Theorem 2.2.** Let \((X, d)\) be a cone metric space and let \(G\) and \(T\) be two self-mappings on \(X\) such that \(TX \subseteq GX\) and \(GX\) is compact. Suppose that \(F: P^6 \to \mathbb{R}\) is a continuous function such that (M1) – (M3) are satisfied. Assume that

\[
\frac{1}{2}D(Gx, Tx) < D(Gx, Gy) \implies F\left(d(Tx, Ty), d(Gx, Gy), d(Gx, Tx), d(Gx, Ty), d(Gy, Ty), d(Gy, Tx)\right) < 0
\]

for all \(x, y \in X\). Then \(G\) and \(T\) have at least one point of coincidence. Moreover, if \(F\) satisfies in (M4) and \(G\) and \(T\) are weakly compatible then \(G\) and \(T\) have a unique common fixed point.

**Corollary 2.3.** Let \((X, d)\) be a cone metric space and let \(G\) and \(T\) be two self-mappings on \(X\) such that \(TX \subseteq GX\) and \(GX\) is compact. Assume that

\[
\frac{1}{2}D(Gx, Tx) < D(Gx, Gy) \implies D(Tx, Ty) < aD(Gx, Gy) + bD(Gx, Tx) + cD(Gx, Ty) + dD(Gy, Ty) + eD(Gy, Tx)
\]

for all \(x, y \in X\), where \(a, b, c, d, e \geq 0\), \(a + b + 2c + d = 1\) and \(d \neq 1\). Then \(G\) and \(T\) have at least one point of coincidence. Moreover, if \(e \leq b + c + d\) and \(G\) and \(T\) are weakly compatible then \(G\) and \(T\) have a unique common fixed point.

**Theorem 2.4.** Let \((X, d)\) be a compact cone metric space and \(T\) be a self-mapping on \(X\). Suppose that \(\psi_T: P \to P\) and \(\varphi_T: P^5 \to P\) are two continuous mappings such that (P1) – (P3) are satisfied. Assume that
\[
\frac{1}{2}d(x, Tx) - d(x, y) \notin \text{int} P \implies \\
\psi_p(d(Tx, Ty)) \ll \varphi_p \left( d(x, y), d(x, Tx), d(x, Ty), d(y, Ty), (y, Tx) \right)
\]  
(3)

for all \( x, y \in X \) with \( x \neq y \). Then \( T \) has at least one fixed point. Moreover, if \( \psi_p \) and \( \varphi_p \) satisfy (P4) then \( T \) has a unique fixed point.

**Theorem 2.5.** Let \((X, d)\) be a cone metric space and let \( G \) and \( T \) be two self-mappings on \( X \) such that \( TX \subseteq GX \) and \( GX \) is compact. Suppose that \( \psi_p : P \to P \) and \( \varphi_p : P^5 \to P \) are two continuous mappings satisfying (P1) - (P3). Assume that
\[
\frac{1}{2}d(Gx, Tx) - d(Gx, Gy) \notin \text{int} P \implies \\
\psi_p(d(Tx, Ty)) \ll \varphi_p \left( d(Gx, Gy), d(Gx, Tx), d(Gx, Ty), d(Gy, Ty), d(Gy, Tx) \right)
\]

for all \( x, y \in X \) with \( x \neq y \). Then \( G \) and \( T \) have at least one point of coincidence. Moreover, if \( \psi_p \) and \( \varphi_p \) satisfy (P4) and \( G \) and \( T \) are weakly compatible then \( G \) and \( T \) have a unique common fixed point.

**Corollary 2.6.** Let \((X, d)\) be a cone metric space and let \( F \) and \( T \) be two self-mappings on \( X \) such that \( TX \subseteq FX \) and \( FX \) is compact. Assume that
\[
\frac{1}{2}d(Fx, Tx) - d(Fx, Fy) \notin \text{int} P \implies d(Tx, Ty) \ll M(x, y)
\]
for all \( x, y \in X \) with \( x \neq y \), where

\[
M(x, y) = Ad(Fx, Fy) + Bd(Fx, Tx) + Cd(Fx, Ty) + Bd(Fy, Ty) + Ed(Fy, Tx)
\]

and \( A, B, C, D, E \geq 0 \), \( A + B + 2C + D = 1 \) and \( D \neq 1 \). Then \( F \) and \( T \) have at least one point of coincidence. Moreover, if \( E \leq B + C + D \) and \( F \) and \( T \) are weakly compatible then \( F \) and \( T \) have a unique common fixed point.

**References**


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Jordan product and operator monotone functions

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Abstract
We show that the symmetrized product $AB + BA$ of two positive operators $A$ and $B$ is positive if and only if $f(A + B) \leq f(A) + f(B)$ for all non-negative operator monotone functions $f$ on $[0, \infty)$ and deduce an operator inequality. We also give a necessary and sufficient condition for that the composition $f \circ g$ of an operator convex function $f$ on $[0, \infty)$ and a non-negative operator monotone function $g$ on an interval $(a, b)$ is operator monotone and give some applications.

Keywords: Operator monotone function; Jordan product; operator convex function; subadditivity; composition of functions

Mathematics Subject Classification: Primary 47A63; Secondary 47B10, 47A30.

1 Introduction
An operator $A \in \mathcal{B}(H)$ is called positive if $\langle Ax, x \rangle \geq 0$ holds for every $x \in H$ and then we write $A \geq 0$. The set of all positive operators on $H$ is denoted by $\mathcal{B}(H)_+$. The symmetrized product of two operators $A, B \in \mathcal{B}(H)$ is defined by $S(A, B) = AB + BA$. In general, the symmetrized product of two operators $A, B \in \mathcal{B}(H)_+$ is not positive. For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then $AB + BA$ is not a positive operator. Gustafson [3] showed that if $0 \leq m \leq A \leq M$ and $0 \leq n \leq B \leq N$, then
\[ mn - \frac{(M - m)(N - n)}{8} \leq \frac{1}{2} S(A, B). \] (1)

A continuous real valued function $f$ defined on an interval $J$ is called operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for all self adjoint operators $A, B$ with spectra in $J$. Also if $f(\lambda A + (1 - \lambda)B) \leq f(A) + (1 - \lambda)f(B)$ for all $\lambda \in [0; 1]$, then $f$ is said to be operator convex function on the interval $J$. If $f$ is a non-negative operator monotone function on $[0, \infty)$, then the subadditivity of $f$ does not hold in general. Aujla and Bourin [1] showed that if $A, B \geq 0$ are matrices and $f : [0, \infty) \rightarrow [0, \infty)$ is a concave function, i.e. $-f$ is convex, then there exist unitaries $U, V$ such that
\[ f(A + B) \leq U f(A) U^* + V f(B) V^*. \]

Throughout this paper $\mathcal{M}(H)$ denotes the set of all $(A, B) \in \mathcal{B}(H)_+ \times \mathcal{B}(H)_+$ for which $f(A + B) \leq f(A) + f(B)$ for all non-negative operator monotone functions $f$ on $[0, \infty)$.

2 The results

Theorem 2.1. Let $A, B \in \mathcal{B}(H)_+$. Then $AB + BA$ is positive if and only if $f(A + B) \leq f(A) + f(B)$ for all non-negative operator monotone functions $f$ on $[0, \infty)$. 

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If \( A, B \in B(H) \), commute, then \( AB = (A^{1/2}B^{1/2})^2 \geq 0 \). The above theorem therefore shows that \( f(A + B) \leq f(A) + f(B) \) for any non-negative operator monotone function \( f \) on \([0, \infty)\). In particular, for each \( n \in \mathbb{N} \) we have \( f(nA) \leq nf(A) \).

**Corollary 2.2.** Let \( 0 \leq m \leq A \leq M, 0 \leq n \leq B \leq N \) and \( f \) be a non-negative operator monotone function on \([0, \infty)\). If \((M - m)(N - n) \leq 8mn\), then
\[
 f(A + B) \leq f(A) + f(B).
\]

**Corollary 2.3.** Let \( 0 \leq p \leq \frac{1}{2} \) and \( f \) be a non-negative operator monotone function on \([0, \infty)\). Then for positive operators \( A \) and \( B \) with \( A \leq B \) it holds that
\[
 f(B^p) \leq f \left( \frac{B^p + A^p}{2} \right) + f \left( \frac{B^p - A^p}{2} \right).
\]

**Theorem 2.4.** Let \( f \) be a non-negative operator monotone function on \([0, \infty)\). Let \( A \) and \( A_i \) (\( 1 \leq i \leq n \)) be positive operators with spectra in \([\lambda, (1 + 2\sqrt{2})\lambda]\) for some \( \lambda \in \mathbb{R}_+ \). Then

(i) For every isometry \( C \in B(H) \),
\[
 f(C^*AC) \leq 2C^*f \left( \frac{A}{2} \right)C.
\]

(ii) For operators \( C_i \) (\( i = 1, \cdots, n \)) with \( \sum_{i=1}^n C_i^*C_i = I \),
\[
 f \left( \sum_{i=1}^n C_i^*A_iC_i \right) \leq 2 \sum_{i=1}^n C_i^*f \left( \frac{A}{2} \right)C_i.
\]

**Corollary 2.5.** Let \( \lambda \in \mathbb{R}_+ \) and \( w_i \in [\lambda, (1 + 2\sqrt{2})\lambda] \) (\( i = 1, \cdots, n \)). Let \( f \) be a non-negative operator monotone function on \([0, \infty)\) and \( A_i \) be positive operators such that \( \sum_{i=1}^n A_i = I \). Then
\[
 f(\sum_{i=1}^n w_iA_i) \leq 2 \sum_{i=1}^n \frac{w_i}{2}A_i.
\]

**Theorem 2.6.** Let \( A, B \in B(H) \). Then \( B^2 \leq A^2 \) if and only if for each operator convex function \( f \) on \([0, \infty)\) with \( f'(0) \geq 0 \) it holds that
\[
 f(B) \leq f(A)
\]

**Theorem 2.7.** Let \( g \) be a non-negative operator monotone function on an interval \((a, b)\). Let \( g(z) = u(z) + iv(z) \) be its analytic continuation to the upper half-plane. Then for each operator convex function \( f \) on \([0, \infty)\) with \( f'(0) \geq 0 \), \( f \circ g \) is operator monotone on \((a, b)\) if and only if \( u(z) \geq 0 \) on the upper half-plane.

**Remark 2.8.** If \( g \) is a non-negative operator monotone function on \((a, b)\) not a zero constant function and \( u(z) \geq 0 \) on the domain of \( g \), then \( u(z) > 0 \) on this domain. To see this note that:

(i) If for some \( t_0 \in (a, b) \) we have \( g(t_0) = 0 \), then there exists \( R > 0 \) such that \( g \) can be represented as \( g(t) = \sum_{n=1}^{\infty} \frac{g^{(n)}(t_0)}{n!}(t-t_0)^n \) for all \( t \in (t_0 - R, t_0 + R) \) /p. 6/ [Hamed Najafi]. Since \( g \) is a non-negative monotone function, \( g(t) = 0 \) for all \( t \leq t_0 \). Hence \( g^{(n)}(t_0) = 0 \) for all \( n \), so \( g \) is zero on the neighborhood \((t_0 - R, t_0 + R)\) of \( t_0 \). Thus \( \{ t : g(t) = 0 \} \) is clopen. Hence \( g \) is zero on \((a, b)\) contradicting the assumption above. Thus \( g(t) > 0 \) on \((a, b)\).

(ii) If \( g \neq 0 \) is a constant function, then clearly \( u(z) > 0 \).

(iii) If \( g \) is not a constant function, then by the open mapping theorem for non-constant analytic functions, \( u \) maps the upper half-plane into \( \{ z : \text{Im}z > 0 \text{ and } u(z) > 0 \} \).
**Corollary 2.9.** Let $0 \leq p \leq \frac{1}{2}$ and let $f$ be an operator convex function on $[0, \infty)$ with $f'_{+}(0) \geq 0$. If $B \leq A$, then
\[ f(B^{p}) \leq f(A^{p}). \]

**Corollary 2.10.** Let $0 \leq p \leq \frac{1}{2}$ and $f$ be a non-negative operator monotone function on $[0, \infty)$. Then for positive operators $A$ and $B$ with $B \leq A$,

(i) $B^{p}f(B^{p}) \leq A^{p}f(A^{p})$;

(ii) If $f$ is strictly positive on $(0, \infty)$ and $A, B$ are invertible, then $A^{p^{-1}}f(A^{p}) \leq B^{p^{-1}}f(B^{p})$.

**References**


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A property of nilpotent ideals in certain Banach algebras

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Abstract
We show that nilpotent ideals in approximately amenable or pseudo-amenable Banach algebras cannot have a certain property, and hence cannot be boundedly approximately complemented in those Banach algebras.

Keywords: Bounded approximately complemented subspaces, Pseudo-amenability, Approximate amenability, Biprojectivity.

Mathematics Subject Classification: 46H20, 46B28

1 Introduction
Let \( \mathcal{A} \) be a Banach algebra and \( \mathcal{X} \) a Banach \( \mathcal{A} \)-bimodule. A derivation from \( \mathcal{A} \) into \( \mathcal{X} \) satisfying \( D(ab) = a \cdot D(b) + D(a) \cdot b \) for all \( a, b \in \mathcal{A} \). For each \( x \in \mathcal{X} \), the derivation \( D(a) = a \cdot x - x \cdot a \) is called an inner derivation, denoted by \( \text{ad}_x \). A Banach algebra \( \mathcal{A} \) is amenable if every derivation \( D : \mathcal{A} \to \mathcal{X}^* \) is inner for each Banach \( \mathcal{A} \)-bimodule \( \mathcal{X} \).

A derivation \( D : \mathcal{A} \to \mathcal{X}^* \) is called approximately inner if there exists a net \((x_\alpha) \subseteq \mathcal{X}\) such that \( D(a) = \lim_\alpha \text{ad}_{x_\alpha}(a) \) for all \( a \in \mathcal{A} \). A Banach algebra \( \mathcal{A} \) is approximately amenable if every derivation \( D : \mathcal{A} \to \mathcal{X}^* \) is approximately inner.

A Banach algebra \( \mathcal{A} \) is pseudo-amenable if there is a net \((m_\alpha) \subseteq \mathcal{A} \hat{\otimes} \mathcal{A} \), called an approximate diagonal, such that \( a \cdot m_\alpha - m_\alpha \cdot a \to 0 \) and \((\Delta m_\alpha) a \to a \) for each \( a \in \mathcal{A} \).

In 1989, Curtis and Loy have shown that non-zero nilpotent ideals in amenable Banach algebras must be infinite dimensional [2]. In 1994, Loy and Willis investigated the relation between existence of the approximation property and existence of nilpotent ideals in amenable and biprojective Banach algebras [8]. Related to the conjectures in [8], in 1999, Zhang defined concepts of approximately complemented subspaces of normed spaces and approximately biprojective Banach algebras. Using these concepts he proved that any approximately biprojective Banach algebra with left and right approximate identities does not have a nontrivial nilpotent ideal whose closure is approximately complemented [10].

In this paper, we introduce a new notion of bounded approximate biprojectivity and give another property of nilpotent ideals related to the conjectures raised by Loy and Willis in [8]. This property allows us to generalize results in [8] and [10] to a large class of Banach algebras containing approximately amenable and pseudo-amenable Banach algebras. In particular, we show that in approximately biprojective Banach algebras, nilpotent ideals cannot have a special property, called the “property \( \mathcal{B} \)”. As a consequence, we show that they cannot be boundedly approximately complemented in those Banach algebras.
2 Main Results

Definition 2.1. A Banach algebra $A$ is called approximately biprojective if there exists a net $(\rho_\alpha) \subseteq B(A, \hat{A}\hat{A})$ such that

(i) $\Delta \circ \rho_\alpha(a) \rightarrow a$ for all $a \in A$;
(ii) $\rho_\alpha(b) - \rho_\alpha(ab) \rightarrow 0$ for all $a, b \in A$;
(iii) $\rho_\alpha(a) \cdot b - \rho_\alpha(ab) \rightarrow 0$ for all $a, b \in A$.

Moreover, if the net $(\rho_\alpha)$ can be chosen bounded, we say that $A$ is boundedly approximately biprojective.

Definition 2.2. Let $I$ be a closed ideal in a Banach algebra $A$, and let $\iota : I \rightarrow A$ be the inclusion map. We say that $I$ has the property $B$ if the following assertion is true:

$B$ If $(q_n) \subseteq I \hat{A}/\hat{J}$ is a sequence such that $(\iota \otimes I_A/I)_n(q_n) \rightarrow 0$ for some closed ideal $J$ of $A$ with $J \subseteq I$ and $J^2 = 0$, then $q_n \rightarrow 0$.

Theorem 2.3. Let $A$ be a biflat Banach algebra. Then $A$ is approximately biprojective.

Proof. This is a consequence of Mazur's theorem, as it is used in [5, Theorem 2.1 Hasan Pourmahmood-Aghababa].

Proposition 2.4. Let $A$ be an approximately amenable or a pseudo-amenable Banach algebra. Then $A$ is approximately biprojective.

Proof. Let $A$ be approximately amenable. Then by [5, Theorem 2.1 Hasan Pourmahmood-Aghababa] and [4, Proposition 2.6 Hasan Pourmahmood-Aghababa] there are nets $(m_\alpha) \subseteq A\hat{A}$, $(F_\alpha)$, $(G_\alpha) \subseteq A$ such that

(i) $a \cdot m_\alpha - m_\alpha \cdot a + F_\alpha \otimes a - a \otimes G_\alpha \rightarrow 0$;
(ii) $a F_\alpha \rightarrow a, G_\alpha a \rightarrow a$; and
(iii) $\Delta(m_\alpha) - F_\alpha a - G_\alpha a \rightarrow 0$.

Define $\rho_\alpha : A \rightarrow A\hat{A}$ by $\rho_\alpha(a) = m_\alpha \cdot a - F_\alpha \otimes a$ for all $a \in A$. It can be easily seen that $(\rho_\alpha)$ satisfies Definition 13. Hence $A$ is approximately biprojective.

If $A$ is pseudo-amenable with approximate diagonal $(m_\alpha)$, then the net $(\rho_\alpha)$ with $\rho_\alpha(a) = a \cdot m_\alpha$ satisfies Definition 13.

Lemma 2.5. Suppose that $A$ is an approximately biprojective Banach algebra, $J$ and $N$ are closed ideals in $A$ such that $J \subseteq N$ and $NJ = \{0\}$ ($JN = \{0\}$). Suppose also that $N$ has the property $B$, and $A$ has a left (right) approximate identity for $J$, or $A$ has an element which is not a left (right) zero-divisor. Then $JA = \{0\}$ ($JA = \{0\}$).

Proof. We show that $JA = \{0\}$, the other case is similar. Let $(\rho_{\alpha})_{\alpha \in I} \subseteq B(A, \hat{A}\hat{A})$ be a net satisfying Definition 13. Let $\iota : N \rightarrow A$ and $\varrho : A \rightarrow N/\hat{J}$ be the inclusion and quotient maps, respectively. Since $\iota N = 0$, the operator $p : N \hat{A}/\hat{J} \rightarrow N$ determined by $p(b \otimes (c + J)) = bc$ is well defined.

Suppose to the contrary that $JA \neq \{0\}$. Using the ideal $J$ (note that $J^2 = \{0\}$), we show that $N$ does not have the property $B$. Since $JA \neq \{0\}$, there exist $a \in A$ and $j \in J$ with $ja \neq 0$. We can find $u \in A$ such that $uJa \neq 0$, by our assumption. Let $L_b$ and $R_b : A \rightarrow N$ be the left and
right multiplication maps by \( b \in \mathcal{N} \), respectively. For each \( \alpha \) set \( v_\alpha = (\mathcal{L}_{(uj)} \otimes q) \circ \rho_\alpha(a) \) which belongs to \( \mathcal{N} \hat{\otimes} \mathcal{A}/\mathcal{J} \). Since \( q \circ i \circ \mathcal{R}_{(j,a)} = 0 \), by condition (iii) of Definition 13 we have

\[
\lim_\alpha (I_A \otimes q)(\rho_\alpha(u)ja) = \lim_\alpha (I_A \otimes q)(\rho_\alpha(u)ja)
= \lim_\alpha (I_A \otimes q) \circ (I_A \otimes i) \circ (I_A \otimes \mathcal{R}_{(j,a)}) \circ \rho_\alpha(u)
= \lim_\alpha (I_A \otimes (q \circ i \circ \mathcal{R}_{(j,a)}) \circ \rho_\alpha(u)
= 0,
\]

and thus by condition (ii) of Definition 13,

\[
\lim_\alpha (\iota \otimes I_{\mathcal{A}/\mathcal{J}})(v_\alpha) = \lim_\alpha (\iota \otimes I_{\mathcal{A}/\mathcal{J}}) \circ (I_N \otimes q) \circ (\mathcal{L}_{(uj)} \otimes I_A) \circ \rho_\alpha(a)
= \lim_\alpha (I_A \otimes q) \circ (\iota \otimes I_A) \circ (\mathcal{L}_{(uj)} \otimes I_A) \circ \rho_\alpha(a)
= \lim_\alpha (I_A \otimes q)(u_j \rho_\alpha(a))
= \lim_\alpha (I_A \otimes q)(\rho_\alpha(u)ja)
= 0.
\]

It follows that \( \mathcal{N} \) does not have the property \( \mathcal{B} \), a contradiction. \( \square \)

**Theorem 2.6.** Suppose that \( \mathcal{A} \) is an approximately biprojective Banach algebra. If \( \mathcal{A} \) has both left and right approximate identities, then \( \mathcal{A} \) cannot have a non-zero nilpotent ideal with the property \( \mathcal{B} \).

**Proof.** Let \( \mathcal{N} \) be a non-zero nilpotent ideal in \( \mathcal{A} \) with \( \text{Nil}(\mathcal{N}) = n + 1 \) and with the property \( \mathcal{B} \). Let \( \mathcal{J} \subseteq \mathcal{N} \) be the closed linear span of \( \mathcal{N}^n \) which is a non-zero closed ideal in \( \mathcal{A} \) and \( \mathcal{N}/\mathcal{J} = \{0\} \). Since \( \mathcal{A} \) has a left approximate identity, by Lemma 1.3, \( \mathcal{J} \mathcal{A} = \{0\} \) and since \( \mathcal{A} \) has a right approximate identity, we obtain \( \mathcal{J} = \mathcal{J}\mathcal{A} = \{0\} \) which is a contradiction. \( \square \)

A subspace \( \mathcal{Y} \) of a normed space \( \mathcal{X} \) is called boundedly approximately complemented in \( \mathcal{X} \) if there is a bounded net \( (\rho_\alpha) \) of continuous operators from \( \mathcal{X} \) into \( \mathcal{Y} \) such that \( \rho_\alpha(x) \) converges to \( x \) uniformly on compact subsets of \( \mathcal{Y} \).

**Corollary 2.7.** Let \( \mathcal{A} \) be an approximately amenable or a pseudo-amenable Banach algebra. Then by Proposition 7 and Theorem 2.6,

(i) \( \mathcal{A} \) cannot have a non-zero nilpotent ideal with the property \( \mathcal{B} \);

(ii) nilpotent ideals cannot be boundedly approximately complemented in \( \mathcal{A} \);

(iii) \( \mathcal{A} \) has no finite dimensional nilpotent ideal;

(iv) \( \text{rad}(\mathcal{A}) \) cannot be finite dimensional.

**References**


A property of nilpotent ideals in certain Banach algebras


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On introducing a class of semiorthogonal wavelets and their applications

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Abstract

Based on polynomials, we construct a class of wavelets and their duals which have semiorthogonal continuous scaling functions. Since we can compute boundary conditions from their continuity property, we can apply them for approximation aims. As an example we use them for solution of integral equations.

Keywords: Integral equations, B-spline scaling function, dual functions, wavelets

1 Introduction

Wavelet theory is a new, effective and practical subject in different areas of mathematics, physics, engineering and so on. Classical way to construct wavelets goes through MRAs, which is a procedure for constructing wavelets from a scaling function. This procedure provides an orthonormal basis for $L^2(\mathbb{R})$ with vanishing moment and small supports. These properties lead one to think of wavelets and their scaling function as a good candidate for numerical aspects which need fast algorithms and sparse matrix in their routines. On the other hand the theory and application of integral equation is an important subject in many areas of engineering, signal processing, physics and applied mathematics. Often we would like to solve an integral or differential equation which is arisen from a natural or practical problem. Several numerical or analytical methods for approximating the solution of integral equations are known. Some of these methods transform a given integral equation into a system of nonlinear equations, while another, like applying orthonormal bases, reduce them to a linear system of algebraic equations [1, 6, 7]. In this work we apply compactly supported semiorthogonal B-spline father wavelets, specially constructed for the bounded interval $[0, 1]$ to solve the second kind linear Fredholm integral equations of the form

$$g(x) = f(x) + \int_0^1 k(x, t)g(t)dt, \quad 0 \leq x \leq 1,$$

where $f$ and $k$ are given continuous functions and $g$ is an unknown function to be determined. It is known that semiorthogonal wavelets are more better furnished to solving integral equations applications than orthogonal ones. There are different works in this area [1, 5]. In [2] we have constructed B-spline scaling functions of order 5 (that we call them pantic B-splines) and their dual, and we have shown that they provide better approximation results for solution of integral equations, in comparison with less degrees or other kind of scaling functions. Here we use these class of functions for solving different examples in different level. Semiorthogonal compactly supported B-spline wavelets have interesting properties such as, in a bounded interval, they behave
more better and easier than other wavelets in boundary conditions. Based on our method the integral equation will be reduced to a set of algebraic equations by expanding \( g \) in terms of pantic B-spline scaling functions, with unknown coefficients. This coefficients then will be determined from the properties of desired B-splines.

2 Constructing function space

When we deal with semiorthogonal B-spline scaling functions on the entire real line, we should be sure that they are not outside the domain of problem. Here we construct compactly supported B-spline functions on the interval \([0, 1]\). Also we need another condition to be sure that there exist at least one complete inner scaling function, that is B-spline scaling functions of order \( n \) should satisfy \( 2^J \geq n \). We use B-spline order 5 (degree 4), so the pantic B-spline of lowest level, which have to be integer, is \( J = 3 \). This contains all octave levels to \( J \geq 3 \). It is well known that B-spline functions of order \( n \) are determined by \( B_n = B_{n-1} \ast B_1 \) where \( B_1(x) = \chi_{[0,1]} \) and \( \ast \) denotes the convolution of functions. The two scale relation \( \phi_{jk}(x) = \phi(2^j x - k) \) describes all scaling B-spline functions, and characterization of a function with these scaling functions is well-known \([\text{?}]\). Now we introduce the pantic B-spline functions which are computed from characteristic functions and the convolution mentioned above and also applying boundary conditions:

\[
\phi(x) = \begin{cases} 
\frac{-1}{6} x^4 + \frac{1}{24} x^3 - \frac{5}{6} x^2 + \frac{5}{6} x - \frac{5}{24} & 0 \leq x < 1 \\
\frac{1}{24} x^4 - \frac{5}{6} x^3 + \frac{25}{6} x^2 - \frac{125}{6} x + \frac{625}{24} & 1 \leq x < 2 \\
\vdots \\
\frac{1}{24} x^4 - \frac{5}{6} x^3 + \frac{25}{6} x^2 - \frac{125}{6} x + \frac{625}{24} & 4 \leq x < 5 \\
\text{otherwise} 
\end{cases}
\]

So the corresponding scaling function is:

\[
\phi_{jk}(x) = \frac{\{1/24(2^j x - k)^4\}_{\chi_{[k,k+1]}}}{I} + \cdots + \{1/24(2^j x - k)^4 - 5/6(2^j x - k)^3 + 25/4(2^j x - k)^2 - 125/6(2^j x - k) + 625/24\}_{\chi_{[k+4,k+5]}} 
\]

\[(2)\]

otherwise they are zero. Their left and right-hand side boundary conditions are given below. \( x_j \) denotes \( 2^j x \).

\( k = -4 \):

\[
\phi_{j,k}(x) = \begin{cases} 
V & 0 \leq x_j < 1 \\
0 & \text{otherwise,} 
\end{cases}
\]

\( k = 2^j - 2 \):

\[
\phi_{j,k}(x) = \begin{cases} 
II & k + 1 \leq x_j < k + 2 \\
I & k \leq x_j < k + 1 \\
0 & \text{otherwise,} 
\end{cases}
\]

\[
\vdots 
\]
we have which is used for finding dual functions. Now we are ready to find the desired algebraic equations. Let

\[ f(x) = \sum_{k=-4}^{2^M-1} a_k \phi_{Mk} = A \Phi_M. \]  

(1)

Then we have the multiwavelet \( \Phi_M \) such that

\[ \Phi_M = [\phi_{M,-4}, \phi_{M,-3}, \ldots, \phi_{M,2^M-1}]^T \]

(2)

and \( M \) is an arbitrary integer such that \( M \geq J \). Let matrix \( \tilde{\Phi}_M \), denotes the dual of \( \Phi_M \). So \( \tilde{\Phi}_M = [\tilde{\phi}_{M,-4}, \tilde{\phi}_{M,-3}, \ldots, \tilde{\phi}_{M,2^M-1}]^T \). If \( \phi_{M,k} \) are the dual functions, then from semiorthogonality condition

\[ \int_0^1 \tilde{\Phi}_M \Phi_M^T dx = I_{2^M+3}, \]

we have

\[ a_k = \int_0^1 f(x) \tilde{\phi}_{M,k}(x) dx, \quad k = -3, \ldots, 2^M - 1. \]

(2)

Let

\[ P_M = \int_0^1 \Phi_M \Phi_M^T \]

(2)

where \((P_M)_{ij} = \int_0^1 \phi_{M,i} \phi_{M,j}^T dx\) is \(ij\)-th entry of matrix \( P_M \). First we compute \( \tilde{\Phi}_M = (P_M)^{-1} \Phi_M \), which is used for finding dual functions. Now we are ready to find the desired algebraic equations. Considering Fredholm integral equation (1), first we expand \( y \) with scaling functions i.e.

\[ g(x) = A_y \Phi_M(x) \]

(2)

where \( \Phi_M \) is given in (2) and \( A_y \) is a \( 1 \times 2^{M+1} \) matrix of unknown coefficients, similar to \( A \). Also we describe \( f(x) \) and \( k(x,t) \) by B-spline dual functions \( f(x) = A_f \tilde{\Phi}_M \), and \( k(x,t) = \tilde{\Phi}_M(t) \Gamma \tilde{\Phi}_M(x) \) on which \( \Gamma_{ij} = \int_0^1 (\int_0^1 k(x,t) \tilde{\phi}_i(t) dt) \tilde{\phi}_j(x) dx \), and finally we get

\[ \int_0^1 k(x,t) g(t) dt = \int_0^1 \tilde{\Phi}_M^T(t) \Gamma \tilde{\Phi}_M(x) A_y \Phi_M(t) dt = A_y \tilde{\Phi}_M(x). \]

So by applying all these equations in (1), corresponding integral equation will transform to \( A_y \Phi_M(x) = A_y \Gamma \tilde{\Phi}_M(x) + A_f \tilde{\Phi}_M(x) \). Now we multiply \( \tilde{\Phi}_M(x) \) by this equation and integrate with respect to \( x \), so \( A_y P_M \Gamma = A_y \), or \( A_y = A_f (P_M - \Gamma)^{-1} \). The last equation provides desired algebraic equations for solving \( g(x) = A_y \Phi_M(x) \).

3 Main Result

As an example we compute matrix \( P_4 \) according to above algorithm, and then we will apply it for a known integral equation.

\[
\begin{bmatrix}
0.0063 & 0.0135 & \ldots & 0 \\
0.0135 & \ldots & 0.0625 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0.0135 & 0.0063
\end{bmatrix}
\]

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First we compute $A_f$ and $\Gamma$, then $A_g$ is in hand. Finally $g(x) = A_g \Phi_M$.

Numerical results are computed for some integral equation via following example.

**Example 3.1.** Let us consider the integral equation

$$g(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{g(t)}{\sqrt{x-t}}dt, 0 \leq x \leq 1$$

With $g(x) = x^2$ as exact solution.

**References**


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Multipliers of controlled $G$-frames

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Abstract
Multipliers have been recently introduced by P. Balazs as operators for Bessel sequences and frames in Hilbert spaces. These are operators that combine (frame-like) analysis, a multiplication with a fixed sequence (called the symbol) and synthesis. Weighted and controlled frames have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator. Also $g$-frames are the most popular generalization of frames that include almost all of the frame extensions. In this manuscript multiplier operator for controlled $g$-frames will be defined and some of its properties will be shown.

Keywords: frame; $g$-frame; $g$-Bessel; $g$-Riesz basis; $g$-orthonormal basis; multiplier; Schatten $p$-class; Hilbert-Schmidt; trace class; controlled frames; weighted frame; controlled $g$-frame, $(C, C')$-controlled $g$-frame, $(C, C')$-controlled $g$-multiplier operator.

Mathematics Subject Classification: 42C15, 41A58, 47A58.

1 Introduction

Gabor multipliers [5] led to the introduction of Bessel and frame multipliers for abstract Hilbert spaces $H_1$ and $H_2$. These operators are defined by a fixed multiplication pattern (the symbol) which is inserted between the analysis and synthesis operators.

Definition 1.1. Let $H_1$ and $H_2$ be Hilbert spaces, let $(\psi_k) \subseteq H_1$ and $(\phi_k) \subseteq H_2$ be Bessel sequences. Fix $m = (m_k) \in l^\infty$. The operator $M_{m, (\phi_k), (\psi_k)} : H_1 \to H_2$ defined by

$$ M_{m, (\phi_k), (\psi_k)}(f) = \sum_k m_k \langle f, \psi_k \rangle \phi_k \quad \forall f \in H $$

is called the Bessel multiplier for the Bessel sequences $(\psi_k)$ and $(\phi_k)$. The sequence $m$ is called the symbol of $M$.

Several basic properties of these operators were investigated in [1]. Multipliers are not only interesting from a theoretical point of view, but they are also used in applications, in particular in the field of audio and acoustic. They have been investigated for $g$-frames [9], $p$-frames in Banach spaces [10], and for Banach frames [4].

1.1 $G$-Frames

$G$-frames, introduced by W. Sun in [11] and improved in [8, 6, 7], are a natural generalization of frames which cover many other extensions of frames.
Definition 1.2. A sequence $\Lambda = \{ \Lambda_i : i \in I \}$ is called generalized frame, or simply a \textit{g-frame}, for $\mathcal{H}$ with respect to $\{ \mathcal{H}_i : i \in I \}$ if there exist constants $A > 0$ and $B < \infty$ such that
\[
A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.
\] The numbers $A$ and $B$ are called g-frame bounds.

$\Lambda = \{ \Lambda_i : i \in I \}$ is called \textit{tight g-frame} if $A = B$ and Parseval g-frame if $A = B = 1$. If the second inequality in (1) holds, the sequence is called g-Bessel sequence.

The operator $S_\Lambda f = \sum_i \Lambda_i^* \Lambda_i f$ is called the g-frame operator of $\Lambda = \{ \Lambda_i : i \in I \}$ and it is known for a g-frame $\Lambda$ that $S_\Lambda$ is a positive and invertible and $AI \leq S_\Lambda \leq BI$, and every $f \in \mathcal{H}$ has an expansion $f = \sum_i \Lambda_i^* S_\Lambda^{-1} f$.

1.2 Controlled g-frames

Weighted and controlled frames have been introduced recently to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces [2], however they are used earlier in [3] for spherical wavelets. In this section, the concepts of controlled frames and controlled Bessel sequences will be extended to g-frames and we will show that controlled g-frames are equivalent g-frames.

Definition 1.3. Let $C, C' \in \mathcal{GL}^+(\mathcal{H})$. The family $\Lambda = \{ \Lambda_i : i \in I \}$ will be called a $(C, C')$-controlled g-frame for $\mathcal{H}$, if $\Lambda$ is a g-Bessel sequence and there exist constants $A > 0$ and $B < \infty$ such that
\[
A\|f\|^2 \leq \sum_{i \in I} \langle \Lambda_i C f, \Lambda_i C' f \rangle \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.
\]

$A$ and $B$ will be called controlled frame bounds. If $C' = I$, we call $\Lambda = \{ \Lambda_i \}$ a $C$-controlled g-frame for $\mathcal{H}$ with bounds $A$ and $B$. If the second part of the above inequality holds, it will be called $(C, C')$-controlled g-Bessel sequence with bound $B$.

We call the $(C, C)$-controlled Bessel sequence and $(C, C)$-controlled g-frame, $C^2$-controlled Bessel sequence and $C^2$-controlled g-frame with bounds $A$, $B$.

For a $(C, C')$-controlled Bessel sequence $\Lambda = \{ \Lambda_i : i \in I \}$, the operator
\[
L_{C\Lambda C'} : \mathcal{H} \to \mathcal{H}, \quad L_{C\Lambda C'} f := \sum_{i \in I} C' \Lambda_i^* C f,
\]
is called the $(C, C')$-controlled Bessel sequence operator, also $L_{C\Lambda C'} = CS_\Lambda C'$. It follows from the definition that for a g-frame, this operator is positive and invertible and $AI \leq L_{C\Lambda C'} \leq BI$. Also, if $C$ and $C'$ commute with each other, then $C', C'^{-1}, C, C^{-1}$ commute with $L_{C\Lambda C'}, CS_\Lambda, S_\Lambda C'$. The following proposition shows that any g-frame is a controlled g-frame and versa. This is the most important result of weighted and controlled g-frame in the sense of precondition.

Proposition 1.4. Let $C \in \mathcal{GL}^+(\mathcal{H})$. The family $\Lambda = \{ \Lambda_i : i \in I \}$ is a g-frame if and only if $\Lambda$ is a $C^2$-controlled g-frame.

Proposition 1.5. Assume that $\Lambda = \{ \Lambda_i : i \in I \}$ is a g-frame and $C, C' \in \mathcal{GL}^+(\mathcal{H})$, which commute with each other and commute with $S_\Lambda$. Then $\Lambda = \{ \Lambda_i : i \in I \}$ is a $(C, C')$-controlled g-frame.

2 Multipliers of Controlled g-Frame

Extending the concept of multipliers of frames, in this section, we will define controlled g-frame’s multiplier for $C$-controlled g-frames in Hilbert spaces. The definition of general case $(C, C')$-controlled g-frames goes smooth.
**Definition 2.1.** Let \( C, C' \in GL^+ (\mathcal{H}) \) and \( \Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \} \), \( \Theta = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) be \( C^2 \) and \( C' \)-controlled \( g \)-Bessel sequences for \( \mathcal{H} \), respectively. Let \( m \in \ell^\infty \). The operator \( M_m C \Theta \Lambda C' : \mathcal{H} \to \mathcal{H} \) defined by

\[
M_m C \Theta \Lambda C' f := \sum_{i \in I} m_i C \Theta_i^* \Lambda_i C' f, \quad \forall f \in \mathcal{H}
\]

is called the \((C, C')\)-controlled multiplier operator with symbol \( m \).

**Theorem 2.2.** Let \( \Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) and \( \Theta = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) be controlled \( g \)-Bessel sequences for \( \mathcal{H} \). If \( m = (m_i) \in \ell^p \) and \( (\text{dim} \mathcal{H}_i)_{i \in I} \in \ell^\infty \), then \( M_m C \Theta \Lambda C' \) is a Schatten \( p \)-class operator.

And

**Corollary 2.3.** Let \( \Lambda = \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) and \( \Theta = \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) be controlled \( g \)-Bessel sequences for \( \mathcal{H} \).

1. If \( m = (m_i) \in \ell^1 \) and \( (\text{dim} \mathcal{H}_i)_{i \in I} \in \ell^\infty \), then \( M_m C \Theta \Lambda C' \) is a trace-class operator.
2. If \( m = (m_i) \in \ell^2 \) and \( (\text{dim} \mathcal{H}_i)_{i \in I} \in \ell^\infty \), then \( M_m C \Theta \Lambda C' \) is a Hilbert-Schmit operator.

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Extension of \( p g \)-frames via Bochner spaces

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Abstract
In this talk we introduce the continuous version of \( p g \)-frames for Banach spaces, say Bochner \( p g \)-frames. Then we characterize the Bochner \( p g \)-frames and specify the optimal bounds of a Bochner \( p g \)-frame. Also, we define a Bochner \( q g \)-Riesz basis and verify the relations between Bochner \( p g \)-frames and Bochner \( q g \)-Riesz bases. Finally, we discuss the perturbation of Bochner \( p g \)-frames.

Keywords: Banach space, Hilbert space, Frame, Bochner measurable, Bochner \( pg \)-frame, Bochner \( pg \)-Bessel family, Bochner \( q g \)-Riesz basis.

Mathematics Subject Classification: 42C15

1 Introduction

The concept of frames (discrete frames) in Hilbert spaces has been introduced by Duffin and Schaeffer [5] in 1952 to study some deep problems in non-harmonic Fourier series. After the fundamental paper [3] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames. Frames play a fundamental role in signal processing, image and data compression and sampling theory. They provided an alternative to orthonormal bases, and have the advantage of possessing a certain degree of redundancy. A discrete frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements. For more details about discrete frames see [2].

Throughout this manuscript, \( X \) and \( H \) will be a Banach space and a Hilbert space, respectively, and \( \{H_\omega\}_{\omega \in \Omega} \) is a family of Hilbert spaces.

Definition 1.1. A function \( f : \Omega \to X \) is called Bochner measurable if there exists a sequence of simple functions \( \{f_n\}_{n=1}^\infty \) such that \( \lim_{n \to \infty} \|f_n(\omega) - f(\omega)\| = 0 \), a.e. \( [\mu] \).

Remark 1.2. Suppose that \( (\Omega, \Sigma, \mu) \) is a measure space and \( X^* \) has the Radon-Nikodym property. Let \( 1 \leq p \leq \infty \). The Bochner space of \( L^p(\mu, X) \) is defined to be the Banach space of (equivalence classes of) \( X \)-valued Bochner measurable functions \( F \) from \( \Omega \) to \( X \) for which the norms

\[
\|F\|_p = \left( \int_{\Omega} \|F(\omega)\|^p d\mu(\omega) \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty
\]

\[
\|F\|_\infty = \text{ess sup}_{\omega \in \Omega} \|F(\omega)\|, \quad p = \infty
\]

are finite. In [4], [1] and [6, p.51] it is proved that if \( 1 \leq p < \infty \) and \( q \) is such that \( \frac{1}{p} + \frac{1}{q} = 1 \), then \( L^q(\mu, X^*) \) is isometrically isomorphic to \( (L^p(\mu, X))^* \) if and only if \( X^* \) has the Radon-Nikodym property. This isometric isomorphism is the mapping

\[
\psi : L^q(\mu, X^*) \to (L^p(\mu, X))^*
\]
where the mapping $\psi(g)$ is defined on $L^p(\mu, X)$ by

$$
\psi(g)(f) = \int_{\Omega} g(\omega)(f(\omega))d\mu(\omega), \quad f \in L^p(\mu, X).
$$

So for all $f \in L^p(\mu, X)$ and $g \in L^q(\mu, X^*)$ we have

$$
< f, \psi(g) > = \int_{\Omega} < f(\omega), g(\omega) > d\mu(\omega).
$$

In the following, we use the notation $< f, g >$ instead of $< f, \psi(g) >$, so for all $f \in L^p(\mu, X)$ and $g \in L^q(\mu, X^*)$

$$
< f, g > = \int_{\Omega} < f(\omega), g(\omega) > d\mu(\omega).
$$

## 2 Bochner pg-Frames and Bochner qg-Riesz bases

**Definition 2.1.** Let $1 < p < \infty$. The family $\{\Lambda_\omega \in B(X, H_\omega) : \omega \in \Omega\}$ is a Bochner pg-frame for $X$ with respect to $\{H_\omega\}_{\omega \in \Omega}$ if:

(i) For each $x \in X$, $\omega \mapsto \Lambda_\omega(x)$ is Bochner measurable,

(ii) there exist positive constants $A$ and $B$ such that

$$
A\|x\| \leq \left( \int_{\Omega} \|\Lambda_\omega(x)\|^p d\mu(\omega) \right)^{\frac{1}{p}} \leq B\|x\|, \quad x \in X. \quad (1)
$$

$A$ and $B$ are called the lower and upper Bochner pg-frame bounds, respectively. We call $\{\Lambda_\omega\}_{\omega \in \Omega}$ a tight Bochner pg-frame if $A$ and $B$ can be chosen such that $A = B$, and a Parseval Bochner pg-frame if $A$ and $B$ can be chosen such that $A = B = 1$. If for each $\omega \in \Omega$, $H_\omega = H$ then $\{\Lambda_\omega\}_{\omega \in \Omega}$ is called a Bochner pg-frame for $X$ with respect to $H$. A family $\{\Lambda_\omega \in B(X, H_\omega) : \omega \in \Omega\}$ is called a Bochner pg-Bessel family for $X$ with respect to $\{H_\omega\}_{\omega \in \Omega}$ if the right inequality in (1) holds. In this case, $B$ is called the Bessel bound.

In the following some corresponding operators for a Bochner pg-frame are defined.

**Definition 2.2.** Let $\{\Lambda_\omega\}_{\omega \in \Omega}$ be a Bochner pg-Bessel family for $X$ with respect to $\{H_\omega\}_{\omega \in \Omega}$ and $q$ be the conjugate exponent of $p$. We define the operators $T$ and $U$, by

$$
T : L^q(\mu, \oplus_{\omega \in \Omega} H_\omega) \longrightarrow X^*
$$

$$
< x, TG > = \int_{\Omega} < \Lambda_\omega(x), G(\omega) > d\mu(\omega), \quad x \in X, \ G \in L^q(\mu, \oplus_{\omega \in \Omega} H_\omega), \quad (2)
$$

$$
U : X \longrightarrow L^p(\mu, \oplus_{\omega \in \Omega} H_\omega)
$$

$$
< Ux, G > = \int_{\Omega} < \Lambda_\omega(x), G(\omega) > d\mu(\omega), \quad x \in X, \ G \in L^q(\mu, \oplus_{\omega \in \Omega} H_\omega). \quad (3)
$$

The operators $T$ and $U$ are called the synthesis and analysis operators of $\{\Lambda_\omega\}_{\omega \in \Omega}$, respectively.

It is easy to show that these operators are well-defined and bounded.

**Definition 2.3.** Let $1 < q < \infty$. A family $\{\Lambda_\omega \in B(X, H_\omega) : \omega \in \Omega\}$ is called a Bochner qg-Riesz basis for $X^*$ with respect to $\{H_\omega\}_{\omega \in \Omega}$ if:

(i) $\{x : \Lambda_\omega x = 0, \text{ a.e. } [\mu] = \{0\}\}$.

(ii) For each $x \in X$, $\omega \mapsto \Lambda_\omega(x)$ is Bochner measurable and the operator $T$ defined by (2) is well-defined and there are positive constants $A$ and $B$ such that

$$
A\|G\|_q \leq \|TG\| \leq B\|G\|_q, \quad G \in L^q(\mu, \oplus_{\omega \in \Omega} H_\omega).
$$

$A$ and $B$ are called the lower and upper Bochner qg-Riesz basis bounds of $\{\Lambda_\omega\}_{\omega \in \Omega}$, respectively.
3 Main Results

The following theorem characterizes Bochner pg-frames by the synthesis operator.

**Theorem 3.1.** Consider the family \( \{ \Lambda_\omega \in B(X, H_\omega) : \omega \in \Omega \} \).
(i) Let \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) be a Bochner pg-frame for \( X \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \). Then the operator \( T \) defined by (2) is a surjective bounded operator.
(ii) Let \( (\Omega, \Sigma, \mu) \) be a measure space where \( \mu \) is \( \sigma \)-finite and for each \( x \in X \), \( \omega \mapsto \Lambda_\omega(x) \) be Bochner measurable. Let the operator \( T \) defined by (2) be a surjective bounded operator. Then \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a Bochner pg-frame for \( X \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \).

The next theorem gives us the optimal bounds of a Bochner pg-frame.

**Theorem 3.2.** Let \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) be a Bochner pg-frame for \( X \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \). Then \( \|T\| \) and \( \|\hat{U}\| \) are the optimal upper and lower Bochner pg-frame bounds of \( \{ \Lambda_\omega \}_{\omega \in \Omega} \), respectively, which \( \hat{U} \) is the inverse of \( U \) on \( R_U \) and \( T \), \( U \) are synthesis and analysis operators of \( \{ \Lambda_\omega \}_{\omega \in \Omega} \), respectively.

The next theorem shows that we can find some conditions that a Bochner pg-Riesz basis is a Bochner pg-frame.

**Theorem 3.3.** Suppose \((\Omega, \Sigma, \mu)\) is a measure space where \( \mu \) is \( \sigma \)-finite and consider the family \( \{ \Lambda_\omega \in B(X, H_\omega) : \omega \in \Omega \} \).
(i) Assume that for each \( x \in X \), \( \omega \mapsto \Lambda_\omega(x) \) is Bochner measurable. \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a Bochner pg-Riesz basis for \( X^* \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) if and only if the operator \( T \) defined by (2) is an invertible bounded operator from \( L^p(\mu, \oplus_{\omega \in \Omega} H_\omega) \) onto \( X^* \).
(ii) Let \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) be a Bochner pg-Riesz basis for \( X^* \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) with optimal upper Bochner pg-Riesz basis bound \( B \). If \( p \) is the conjugate exponent of \( q \) then \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a Bochner pg-frame for \( X \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) with optimal upper Bochner pg-frame bound \( B \).

Now, we state some equivalent conditions for a Bochner pg-frame being a Bochner pg-Riesz basis.

**Theorem 3.4.** Suppose \((\Omega, \Sigma, \mu)\) is a measure space where \( \mu \) is \( \sigma \)-finite. Let \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) be a Bochner pg-frame for \( X \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) with synthesis operator \( T \) and analysis operator \( U \) and \( q \) be conjugate exponent of \( p \). Then the following statements are equivalent:
(i) \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a Bochner pg-Riesz basis for \( X^* \).
(ii) \( T \) is injective.
(iii) \( R_U = L^p(\mu, \oplus_{\omega \in \Omega} H_\omega) \).

In the following theorem we discuss the perturbation of Bochner pg-frames.

**Theorem 3.5.** Suppose \((\Omega, \Sigma, \mu)\) is a measure space where \( \mu \) is \( \sigma \)-finite. Let \( \{ \Lambda_\omega \}_{\omega \in \Omega} \) be a Bochner pg-frame for \( X \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) and \( q \) be the conjugate exponent of \( p \). Let \( \{ \Theta_\omega \in B(X, H_\omega) : \omega \in \Omega \} \) be a family such that for all \( x \in X \), \( \omega \mapsto \Theta_\omega(x) \) is Bochner measurable. Assume that there exist constants \( \lambda_1, \lambda_2, \gamma \) such that \( 0 \leq \lambda_2 < 1 \), \(-\lambda_2 \leq \lambda_1 < 1 \), \( 0 \leq \gamma < (1 - \lambda_1 - 2\lambda_2)A \) and

\[
| \int_\Omega < (\Lambda_\omega - \Theta_\omega)x, G(\omega) > \mu(\omega) | \\
\leq \lambda_1 \int_\Omega < \Lambda_\omega x, G(\omega) > d\mu(\omega) + \lambda_2 \int_\Omega < \Theta_\omega x, G(\omega) > d\mu(\omega) + \gamma \| G \|_q,
\]

for all \( G \in L^q(\mu, \oplus_{\omega \in \Omega} H_\omega) \) and \( x \in X \). Then \( \{ \Theta_\omega \}_{\omega \in \Omega} \) is a Bochner pg-frame for \( X \) with respect to \( \{ H_\omega \}_{\omega \in \Omega} \) with bounds

\[
A \left[ \frac{(1 - \lambda_1 - 2\lambda_2) - \gamma}{1 - \lambda_2} \right] \quad \text{and} \quad B \left[ \frac{1 + \lambda_1 + \frac{\gamma}{2}}{1 - \lambda_2} \right],
\]

where \( A \) and \( B \) are the Bochner pg-frame bounds for \( \{ \Lambda_\omega \}_{\omega \in \Omega} \).
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Metric convexity and $F$-spaces

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Abstract

In this article, first we state the notions of metric convexity of sets in the sense of Manger [1] and in the sense of Soltan [5]. Then we review some properties of convex sets and convex halls in the sense of Soltan. Finally, we study the relation between metric convexity and geometric properties of $F$-spaces.

Keywords: Metric convexity, Metric segment, $F$-space, Strictly convex, Pseudo strictly convex.

Mathematics Subject Classification: 52A01, 54G05, 26A51

1 Introduction

Let $(X,d)$ be a metric space and $x,y \in X$. A metric segment between $x$ and $y$ is defined by:

$[x,y] := \{z \in X : d(x,z) + d(z,y) = d(x,y)\}.$

It is easy to check that the metric segment in any normed space is convex.

Let $(\mathbb{R}^2, \|\cdot\|_2)$, where $\|\cdot\|_2$ is a euclidean norm. Then the metric segment between $x,y \in \mathbb{R}^2$ is the line segment between them. But the metric segment between $x$ and $y$ in $(\mathbb{R}^2, \|\cdot\|_1)$, where $\|\cdot\|_1$ is defined by $\|(x_1,x_2)\|_1 = |x_1| + |x_2|$, is a rectangle with the diametral points $x$ and $y$.

The following statements are some properties of metric segments:

1) If $z \in [x,y]$, then $[x,z] \subset [x,y]$ and $[z,y] \subset [x,y]$.
2) If $z,t \in [x,y]$, then $d(z,t) \leq d(x,y)$. In particular, $\text{diam}[x,y] = d(x,y)$.
3) Suppose $z,t \in [x,y]$, then $t \in [x,z] \Rightarrow z \in [t,y]$.
4) Suppose $z,t \in [x,y]$, if $z \leq t \Rightarrow t \in [z,y]$ then $\leq$ is a partial order. By Zorn’s Lemma $[x,y]$ has a maximal totally ordered subset which is not necessarily unique.

Let $x,y \in X$. For each $0 \leq \lambda \leq 1$, the $\lambda$–section of $[x,y]$ is defined by:

$[x,y]^\lambda = \{z \in [x,y] : d(x,z) = \lambda d(x,y)\}.$

For all $x,y$ in metric space $X$, and $\lambda \in [0,1]$, the following properties of $\lambda$–section are easy to extract:

1) $[x,y]^\lambda = \bigcup_{0 \leq \lambda \leq 1} [x,y]^\lambda$;
2) $[x,y]^\lambda = [y,x]^{1-\lambda}$;

and if $X$ is a normed space

3) $[x,y]^\lambda$ is convex;
4) $[x,y]^\lambda$ is singleton if and only if $X$ is strictly convex.

Definition 1.1. A convex subset of a metric space: $\text{In the sense of Menger}[1]$ (for short, M-convex) is defined by:

"A subset $C \subset X$ is metrically convex if $([x,y] - \{x,y\}) \cap C \neq \emptyset$, for every pair of distinct points.}
A simple example shows that balls in a metric space are not necessarily S-convex.

**Remark 2.2.** It is obvious that $C \subseteq X$ is S-convex if and only if $h(C) = C$ but for an arbitrary subset $C$ of $X$, $h(C)$ is not necessary S-convex.

**Proposition 2.3.** $h(\bigcap_{\alpha} A_{\alpha}) \subseteq \bigcap_{\alpha} h(A_{\alpha})$. In particular, the intersection of a family of S-convex sets is S-convex set.

From the above definitions immediately obtains the following proposition.

**Proposition 2.4.** Let $(X,d)$ be a metric space and $C \subseteq X$. The following statements are hold:
1) $h(C) \subseteq S - co(C)$,
2) $S - co(C) = \bigcup_{n \geq 0} h^n(C)$,
3) $C$ is a S-convex set if and only if $S - co(C) = C$.

**Proposition 2.5.** Suppose that $B_r(x) = \{ y \in X; d(y,x) < r \}$. Then $h(B_r(x)) \subseteq B_{2r}(x)$.

Following example shows that $2r$ in the above proposition is optimal.
Example 2.6. Let $B_1^{||.||_1}(0)$ the unite ball of $(\mathbb{R}^2, ||.||_1)$. It is easy to see $h(B_1^{||.||_1}(0)) = B_1^{||.||_1}(0)$ which is not a subset of $B_1^{||.||_1}(0)$ for any $r < 2$. In general, the metric convex hull of the unite ball of $(\mathbb{R}^n, ||.||_1)$ is the unite ball of $(\mathbb{R}^n, ||.||_\infty)$.

The following theorem is the main result of this article.

**Theorem 2.7.** Suppose that $(X, d)$ be a F-space. The following statements are equivalent:

i) $(X, d)$ is pseudo strictly convex metric space,

ii) Every convex set in $X$ is $S$-convex.

**Proof.** i)$\Rightarrow$ii):

Suppose that $d$ is P.S.C and $C \subset X$ is convex and $x, y \in C$ and $z \in [x, y]$. Then

$$d(x, z) + d(z, y) = d(x, y).$$

By invariance of $d$ we have

$$d(x - z, 0) + d(z - y, 0) = d(x - y, 0).$$

By hypothesis there is $t > 0$ such that $x - z = t(z - y)$ therefore

$$z = \frac{1}{1 + t}x + \frac{t}{1 + t}y.$$ Since $C$ is convex then $z \in C$. It follows $[x, y] \subset C$. So, $C$ is $S$-convex.

ii)$\Rightarrow$i):

Suppose that $d(x + y, 0) = d(x, 0) + d(y, 0)$ that $x \neq 0 \neq y$. We have to show there is $t > 0$ such that $y = tx$. Take

$$C = \{\lambda x - (1 - \lambda)y : \lambda \in [0, 1]\}.$$ Then $C$ is convex. By hypothesis $C$ is $S$-convex. Then for each $u, v \in C$, $[u, v] \subset C$. Since $x, -y \in C$, then $[x, -y] \in C$. We have:

$$d(x - y, x) + d(x - y, -y) = d(x - y - x, 0) + d(x - y + y, 0) = d(0, y) + d(x, 0) = d(x + y, 0) = d(x, -y)$$

It shows $x - y \in [x, -y]$. Therefore $x - y \in C$. So there exists $\lambda \in (0, 1)$ such that $x - y = \lambda x - (1 - \lambda)y$, It follows $y = \frac{1 - \lambda}{\lambda}x$. This complete the proof. \(\square\)

**Remark 2.8.** Since pseudo strict convexity is equivalent to strict convexity in every norm vector space, (see,[2] pp. 180-181), So a norm vector space is strictly convex if and only if every convex set is $S$-convex.

**Remark 2.9.** It is easy to see that every $S$-convex set in a norm vector space is convex. But the following example shows that it is not true in F-spaces.

**Example 2.10.** Suppose that $(X, ||.||)$ is Banach space. Then $(X, d)$, where $d(x, y) = \ln(1 + ||x - y||)$ is obviously a F-space. Let $x, y \in X$. Take $z \in [x, y]$. Then

$$\ln(1 + ||x - z||) + \ln(1 + ||y - z||) = \ln(1 + ||x - y||) \Rightarrow 1 + ||x - z|| + ||z - y|| + ||x - z|| + ||z - y|| = 1 + ||x - y||.$$

It follows that $z = x$ or $z = y$. Then $[x, y] = \{x, y\}$. Then $\{x, y\}$ is $S$-convex but it is not convex in usual sense.

**References**


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Weighted composition operators between growth spaces and logarithmic growth space

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Abstract
Let $\Omega$ be a bounded, circular and strictly convex domain with the boundary of class $C^2$ in $\mathbb{C}^n$. Denote by $\mathcal{H}(\Omega)$ the space of all holomorphic functions on $\Omega$. For $\psi \in \mathcal{H}(\Omega)$ and a holomorphic self map $\varphi$ of $\Omega$, put $\psi C \varphi f = \psi(f \circ \varphi)$ for $f \in \mathcal{H}(\Omega)$. We characterize those $\psi$ and $\varphi$ for which the weighted composition operator $\psi C \varphi$ is a bounded operator between the growth space $A^{-\alpha}(\Omega)$, $\alpha > 0$, and the logarithmic growth space $A^{-\log}(\Omega)$. Also, we study the boundedness and compactness of weighted composition operators from $A^{-\log}(\Omega)$ to Bergman spaces in terms of $p$-Carleson measure.

Keywords: Weighted composition operator, Growth space, Logarithmic growth space, Bergman space, $p$-Carleson measure.

Mathematics Subject Classification: 47B38, 30H20.

1 Introduction
Let $n \in \mathbb{N}$ and $\Omega$ be a bounded, circular and strictly convex domain with the boundary of class $C^2$ in $\mathbb{C}^n$. We recall that a domain $\Omega \subset \mathbb{C}^n$ is strictly convex if it is convex and contains the open line segment $(z, w)$ for each pair of boundary points $z, w \in \partial \Omega$. Thus a convex domain is strictly convex if its boundary does not contain a line segment. The unit ball $\mathbb{B}_n = \{ z \in \mathbb{C}^n : |z| < 1 \}$ is an example of this kind of domain.

Let $r_{\Omega}(z)$ denote the Minkowski functional on $\Omega$, that is

$$ r_{\Omega}(z) = \inf\{ \lambda > 0 : \lambda^{-1} z \in \Omega \}. $$

Clearly $r_{\Omega}(z) < 1$ for all $z \in \Omega$. In the special case $\Omega = \mathbb{B}_n$, we have $r_{\Omega}(z) = |z|$.

Let $\mathcal{H}(\Omega)$ denote the space of all holomorphic functions on $\Omega$. Given $\alpha > 0$, the growth space $A^{-\alpha}(\Omega)$ is the space of all $f \in \mathcal{H}(\Omega)$ for which

$$ \| f \|_{-\alpha} = \sup_{z \in \Omega} (1 - r_{\Omega}(z))^{-\alpha} |f(z)| < \infty. $$

The space $A^{-\alpha}(\Omega)$, is a Banach space with the norm $\| \cdot \|_{-\alpha}$. In the special case $\Omega = \mathbb{B}_n$, the growth space $A^{-\alpha}(\mathbb{B}_n)$ is the standard weighted Banach space

$$ H^\infty_\alpha(\mathbb{B}_n) = \{ f \in \mathcal{H}(\mathbb{B}_n) : \sup_{z \in \Omega} (1 - |z|)^{-\alpha} |f(z)| < \infty \}. $$

For more details about spaces of this type we refer to [1].
The logarithmic growth space $A^{-\log}(\Omega)$ consists of those $f \in \mathcal{H}(\Omega)$ for which

$$\|f\|_{-\log} = \sup_{z \in \Omega} \frac{|f(z)|}{\log \frac{e}{1 - r_\Omega(z)}} < \infty.$$ 

The logarithmic growth space $A^{-\log}(\Omega)$ is a Banach space with the norm $\|\cdot\|_{-\log}$.

Let $\varphi$ be a holomorphic self map of $\Omega$ and $\psi \in \mathcal{H}(\Omega)$. Then the weighted composition operator $\psi C_\varphi$ on $\mathcal{H}(\Omega)$ is defined by $(\psi C_\varphi f)(z) = \psi(z) f(\varphi(z))$. When $\psi(z) \equiv 1$, the weighted composition operator $1 C_\varphi = C_\varphi$ is called the composition operator. It is easy to see that $\psi C_\varphi$ is linear. The main subject in the study of weighted composition operators is to describe operator theoretic properties of $\psi C_\varphi$ in terms of function theoretic properties of $\varphi$ and $\psi$. Dubtsov gave a characterization of the compact weighted composition operators on growth spaces and logarithmic growth space in the case $\Omega = \mathbb{B}_n$, in [3]. In this paper, we study the boundedness of weighted composition operators between growth and logarithmic growth spaces, also from logarithmic growth space to Bergman spaces.

For $0 < p < \infty$, the Bergman space $A^p(\Omega)$ consists of those $f \in \mathcal{H}(\Omega)$ for which

$$\|f\|^p_{A^p(\Omega)} := \int_{\Omega} |f(z)|^p dA(z) < \infty,$$

where $dA(z)$ is the normalized Lebesgue measure on $\Omega$. It is well known that $A^p(\Omega)$ is a Banach space when $p \geq 1$, and in the case that $0 < p < 1$, it is a complete metric space with the distance $d(f,g) = \|f - g\|^p_{A^p(\Omega)}$.

For $0 < p < \infty$, a positive Borel measure $\mu$ on $\Omega$ is called $p$-Carleson measure for $A^{-\log}(\Omega)$ provided that the identity operator, $I : A^{-\log}(\Omega) \to L^p(\Omega, \mu)$ is a bounded operator. The following characterization was proved in [2].

**Lemma 1.1.** Let $0 < p < \infty$ and $\mu$ be a positive Borel measure on $\Omega$. Then the following statements are equivalent.

(i) The measure $\mu$ is a $p$-Carleson measure for $A^{-\log}(\Omega)$.

(ii) $\int_{\Omega} (\log \frac{e}{1 - r_\Omega(z)})^p d\mu(z) < \infty.$

(iii) $I : A^{-\log}(\Omega) \to L^p(\Omega, \mu)$ is a compact operator.

To prove the main results of this paper we need the following lemma.

**Lemma 1.2.** [2, Lemma 3.3SH. Rezaei and H. Mahyar] There exists $M = M(\Omega) \in \mathbb{N}$ such that the following properties hold.

(i) For $\alpha > 0$, there exist functions $f_m \in A^{-\alpha}(\Omega)$, $0 \leq m \leq M$, such that

$$\frac{1}{M} \sum_{m=0}^{M} |f_m(z)| \geq \frac{1}{(1 - r_\Omega(z))^\alpha}, \quad z \in \Omega.$$

(ii) There exist functions $h_m \in A^{-\log}(\Omega)$, $0 \leq m \leq M$, such that

$$\frac{1}{M} \sum_{m=0}^{M} |h_m(z)| \geq \log \frac{e}{1 - r_\Omega(z)}, \quad z \in \Omega.$$
2 Main Results

We characterize the boundedness of weighted composition operators between these spaces.

**Theorem 2.1.** Let $\alpha > 0$, $\psi \in \mathcal{H}(\Omega)$ and $\varphi$ be a holomorphic self map of $\Omega$. Then $\psi C_\varphi(\mathcal{A}^{-\alpha}(\Omega)) \subseteq \mathcal{A}^{-\log}(\Omega)$ if and only if

$$\sup_{z \in \Omega} \frac{|\psi(z)|}{(\log \frac{e}{1-\theta(z)})^{\alpha}(1 - r_\Omega(\varphi(z)))^\alpha} < \infty. \quad (1)$$

**Proof.** Suppose that $\psi C_\varphi(\mathcal{A}^{-\alpha}(\Omega)) \subseteq \mathcal{A}^{-\log}(\Omega)$. Using Lemma 1.2, there exist functions $f_m \in \mathcal{A}^{-\alpha}(\Omega)$, $0 \leq m \leq M$, satisfying

$$\sum_{m=0}^{M} |f_m(z)| \geq \frac{1}{(1 - r_\Omega(z))^\alpha}, \quad z \in \Omega. \quad (2)$$

Setting $z = \varphi(w)$ in (2), we get

$$\sum_{m=0}^{M} |f_m(\varphi(w))| \geq \frac{1}{(1 - r_\Omega(\varphi(w)))^\alpha}, \quad w \in \Omega.$$ 

Then

$$\frac{|\psi(w)|}{\log \frac{e}{1-r_\Omega(\varphi(w))}(1 - r_\Omega(\varphi(w)))^\alpha} \leq \sum_{m=0}^{M} \frac{|\psi(w)||f_m(\varphi(w))|}{\log \frac{e}{1-r_\Omega(\varphi(w))}} = \sum_{m=0}^{M} \frac{|\psi(z)||f_m(z)|}{\log \frac{e}{1-r_\Omega(z)}} \leq \sum_{m=0}^{M} \frac{|\psi(z)||f_m(z)|}{\log \frac{e}{1-r_\Omega(z)}}.$$ 

Since $f_m \in \mathcal{A}^{-\alpha}(\Omega)$ and $\psi C_\varphi(\mathcal{A}^{-\alpha}(\Omega)) \subseteq \mathcal{A}^{-\log}(\Omega)$, then $\psi C_\varphi f_m$ are in $\mathcal{A}^{-\log}(\Omega)$ for $0 \leq m \leq M$. This implies that (1) is satisfied. Conversely for $f \in \mathcal{A}^{-\alpha}(\Omega)$, we have

$$\|\psi C_\varphi f\|_{-\log} = \sup_{z \in \Omega} \frac{|\psi(z)f(\varphi(z))|}{\log \frac{e}{1-r_\Omega(z)}} = \sup_{z \in \Omega} \frac{|\psi(z)|}{(1 - r_\Omega(\varphi(z)))^\alpha \log \frac{e}{1-r_\Omega(z)}}(1 - r_\Omega(\varphi(z)))^\alpha |f(\varphi(z))|$$

$$\leq ||f||_{-\alpha} \sup_{z \in \Omega} \frac{|\psi(z)|}{(1 - r_\Omega(\varphi(z)))^\alpha \log \frac{e}{1-r_\Omega(z)}}, \quad (4)$$

which implies that $\psi C_\varphi(\mathcal{A}^{-\alpha}(\Omega)) \subseteq \mathcal{A}^{-\log}(\Omega)$. □

Moreover, the above argument shows that $\psi C_\varphi : \mathcal{A}^{-\alpha}(\Omega) \to \mathcal{A}^{-\log}(\Omega)$ is, in fact, bounded if and only if (1) is satisfied.

Similarly one can prove the following theorem.

**Theorem 2.2.** Let $\alpha > 0$, $\psi \in \mathcal{H}(\Omega)$ and $\varphi$ be a holomorphic self map of $\Omega$. Then $\psi C_\varphi(\mathcal{A}^{-\log}(\Omega)) \subseteq \mathcal{A}^{-\alpha}(\Omega)$ if and only if

$$\sup_{z \in \Omega} \frac{e}{(1 - r_\Omega(\varphi(z)))^\alpha} |\psi(z)| < \infty.$$
Finally we characterize the boundedness and compactness of weighted composition operators from logarithmic growth space into Bergman spaces. To do this we need the following definitions.

For $p \in [1, \infty)$ and $\psi \in \mathcal{H}(\Omega)$, we define the weighted Lebesgue measure $\mu_p^\psi$ by

$$d\mu_p^\psi(z) = |\psi(z)|^p dA(z).$$

Also for any holomorphic self-map $\varphi$ of $\Omega$, we define another measure $\mu_p^{\psi, \varphi} = \mu_p^\psi \circ \varphi^{-1}$ and call it the pull-back measure of $\mu_p^\psi$ induced by $\varphi$.

**Theorem 2.3.** Let $p \in [1, \infty)$, $\psi \in \mathcal{H}(\Omega)$ and $\varphi$ be a holomorphic self-map of $\Omega$. Then the following statements are equivalent:

(i) $\psi C_\varphi : A^{-\log} \rightarrow A^p(\Omega)$ is compact.

(ii) $\psi C_\varphi : A^{-\log} \rightarrow A^p(\Omega)$ is bounded.

(iii) $\mu_p^{\psi, \varphi}$ is a $p$-Carleson measure for $A^{-\log}(\Omega)$.

(iv) $\int_{\Omega} (\log \frac{c}{r_\Omega(\varphi(z))})^p |\psi(z)|^p dA(z) < \infty$.

(v) $I : A^{-\log} \rightarrow L^p(\Omega, \mu_p^{\psi, \varphi})$ is a compact operator.

**References**


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Near identity pair frames

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Abstract
Christensen and Laugensen[1] introduced the notion of “approximately dual frames” in the context of ordinary Bessel sequences. Here we present a new notion similar to that which is a more general context. Also some results in [1] are generalized.

Keywords: approximately dual frame, near identity pair frame, positively near identity pair frame

Mathematics Subject Classification: 42C15

1 Introduction

In a separable Hilbert space $H$, a sequence $\{f_n\}$ is said to be a frame for $H$ if there exist positive constants $A$, $B$ such that for every $f \in H$,

$$A \| f \|^2 \leq \sum_{n} |< f, f_n >|^2 \leq B \| f \|^2.$$  

Constants $A$ and $B$ are called lower and upper frame bounds, respectively. If the upper inequality holds, the sequence will be called a Bessel sequence. To each Bessel sequence there corresponds an operator $T : H \rightarrow l^2$ such that for each $f \in H$, $T(f) = \{< f, f_n >\}_n$. $T$ is a well-defined and bounded operator with the adjoint $T^* : l^2 \rightarrow H$, with $T^*(\{c_n\}) = \sum_n c_n f_n$. The operators $T$ and $T^*$ are called the analysis and synthesis operators of $\{f_n\}$, respectively. The operator $S = T^*T : H \rightarrow H$ is called the frame operator. $S$ is a bounded, invertible and positive operator. Let $\{f_n\}$ be a frame in $H$ and $\{g_n\}$ be any sequence in $H$. $\{g_n\}$ is said to be a dual of $\{f_n\}$ if for every $f \in H$, $f = \sum_n < f, f_n > g_n = \sum_n < f, g_n > f_n$. In recent years many different kinds of frames are studied. Generalized frames are of these kinds.

Definition 1.1. A sequence $\Lambda = \{\Lambda_n\}$ of bounded operators in $B(H)$ is called a generalized frame (abbreviated as g-frame) for $H$ if there exist positive constants $A$, $B$ such that for every $f \in H$,

$$A \| f \|^2 \leq \sum_{n} \| \Lambda_n f \|^2 \leq B \| f \|^2.$$  

Constants $A$ and $B$ are called g-frame bounds. If the right hand side inequality holds, $\Lambda$ is said to be a g-Bessel sequence. For simplicity, in this article we omit the prefix “g” and only we call them frames (or Bessels)

Proposition 1.2. A sequence $\Lambda$ in $B(H)$ is a Bessel sequence for $H$ iff the operator

$$S_\Lambda : f \mapsto \sum_n \Lambda_n^* \Lambda_n f,$$

is a well-defined operator from $H$ into $H$.  

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2 Main Result

Theorem 2.1. For a sequence $\Lambda$ of bounded operators on $H$ and its corresponding operator $S_\Lambda$, the followings are equivalent.

(1) $\Lambda$ is a frame.

(2) $S_\Lambda$ is well-defined and there exists $\alpha \in (0, \infty)$ such that

$$\| I - \alpha S \| < 1.$$ 

(3) $S_\Lambda$ is well-defined, bounded and invertible.

Now we state our new results. First we need to give a new definition.

Definition 2.2. For two sequences $\Lambda = \{\Lambda_n\}$ and $\Gamma = \{\Gamma_n\}$ of bounded operators and a sequence $m = \{m_n\}$ of scalars, we say that the triple $(m, \Gamma, \Lambda)$ is an $m$-pair Bessel for $H$ if the operator

$$S_{m\Gamma\Lambda} : H \to H, \quad S_{m\Gamma\Lambda}f = \sum_n m_n \Gamma^*_n \Lambda_n f,$$

is well-defined.

An $m$-pair frame is an $m$-pair Bessel for which the operator $S_{m\Gamma\Lambda}$ is invertible.

Christensen and Laugesen[1] introduced the notion of “approximately dual frame” as below. Suppose that $\{f_k\}$ and $\{g_k\}$ are Bessel sequences in $H$; we will denote their synthesis operators by $T : l^2 \to H$ and $U : l^2 \to H$, respectively. The operators $TU^* : H \to H$ and $UT^* : H \to H$ will be called mixed frame operators. Recall that $\{f_k\}$ and $\{g_k\}$ are dual frames when $TU^* = I$ or $UT^* = I$.

Definition 2.3. Bessel sequences $\{f_k\}$ and $\{g_k\}$ are said to be approximately dual frames if $\| I - TU^* \| < 1$ or $\| I - UT^* \| < 1$.

Subject to our new definition of $m$-pair frames, we give our new definition.

Definition 2.4. Let $\Gamma$, $\Lambda$, and $m$ be as above. A pair Bessel $(m, \Gamma, \Lambda)$ will be called a near identity pair frame if there exist a nonzero $\alpha \in \mathbb{C}$ such that

$$\| I - \alpha S_{m\Gamma\Lambda} \| < 1.$$ 

$(m, \Gamma, \Lambda)$ will be called a positively near identity pair frame if $\alpha \in (0, \infty)$ and $S_{m\Gamma\Lambda}$ is self-adjoint.

Now we state our new results.

Theorem 2.5. Let $\Gamma$, $\Lambda$, and $m$ be as above. $(m, \Gamma, \Lambda)$ is a positively near identity pair frame iff there are constants $A, B > 0$ such that

$$A \leq S_{m\Gamma\Lambda} \leq B.$$ 

Theorem 2.6. Every near identity pair frame, is a pair frame.

References


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The solution set of variational inequality related to the $B$-pseudomonotone mappings

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Abstract

In this paper, we give an extra comparison between $B$-pseudomonotone and $F$-hemicontinuous mapping on Banach spaces. Also various existence results on variational inequalities problems are given and it is shown that the solution set of variational inequality related to $B$-pseudomonotone and $F$-hemicontinuous mapping is weakly closed, and weakly compact when the mapping satisfies in some coercive conditions.

Keywords: Variational inequality, Brezis-Pseudomonotonicity, Fan-hemicontinuity.

Mathematics Subject Classification: 47H05

1 Introduction

The notion of a monotone map is generalized to that of pseudomonotone map in both topological and algebraic cases. The topological pseudomonotonicity was introduced by Brezis [1] in 1968 for nonlinear operators and it contains many monotone-like operators which were used by Minty in order to obtain existence theorems for quasi-linear elliptic and parabolic partial differential equations. The algebraic pseudomonotonicity was introduced by Karamardian and has been frequently used in optimization problems.

Recently, Maugeri and Raciti in [3] have shown that if $K$ is convex and closed and the mapping $A$ is $F$-hemicontinuous map, then $A$ is also $B$-pseudomonotone. As, we generalize some results in [3], we also give an extra comparison between $B$-pseudomonotone and $F$-hemicontinuous and analyze various existence results.

Let $X$ be a real reflexive Banach space with topological dual $E^*$ and $K$ a non-empty subset of $E$. Let $T : K \to E^*$ be a mapping. The variational inequality problem (VIP) is to find $u \in K$ such that $< T(u), v - u > \geq 0$, for all $v \in K$. We will show that if $A$ is $B$-pseudomonotone or $F$-hemicontinuous then the solution set of VIP, denoted by $S_{VIP}$, is weakly closed and also becomes weakly compact when the mapping $A$ satisfies in some coercive conditions.

2 Preliminaries and some results

**Definition 2.1.** [3] A map $A$ from $K$ to $E^*$ is called pseudomonotone in the sense of Brezis ($B$-pseudomonotone) if and only if

(a) For each sequence $u_n$ converges weakly to $u$ ($u_n \rightharpoonup u$) in $K$ and such that
lim \sup < Au_n, u_n - u > \leq 0 \text{ it results that: }
\lim \inf < Au_n, u_n - v >\geq< Au, u - v > , \quad \forall v \in K.
(b) For each \( v \in K \) the function \( u \mapsto < Au, u - v > \) is lower bounded on bounded subsets of \( K \).

Remark 2.2. In Definition 2.1, if \( \limsup < Au_n, u_n - u > \leq 0 \) with \( u_n \to u \), then we must have
\( \lim < Au_n, u_n - u > \geq 0 \). Indeed, from Condition (a), we have
\( \lim \inf < Au_n, u_n - v >\geq< Au, u - v > , \quad \forall v \in K. \)
Now, choose \( v = u \). Then \( \lim \inf < Au_n, u_n - u >\geq< Au, u - u > = 0 \), and so \( \lim < Au_n, u_n - u > \geq 0 \). Thus, it is reasonable to assume \( \limsup < Au_n, u_n - u > = 0 \) in the definition of \( B \)-pseudomonotonicity. These facts help us to recognize \( B \)-pseudomonotonicity of a mapping. For example: let \( f(x) = \sin(\frac{x}{2}) - \frac{1}{x} \) for \( x \neq 0 \) and \( f(x) = 0 \) for \( x = 0 \). As it is stated, for \( u = 0 \), \( \lim_{x \to 0} f(x) = -1 \), so the mapping \( f \) is not \( B \)-pseudomonotone.

Definition 2.3. [3] Let \( A : K \to E^* \) be a mapping from \( K \) to \( E^* \) is called \( F \)-hemicontinuous if and only if for all \( v \in K \) the function \( u \mapsto < Au, u - v > \) is weakly lower semicontinuous on \( K \).

Theorem 2.4. Let \( E \) be real reflexive Banach space and \( A : K \to E^* \) be an \( F \)-hemicontinuous mapping, then \( A \) is \( B \)-pseudomonotone.

Proof. By the \( F \)-hemicontinuity assumption, Condition (a) does hold. For Condition (b), by the definition of \( F \)-hemicontinuity, the function \( u \mapsto < Au, u - v > \) is real valued and weakly lower semicontinuous on \( K \). Let \( F \subseteq K \) be a bounded set, then \( \overline{F}^w \) is weakly compact. Thus the function \( u \mapsto < Au, u - v > \) attains a minimum value on \( \overline{F}^w \). Therefore, image of the bounded set under the \( F \)-hemicontinuous mapping is lower bounded, which implies that the mapping \( A \) satisfies Condition (b) of \( B \)-pseudomonotonicity.

Proposition 2.5. [1] Let \( K \) be an open subset of finite dimensional space \( E \) and \( A \) be a mapping from \( K \) to \( E^* \). Then \( A \) is continuous if and only if \( A \) is \( B \)-pseudomonotone.

Proposition 2.6. [3] Let \( K \) be a subset of finite dimensional space \( E \) and \( A \) be a continuous mapping from \( K \) to \( E^* \). Then \( A \) is \( F \)-hemicontinuous and \( B \)-pseudomonotone.

Corollary 2.7. Let \( E \) be a finite dimension space and \( K \subseteq E \) be an open subset. Applying Proposition 1.3, 3 and Theorem 7, it is obvious that all \( F \)-hemicontinuity, \( B \)-pseudomonotonicity and continuity assumptions on \( E \) are equivalent.

3 The Main results

Let us consider the variational inequality related to merely convex and closed subset \( K \) of real reflexive Banach space \( E \). The existence theorems with coercive conditions related to \( B \)-pseudomonotonicity and \( F \)-hemicontinuity are given in the following.

Theorem 3.1. [3] Let \( K \) be a nonempty convex and weakly compact subset of \( E \) and \( A \) a \( B \)-pseudomonotone mapping from \( K \) to \( E^* \). Then \( \text{(VIP)} \) admits solutions.

Theorem 3.2. [3] Let \( K \) be a nonempty, convex and weakly compact subset of \( E \) and \( A : K \to E^* \) an \( F \)-hemicontinuous mapping. Then \( \text{(VIP)} \) admits solutions.

Theorem 3.3. [1] Let \( A : K \to E^* \) be \( B \)-pseudomonotone and let \( K \) be nonempty closed and convex. Moreover, suppose \( A \) satisfies the following condition,
(C1) There exists \( u_0 \in K \) such that:
\[
\lim_{\|u\| \to \infty, u \in K} \frac{< Au, u - u_0 >}{\|u\|} = +\infty.
\]
Then \( \text{(VIP)} \) admits solutions.
Theorem 3.4. [2]. Let $A : K \rightarrow E^*$ be $F$-hemicontinuous and let $K$ be a nonempty closed and convex subset of $E$. Moreover, let us suppose that $A$ satisfies the following condition (C2) There exist $K_1 \subseteq K$ nonempty weakly compact and $K_2 \subseteq K$ compact such that for every $v \in K \setminus K_1$ there exists $w \in K_2$ such that $<Av, v - w> > 0$. Then (VIP) admits solutions.

In [3], the authors have shown that, Condition $C_1$ implies Condition $C_2$, But the converse is not true, in general. The following example shows this fact.

Example 3.5. Let $E = \mathbb{R}$, $K = [1, \infty)$, $K_1 = [1, 2)$, $K_2 = 1$ and for all $u \in K$, $A(u) = \frac{1}{u}$. Then for all $v \in K \setminus K_1$, $w \in K_2$, $<Av, v - w> = (\frac{1}{v})(v - 1) > 0$. The Condition $C_2$ is true, whereas $C_1$ is not established.

Finally, we will show that if $A$ is $B$-pseudomonotone, then $S_{VIP}$ is weakly closed and also becomes weakly compact when the mapping $A$ satisfies in some coercive conditions.

Proposition 3.6. Let $A : K \rightarrow E^*$ a $B$-pseudomonotone mapping, where $K$ is a closed and convex subset of $E$. Moreover, suppose $A$ satisfies Condition (C1). Then $S_{VIP}$ is weakly closed.

Proof. Let $\{u_n\} \subseteq S_{VIP}$ be a weakly convergent sequence to $u$. Since $u_n \in S_{VIP}$, so $<Au_n, v - u_n> \geq 0$ for all $n \in \mathbb{N}$. It means, $<Au_n, u_n - v> \leq 0$, and therefore, $\limsup <Au_n, u_n - v> \leq 0$. Since $A$ is $B$-pseudomonotone, we have

$$\liminf <Au_n, u_n - v> \geq <Au, u - v>, \quad \forall v \in K.$$ 

On the other hand, $\liminf <Au_n, u_n - v> \leq 0$ for all $v \in K$, so $\exists \limsup <Au_n, u_n - v> \geq <Au, u - v>, \quad \forall v \in K$. Hence, $<Au, u - v> \leq 0$. Therefore, $<Au, v - u> \geq 0$ i.e., $u \in S_{VIP}$ and so $S_{VIP}$ is closed.

Corollary 3.7. Let $A : K \rightarrow E^*$ be a $B$-pseudomonotone mapping, where $K$ is a bounded, closed and convex subset of $E$. Then $S_{VIP}$ is weakly compact.

Theorem 3.8. Let $A : K \rightarrow E^*$ be a $F$-hemicontinuous mapping, where $K$ is a closed and convex subset of $E$. Moreover, with Condition $C_1$, the $S_{VIP}$ is weakly compact.

Proof. By Condition $C_1$, we get

$$\exists C, R > 0 : \quad <Au, u - u_0> > 0, \quad \forall u \in K \setminus B(0, R).$$

It is clear that $u_0 \in B(0, R)$. Let $K_1 = \{u \in B(0, R) : <Au, u - u_0> > 0\}$. Obviously, $u_0 \in K_1$, so $K_1$ is nonempty. We show that $K_1$ is weakly compact. Let $\{u_n\} \subseteq K_1$ which $u_n \rightarrow u$. Since $A$ is $F$-hemicontinuous, so

$$\liminf <Au_n, u_n - v> > <Au, u - v> \quad \forall v \in K.$$ 

Now, choose $v = u_0$. Since $\{u_n\} \subseteq K_1$, we have

$$0 \geq \liminf <Au_n, u_n - u_0> > <Au, u - u_0> \quad \forall v \in K.$$ 

So $u \in K_1$, hence $K_1$ is a weakly closed. Also, $K_1 \subseteq B(0, R)$ where $B(0, R)$ is weakly compact. Hence $K_1$ is a weakly compact. Now for $u \in K \setminus K_1$, define $K_u$ as $\text{co}(K_1 \cup u)$ (co$\text{(}$K_1 \cup u\text{)}$ stands for convex hull of $K_1$ and $u$), then $K = \bigcup_{u \in K \setminus K_1} K_u$. Let $S_u = \{v \in K_u : <Au, v - u> > 0, \quad \forall v \in K_u\}$. Because $S_u$ is a solution set of VIP related to $A$ and $K_u$, by Corollary 3.7, $S_u$ is weakly compact and nonempty. Now, it suffices to show that $\bigcap_{u \in K \setminus K_1} S_u \neq \emptyset$. Obviously, $S_{VIP} = \bigcap_{u \in K \setminus K_1} S_u$. Therefore, weakly compactness of $S_u$ shows that $S_{VIP}$ is weakly compact. Let $\{u_i\}_{i=1}^n \subseteq K \setminus K_1$, we show that $\bigcap_{i=1}^n S_{u_i} \neq \emptyset$. Take $K_2 = \text{co}(K_1 \cup \{u_i\}_{i=1}^n)$. Since $K_1$ is a weakly compact and $\{u_i\}_{i=1}^n$ is a finite set, so $K_2$ is also a weakly compact. By Theorem 1.3, there exists $u \in K_2$ such that

$$<Au, u - v> \leq 0, \quad \forall v \in K_2.$$
Now, choose \( v = u_0 \in K_1 \subseteq K_2 \). We get \( < Au, u - u_0 > \leq 0 \), so \( u \in K_1 \). Hence \( u \in K_{u_i} \) for all \( i = 1, 2, 3, \ldots, n \). On the other hand, \( K_{u_i} \subseteq K_2 \) for all \( i = 1, 2, 3, \ldots, n \), therefore for all \( v \in K_{u_i} \), 

\[
<v, Au - u_0> \leq 0
\]

Thus \( u \in K_{u_i} \), such that for \( v \in K_{u_i} \), \( < Au, u - v > \leq 0 \) for all \( i = 1, 2, 3, \ldots, n \). So \( u \in S_{u_i} \) for all \( i = 1, 2, 3, \ldots, n \), and \( u \in \bigcap_{i=1}^{n} S_{u_i} \). Therefore, by finite intersection theorem, \( \bigcap_{u \in K \setminus K_1} S_u \neq \emptyset \). \( \square \)

**Proposition 3.9.** Let \( A: K \to E^* \) a \( B \)-pseudomonotone mapping, where \( K \) is a closed and convex subset of \( E \). Moreover, with Condition \( C_2 \), \( S_{VIP} \) is weakly compact.

### References


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Hardy inequality and its applications in control theory

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Abstract
In this paper we study Hardy inequality and some extensions of it. Also, we discuss about the optimal constant in these inequalities and the importance of it. Furthermore, we prove an Improved Hardy inequality which plays an essential role in the proof of well-posedness and null controllability of a class of one dimensional degenerate-singular parabolic equations.

Keywords: Hardy inequality, Improved Hardy inequality, Optimal constant, null controllability.

Mathematics Subject Classification: 35K65, 93B05, 93B07.

1 Introduction
Hardy inequality states that if \( a_1, a_2, a_3, \ldots \) is a sequence of non-negative real numbers which is not identically zero, then for every real number \( p > 1 \) one has

\[
\sum_{n=1}^{\infty} \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p.
\]

An integral version of Hardy’s inequality states if \( f \) is an integrable function with non-negative values, then

\[
\int_0^\infty \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^p \, dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty f(x)^p \, dx.
\]

Equality holds if and only if \( f(x) = 0 \) a.e. Also, the constant is the best possible.

A generalization of (1) was proved by Beesack in 1961: If \( f(x) \geq 0 \), then

\[
\int_0^\infty x^{-r} F(x)^p \, dx \leq \left( \frac{p}{|r-1|} \right)^p \int_0^\infty x^{-r} (xf(x))^p \, dx,
\]

where \( p > 1, \ r \neq 1 \) and

\[
F(x) = \int_0^x f(t) \, dt, \quad \text{for} \ r > 1,
\]

\[
F(x) = \int_x^\infty f(t) \, dt, \quad \text{for} \ r > 1.
\]

Equality holds if and only if \( f(x) = 0 \) a.e. Again, the constant is the best possible.

Now, the basic form of a Hardy inequality for the maps of \( H^1_0(0,1) \) is

\[
\forall u \in H^1_0(0,1), \quad \int_0^1 u^2 \, dx \geq \frac{1}{4} \int_0^1 u^2 \, x^2,
\]
and the constant $\frac{1}{4}$ is optimal. One can generalize (4) in the following way: For $\alpha \in [0, 2)$,

$$
\int_0^1 x^\alpha u_x^2 \, dx \geq \lambda(\alpha) \left( \frac{1-\alpha}{2} \right)^2 \int_0^1 \frac{u^2}{x^{2-\alpha}},
$$

(5)

for all $u \in C^\infty_c(0, 1)$. Note that $\lambda(\alpha) = \left( \frac{1-\alpha}{2} \right)^2$, and again this constant is optimal.

Note that the Hardy inequality (5) plays a fundamental role in the proof of well-posedness and controllability of some classes of degenerate parabolic equations. But for the investigation of degenerate-singular parabolic equations, the above inequality is no more sufficient. We need the following improved Hardy inequality that completes (5): (For the proof see [5]).

**Theorem 1.1.** Let $\alpha \in [0, 2)$ be given. For all $n > 0$ and $\gamma < 2 - \alpha$, there exists some positive constant $C_0 = C_0(\alpha, \gamma, n)$ such that, for all $u \in C^\infty_c(0, 1)$, the following inequality holds:

$$
\int_0^1 x^\alpha u_x^2 \, dx + C_0 \int_0^1 u^2 \, dx \geq \lambda(\alpha) \int_0^1 \frac{u^2}{x^{2-\alpha}} + n \int_0^1 \frac{u^2}{x^{\gamma}}.
$$

(6)

Besides $C_0(\alpha, \gamma, n)$ is explicitly given by

$$
C_0(\alpha, \gamma, n) = (n + 1) \left( \frac{\gamma}{\alpha+\gamma} \right) \left( \frac{2-\alpha+\gamma}{2-\alpha-\gamma} \right) \frac{4\gamma}{(2-\alpha)^2-\gamma^2}^{\frac{\gamma}{\alpha+\gamma}}.
$$

The optimal constant $\lambda(\alpha)$ is important when one studies the critical case in the related degenerate-singular parabolic equation (For the details see [5]). Now, under some assumptions on the degeneracy coefficient $a(\cdot)$, we prove some generalizations of (5) and (6). In fact, we make the following assumption.

**Hypothesis.** Suppose that the degeneracy coefficient $a(\cdot)$ satisfies the following conditions:

(i) $a \in C([0, 1]) \cap C^1((0, 1))$, $a(x) > 0$ in $(0, 1]$ and $a(0) = 0$.

(ii) $\limsup_{x \to 0} \frac{x a'(x)}{a(x)} = \alpha$ where $\alpha \in [0, 2)$.

(iii) If $\alpha \in [1, 2)$, there exist $m > 0$ and $\delta_0 > 0$ such that for every $x \in [0, \delta_0]$, we have

$$
a(x) \geq m \sup_{0 \leq y \leq x} a(y).
$$

Note that the simplest example for a such $a(\cdot)$ is $x^\alpha$, which has been considered in [5]. Furthermore, if $a(\cdot)$ is nondecreasing near zero, then it satisfies (iii), whereas there exist some nondecreasing examples. For instance, consider $a(x) = x(1 + \sin^2 \frac{x}{2})$, and the function $a(x) = x^\alpha \exp(\sin \frac{x}{2})$ for $\alpha \geq 0$ is another example. First, we need a functional setting to study our problem. Indeed a natural functional setting involves the space

$$
H^1_a(0, 1) := \{ u \in L^2(0, 1) \cap H^1_{loc}((0, 1)) : \int_0^1 a(x) u_x^2 \, dx < \infty \}.
$$

Also, depending on the value of $\alpha$ we consider the following subspaces

**Definition 1.2.** (i) For $0 \leq \alpha < 1$, we define

$$
H^1_{a, 0}(0, 1) := \{ u \in H^1_a(0, 1) : u(0) = u(1) = 0 \},
$$

(ii) For $1 \leq \alpha < 2$, we let

$$
H^1_{a, 0}(0, 1) := \{ u \in H^1_a(0, 1) : u(1) = 0 \}.
$$

Now, we have the followings (see [2]).
Lemma 1.3. Suppose $\alpha \in [0, 2)$ and $\beta < 2 - \alpha$. There exists an optimal constant $\lambda^*(a, \beta) > 0$ such that for every $u \in H^1_{a,0}(0,1)$, we have
\[
\int_0^1 a(x)u^2_x dx \geq \lambda^*(a, \beta) \int_0^1 \frac{u^2}{x^{\beta}} dx.
\] (7)

Theorem 1.4. Let $\beta < 2 - \alpha$. There exists some $\lambda^* = \lambda^*(a, \beta)$ such that for all $n > 0$ and $\gamma < 2 - \alpha$, there exists some positive constant $C_0 = C_0(a, \alpha, \beta, \gamma, n) > 0$ so that, for all $u \in H^1_{a,0}(0,1)$, the following inequality holds:
\[
\int_0^1 a(x)u^2_x dx + C_0 \int_0^1 u^2 dx \geq \lambda^* \int_0^1 \frac{u^2}{x^{\beta}} dx + n \int_0^1 \frac{u^2}{x^\gamma} dx.
\] (8)

Remark 1.5. We know the inequality is false in the case $\beta > 2 - \alpha$, $\alpha_* = \lim_{x \to 0} \frac{x\alpha'(x)}{a(x)}$. We don’t have any idea for $2 - \alpha < \beta \leq 2 - \alpha_*$. And for the critical case $\beta = 2 - \alpha$, the improved Hardy inequality (8) may be true or false as there exists some examples (see [2]). Hence the case of critical value $\beta = 2 - \alpha$ has still to be considered. Yet, this question leads to the problem of finding the optimal constants in Hardy and Improved Hardy inequalities, which seems to be not easy for general $a(\cdot)$.

2 Applications

Consider the following degenerate-singular system
\[
\begin{aligned}
    u_t - (a(x)u_x)_x - \frac{\lambda}{x^\beta} u &= h \chi_\Omega, & (t, x) &\in Q_T, \\
    u(t, 1) &= 0, & t &\in (0, T), \\
    u(t, 0) &= 0, & \alpha &\in (0, 1), & t &\in (0, T), \\
    (x^\alpha u_x)(t, 0) &= 0, & \alpha &\in [1, 2), & t &\in (0, T), \\
    u(0, x) &= u_0(x), & x &\in (0, 1),
\end{aligned}
\] (9)

where $Q_T := (0, T) \times (0, 1)$. There exist two questions:

1. Given $u_0 \in L^2(0,1)$, $h \in L^2(Q_T)$, does there exist any solution $u \in C^0([0, T]; L^2(0,1))$ of (9)?

2. Given $u_0 \in L^2(0,1)$, can we find a control $h \in L^2(Q_T)$ such that the solution $u$ of (9) satisfies $u(T) \equiv 0$?

Now, set
\[
    Au := (a(x)u_x)_x + \frac{\lambda}{x^\beta} u.
\]

Using (8), one can prove that for suitable $k \geq 0$, the operator $-(A - kI)$ is positive and self-adjoint. (see [2]), so the answer of the question (1) is positive. For the question (2), by the methods used in [2], we derive an observability inequality for the solutions of the adjoint equation. In fact, the Improved Hardy inequality (8) plays an fundamental role in our proof.

3 Main Result

Finally, by standard arguments (see, e.g., [4]), a null controllability result follows.
Theorem 3.1. Let $T > 0$ be given, and let $\omega$ be a nonempty subinterval of $(0, 1)$. Then, for all $u_0 \in L^2(0, 1)$, there exists $h \in L^2((0, T) \times \omega)$ such that the solution of (9) satisfies $u(T) \equiv 0$ in $(0, 1)$.

Furthermore, we have the estimate

$$\|h\|_{L^2((0, T) \times \omega)} \leq C'\|u_0\|_{L^2(0, 1)},$$

for some $C' = C'(T, a, \alpha, \lambda) > 0$.

Note that the simplest example is $a(x) = x^\alpha$, which has been considered in [5]. Also, as shown in [3], our hypothesis is strictly weaker than the hypothesis considered in [1]. So, our result completes and unifies some results of [1] and [5].

References


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Evaluation operators and Gelfand-Phillips property in closed subspaces of some operator spaces

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Abstract

We give some necessary and sufficient conditions for the Gelfand-Phillips property of closed subspace $M$ of some operator spaces, with respect to limited complete continuity of evaluation operators $\phi_x : M \rightarrow Y$ and $\psi_{y^*} : M \rightarrow X^*$, where $\phi_x(T) = Tx$ and $\psi_{y^*}(T) = T^*y^*$, for all $x \in X, y^* \in Y^*$ and $T \in M$.

Keywords: Gelfand-Phillips property, limited set, evaluation operator, limited completely continuous operator.

Mathematics Subject Classification: 47L05

1 Introduction

A subset $A$ of a Banach space $X$ is called limited, if every weak* null sequence $(x^*_n)$ in $X^*$ converges uniformly on $A$. We know that every relatively compact subset of $X$ is limited, but the converse of this assertion, in general, is false. If every limited subset of a Banach space $X$ is relatively compact, then $X$ has the Gelfand-Phillips (GP) property. For example, the classical Banach spaces $c_0$ and $\ell_1$ have the GP property, every Schur space (i.e., weak and norm convergence of sequences in $X$ are coincide), every separable Banach space and spaces containing no copy of $\ell_1$, such as reflexive spaces, have the same property[2].

Let $X$ and $Y$ be arbitrary Banach spaces and $T : X \rightarrow Y$ be a bounded linear operator. We remember that $T$ is compact, if the image of the closed unit ball $B_X$ of $X$ under $T$ is relatively compact in $Y$. Also $T$ is limited, if $T(B_X)$ is limited in $Y$. We denote the class of all bounded linear, compact and limited operators from $X$ to $Y$ by $L(X,Y)$, $K(X,Y)$ and $Li(X,Y)$ respectively. Also we recall from [8] that a bounded linear operator $T : X \rightarrow Y$ is limited completely continuous (lcc) if it carries limited and weakly null sequences in $X$ to norm null ones in $Y$. We denote the class of all limited completely continuous operators from $X$ to $Y$ by $Lcc(X,Y)$.

In this note, by using the concept of limited complete continuity of operators, we give some characterizations for closed subspaces of some operator spaces that have the GP property.

2 Evaluation operators and Gelfand-Phillips property

For each two Banach spaces $X$ and $Y$, by meaning of [3] or [6], let $\mathcal{U}(X,Y)$ be the component of operator ideal $\mathcal{U}$ of operators from $X$ to $Y$. If $M$ is a closed subspace of $\mathcal{U}(X,Y)$; for each arbitrary elements $x \in X$ and $y^* \in Y^*$, the evaluation operators $\phi_x : M \rightarrow Y$ and $\psi_{y^*} : M \rightarrow X^*$ are defined by $\phi_x(T) = Tx$, $\psi_{y^*}(T) = T^*y^*$, $T \in M$.

Also, the point evaluation sets related to $x \in X$ and $y^* \in Y^*$ are the image of the closed unit ball $B_M$ of $M$, under the evaluation operators $\phi_x$ and $\psi_{y^*}$ and denoted by $M_1(x)$ and $M_1(y^*)$ respectively.
In the following, among other things, we give some necessary or sufficient conditions for GP property of some closed subspace $M$ of operator ideals with respect to limited complete continuity of all evaluation operators,

**Theorem 2.1.** For each two Banach spaces $X$ and $Y$, if the closed subspace $M$ of arbitrary operator ideal $I(X,Y)$ has the GP property, then all of the evaluation operators $\phi_x$ and $\psi_y$ are lcc.

Now we will show that the lcc of evaluation operators is a sufficient condition for the GP property of the subspace. For the proof of the following two theorems, we remember two theorem from [1,7,9]:

**Theorem 2.2.** ([1,9]) Let $X$ and $Y$ be Banach spaces and $H$ be a subset of $L(X,Y)$ such that

a. $H(B_X) := \{Tx : T \in H, x \in B_X\}$ is relatively compact,

b. $H^*y^* := \{T^*y^* : T \in H\}$ is relatively compact, for all $y^* \in Y^*$.

Then $H$ is relatively compact.

Recall that a subset $H \subseteq L(X,Y)$ is uniformly completely continuous, if for each weakly null sequence $(x_n)$ in $X$, $\sup_{T \in H} \|Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$.

**Theorem 2.3.** ([7]) If $X$ contains no copy of $\ell_1$, then a subset $H \subseteq K(X,Y)$ is relatively compact if and only if $H$ is uniformly completely continuous and for each $x \in X$, the set $\phi_x(H)$ is relatively compact in $Y$.

**Theorem 2.4.** Suppose that $M$ is a closed linear subspace of $Li(X,Y)$ such that the closed linear span of the set $M(\hat{X}) := \{Tx : T \in M, x \in X\}$ is a GP subspace of $Y$. If all evaluation operators $\psi_y$ are lcc, then $M$ has the GP property.

**Theorem 2.5.** Let $X$ and $Y$ be two Banach spaces such that $X$ containing no copy of $\ell_1$, If $M$ is closed subspace of $K(X,Y)$ such that each evaluation operator $\phi_x$ is lcc on $M$, then $M$ has the GP property.

**Corollary 2.6.** If $X$ and $Y$ are two Banach spaces such that $X$ containing no copy of $\ell_1$ and $Y$ has the GP property, then $K(X,Y)$ has the GP property.

**Theorem 2.7.** Let $X$ and $Y$ be two Banach spaces such that $Y$ has the Schur property. If $M$ is a closed subspace of $L(X,Y)$ such that each evaluation operator $\psi_y$ is lcc on $M$, then $M$ has the GP property.

**Corollary 2.8.** If $X$ and $Y$ are two Banach spaces such that $X^*$ has the GP property and $Y$ has the Schur property, then $L(X,Y)$ has the GP property.

Finally, by a similar technique, we give a sufficient condition for the GP property of closed subspaces of $L_w^*(X^*,Y)$, consisting of all bounded weak*- weak continuous operators from $X^*$ to $Y$, and note that for each operator $T \in L_w^*(X^*,Y)$, the adjoint operator $T^*$ tends $Y^*$ into $X$.

**Theorem 2.9.** Let $X$ and $Y$ be two Banach spaces such that $X$ has the Schur property. If $M$ is a closed subspace of $L_w^*(X^*,Y)$ such that all evaluation operators $\phi_x$ is lcc on $M$, then $M$ has the GP property.

**References**


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New results on operator monotone functions

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Abstract

Let $n, m \in \mathbb{R}$. We show that the family of all operator monotone functions $f$ on an interval $(a, b)$ such that $f(t_0) = n$ and $f(t_0) = m > 0$ for $t_0 \in (a, b)$ is a normal family, and investigate some properties of odd operator monotone functions. We also characterize the odd operator monotone functions and even operator convex functions on a symmetric interval $(-a, a)$. As a consequence, we show that if $f$ is an odd operator monotone function on $(-a, a)$, then $f$ is concave on $(-a; 0)$ and convex on $(0; a)$.

Keywords: Operator monotone function, operator convex function, normal family, integral representation.

Mathematics Subject Classification: Primary 47A63; Secondary 47B10, 47A30.

1 Introduction

Throughout the paper all operators are considered to be in the algebra $\mathcal{B}(H)$ of all bounded linear operators acting on a complex Hilbert space $H$.

A continuous real valued function $f$ defined on an interval $J$ is called operator monotone if $A \geq B$ implies $f(A) \geq f(B)$ for all self-adjoint operators $A, B$ with spectra in $J$. Some structure theorems on operator monotone functions can be found in [1, 4, 5]. A continues function $f$ is called operator convex on $J$ if $f(\alpha A + (1 - \alpha)B) \leq \alpha f(A) + (1 - \alpha)f(B)$ for all $0 \leq \alpha \leq 1$ and all self adjoint operators $A$ and $B$ with spectra in $J$. The Löwner theorem says that a function $f$ is operator monotone on an interval $J$ if and only if $f$ has an analytic continuation (denoted by the same $f$) to the upper half plan $\Pi_+$ such that $f$ maps $\Pi_+$ into itself. It is shown [6, Lemma 2.1] that a differentiable function $f$ on an interval $J$ is operator convex if and only if there exists a point $t_0 \in J$ such that the function

$$g(t) = \begin{cases} \frac{f(t) - f(t_0)}{t - t_0} & \text{if } t \neq t_0 \\ f'(t_0) & \text{if } t = t_0 \end{cases}$$

is operator monotone on $J$.

If $f(t)$ is an operator monotone function on $(a, b)$, then clearly $f\left(\frac{2t - a - b}{b - a}\right)$ is operator monotone on $(-1, 1)$, so in this paper we study the family of operator monotone functions on $(-1, 1)$.

Let $\mathcal{K}$ denote the family of all operator monotone functions on $(-1, 1)$ such that $f(0) = 0$ and $f'(0) = 1$. Hansen and Pedersen [1] showed that $\mathcal{K}$ is a compact convex subset of the space of all functions on $(-1, 1)$ with pointwise convergence topology and that the extreme points of $\mathcal{K}$ are of the form $f_\lambda(t) = \frac{t}{1 - \lambda}$ with $|\lambda| < 1$. They [1] also proved that every $f \in \mathcal{K}$ can be represented as

$$f(t) = \int_{-1}^{1} \frac{t}{1 - \lambda} d\mu(\lambda),$$

where $\mu$ is a measure on $(-1, 1)$. The measure $\mu$ is unique and is called the main measure of $f$. The main measure of $f$ has the following properties:

1. $\mu([-\lambda, \lambda]) = \lambda$ for $0 < \lambda < 1$.
2. $\mu([-\lambda, \lambda]) = 0$ for $\lambda \geq 1$.
3. $\mu([-\lambda, \lambda]) = \infty$ for $\lambda \leq 0$.

The measure $\mu$ is absolutely continuous with respect to the Lebesgue measure on $(-1, 1)$. The density of $\mu$ is called the main density of $f$ and is denoted by $\rho_f$. The main density is a decreasing function on $(-1, 1)$.

The main measure and the main density of a function $f$ are closely related to the properties of $f$. For example, $f(t)$ is operator convex if and only if $\rho_f(t)$ is a decreasing function on $(-1, 1)$.
where \( \mu \) is a probability measure on \([-1, 1]\).

Let \( \Omega \) be a open subset of \( \mathbb{C} \). A set \( \mathcal{F} \subseteq C(\Omega) \) is said to be bounded if for each compact subset \( K \subseteq \Omega \), \( \sup \{||f||_K : f \in \mathcal{F} \} < \infty \). The Montel theorem states that if \( \mathcal{F} \) is a bounded subset of the set \( A(\Omega) \) of all analytic functions on \( \Omega \), then \( \mathcal{F} \) is a normal family, i.e., each sequence \( \{f_n\} \) in \( \mathcal{F} \) has a subsequence \( \{f_{n_k}\} \) converging uniformly on each compact subset of \( \Omega \).

Throughout this section, let \( \Omega = \Pi_+ \cup \Pi_- \cup (-1, 1) \), where \( \Pi_- \) is the lower half plan.

**Theorem 2.1.** The family \( K \) is bounded in \( A(\Omega) \), so it is a normal family.

Let \( \mathcal{G} \) denote the family of all operator convex function on \((-1, 1)\) that \( f(0) = f'(0) = 0 \) and \( f''(0) = 1 \). The last theorem implies that \( \mathcal{G} \) is a normal family in \( A(\Omega) \).

**Proposition 2.2.** Let \( f \in K \) and \( f(-1, 1) \subseteq (-1, 1) \). Then \( f(t) = t \) for each \( t \in (-1, 1) \).

**Remark 2.3.** In Proposition 2 the condition “\( f \) is operator monotone” is indispensable. Indeed, we have a counterexample: \( f(t) = \frac{2}{\pi} \sin(\frac{\pi}{2} t) \) is real analytic and increasing on \((-1, 1)\) with \( f(0) = 0 \), \( f'(0) = 1 \), \( |f(t)| < 1 \), but \( f(t) \neq t \).

**Corollary 2.4.** If \( f \) is an odd operator monotone function on \((-1, 1)\), then \( f(|t|) \geq f'(0)|t| \). Hence \( f(|A|) \geq f'(0)|A| \) for self-adjoint operator \( A \) with \( |A| < 1 \).

**Theorem 2.5.** Let \( f \) be an odd operator monotone function on \((-1, 1)\). Then \( f \) is concave on \((-1, 0)\) and convex on \((0, 1)\).

**Example 2.6.** The function \( f(t) = \tan t \) is well-known as an odd operator monotone function on \((-\pi/2, \pi/2)\). It is actually convex on \((0, \pi/2)\) and concave on \((-\pi/2, 0)\). It follows from Theorem 2.5 that \( \sin t \) is not operator monotone on any open interval including \( t = 0 \), that is a new fact.

**Theorem 2.7.** An odd operator monotone function on \((-1, 1)\) is of the form

\[
f(t) = f'(0) \int_{-1}^{1} \frac{t}{1 - (\lambda t)^2} d\mu(\lambda),
\]

where \( \mu \) is a probability measure on \([-1, 1]\).

**Corollary 2.8.** Any even operator convex function \( f \) on \((-1, 1)\) is of the form

\[
f(t) = f(0) + \frac{f''(0)}{2} \int_{-1}^{1} \frac{t^2}{1 - (\lambda t)^2} d\mu(\lambda),
\]

where \( \mu \) is a probability measure on \([-1, 1]\).

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New results on operator monotone functions


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Biseparating maps on Banach algebras of vector-valued functions

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Abstract
Let $B(X,E)$ and $B(Y,F)$ be vector-valued Banach algebras of continuous functions and $T : B(X,E) \rightarrow B(Y,F)$ be a separating map. We show that, under certain conditions, $\hat{T}$ may be regarded as a weighted composition operator. Furthermore, it is shown that, under some conditions, if $T$ is a bijective separating map then it is biseparating and induces a homeomorphism between the character spaces of $B(X,E)$ and $B(Y,F)$.

Keywords: Separating maps, Disjointness preserving linear maps, Biseparating maps, Weighted composition operators, Banach algebras of vector-valued functions.

Mathematics Subject Classification: 47B48, 47B33, 46J10.

1 Introduction

For a compact Hausdorff space $X$ and a Banach algebra $(E, \|\cdot\|)$ over the scalar field of complex numbers, the space of all continuous maps from $X$ into $E$ is denoted by $C(X,E)$. The uniform norm on $C(X,E)$ is defined by
\[
\|f\|_X = \sup_{x \in X} \|f(x)\|, \quad f \in C(X,E).
\]
It is easy to see that $(C(X,E), \|\cdot\|_X)$ is a Banach algebra. If $E = \mathbb{C}$ we get the classic function algebra $C(X)$ of all continuous complex-valued functions on $X$.

In the sequel, by $B(X,E)$ we mean a Banach algebra which is contained in $C(X,E)$, where $E$ is a commutative unital Banach algebra. Before presenting the definition of separating and biseparating operators, we recall that the cozero set of $f : X \rightarrow E$ is $\text{coz}(f) = \{x \in X : f(x) \neq 0\}$ and the support of $f$, denoted by $\text{supp}(f)$, is the closure of $\text{coz}(f)$ in $X$.

Definition 1.1. For compact Hausdorff spaces $X,Y$ and Banach algebras $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$, a linear map $T : B(X,E) \rightarrow B(Y,F)$ is called separating or disjointness preserving, if the equality $\text{coz}(f) \cap \text{coz}(g) = \emptyset$ implies the equality $\text{coz}(Tf) \cap \text{coz}(Tg) = \emptyset$, whenever $f, g \in B(X,E)$. Equivalently, a linear map $T : B(X,E) \rightarrow B(Y,F)$ is separating if for every $f, g \in B(X,E)$, the equality $fg = 0$ implies the equality $TfTg = 0$. Moreover, $T$ is called biseparating if it is bijective and both $T$ and $T^{-1}$ are separating.

The concept of separating (disjointness preserving) operators, which is also known as Lamperti operators, seems to be used first in 40’s. Since then many mathematicians have developed this
concept. For example, Y. A. Abramovich made some contributions in the context of Banach lattices and vector lattices in [1, 2]. Separating operators between certain Banach algebras of continuous functions have been studied by J. J. Font and S. Hernandez in 1994 and by J. Araujo and K. Jarosz in [3].

The aim of this paper is to extend the results of A. Jimenez-Vargas in [8] to certain Banach algebras of vector-valued functions. In other words, we study conditions under which every separating bijection $T : B(X, E) \to B(Y, F)$ is biseparating.

Our results are extensions of what we have already obtained for separating maps between vector-valued little Lipschitz algebras $lip^*(X, E)$ in [6].

It is also worth mentioning that the method of A. Jimenez-Vargas in [8] is only valid for the scalar-valued Lipschitz algebras, whereas our method leads to the same results for more general classes of vector-valued Banach algebras.

2 Main Result

Before presenting our results, we first require some definitions and notations in the general settings of Banach algebras.

Definition 2.1. For a Banach algebra $(A, \| \cdot \|)$ the character space of $A$, denoted by $M(A)$, is the set of all characters (nonzero complex-valued multiplicative linear functionals) on $A$.

(i) The Gelfand transform of $f \in A$ is the complex-valued function $\hat{f}$ defined by $\hat{f}(\varphi) = \varphi(f)$ on $M(A)$. Moreover, $\hat{A} = \{ \hat{f} : f \in A \}$.

(ii) $A$ is regular if $M(A) \neq \emptyset$ and for every closed subset $F \subseteq M(A)$ and every $\varphi \in M(A) \setminus F$, there exists $f \in A$ such that $\hat{f}(\varphi) = 1$ and $\hat{f}(F) \subseteq \{ 0 \}$. If in addition, this $f$ satisfies $\| \hat{f} \| \leq 1$ then $A$ is called hyper regular.

(iii) $A$ is normal if $M(A) \neq \emptyset$ and for every closed subset $F \subseteq M(A)$ and every compact subset $K \subseteq M(A)$ with $F \cap K = \emptyset$, there exists $f \in A$ such that $\hat{f}(K) \subseteq \{ 1 \}$ and $\hat{f}(F) \subseteq \{ 0 \}$. If in addition, this $f$ satisfies $\| \hat{f} \| \leq 1$ then $A$ is called hyper normal.

(i) A commutative Banach algebra is regular if and only if it is normal. See, for example, [7, Corollary 4.2.9T. Ghasemi Honary, A. Nikou and A. H. Sanatpour] or [4, Proposition 4.1.18T. Ghasemi Honary, A. Nikou and A. H. Sanatpour].

(ii) If $A$ is a regular commutative Banach algebra such that $\hat{A}$ is closed under complex conjugation, then $A$ is hyper regular [7, Corollary 4.2.10T. Ghasemi Honary, A. Nikou and A. H. Sanatpour].

(iii) Every commutative $C^* - algebra$ is regular and hence it is normal. See, for example, [7, Example 4.2.2T. Ghasemi Honary, A. Nikou and A. H. Sanatpour]. Moreover, every commutative $C^* - algebra$ is hyper regular by (ii).

Definition 2.2. Let $A, B$ be Banach algebras and let $T : A \to B$ be a linear map. The map $\hat{T} : \hat{A} \to \hat{B}$ is defined by $\hat{T}\hat{f} = \hat{Tf}$ for every $f \in A$.

If $A$ and $B$ are semisimple Banach algebras, it is easy to check that $T$ is separating if and only if $\hat{T}$ is separating and $T$ is injective (surjective) if and only if $\hat{T}$ is injective (surjective).

In the case that $T : B(X, E) \to B(Y, F)$ is a separating bijection we will show that $\hat{T}$ may be regarded as a weighted composition operator.

The next result plays a crucial role in the proof of our main theorem. This is in fact an extension of our result in [6, Theorem 2.4T. Ghasemi Honary, A. Nikou and A. H. Sanatpour].
Theorem 2.3. Let $X, Y$ be compact Hausdorff spaces, $E, F$ be commutative unital Banach algebras, and $B(X, E), B(Y, F)$ be hyper normal semisimple unital Banach algebras. If $T : B(X, E) \to B(Y, F)$ is a separating bijection then

(i) there exists a disjoint union $M(B(Y, F)) = Y_c \cup Y_0 \cup Y_d$, where $Y_0$ is closed and $Y_d$ is open in $M(B(Y, F))$.

(ii) there exists a continuous map $h : Y_c \cup Y_d \to M(B(X, E))$ such that $h(\psi) \notin \text{supp}(f)$ implies $\hat{T} \hat{f}(\psi) = 0$ for all $f \in B(X, E)$.

(iii) there exists a nonvanishing function $k : Y_c \to \mathbb{C}$ such that $\hat{T} \hat{f}(\psi) = k(\psi) \hat{f}(h(\psi))$ for all $f \in B(X, E)$ and for all $\psi \in Y_c$.

(iv) $\hat{T} \hat{f}(\psi) = 0$ for all $f \in B(X, E)$ and for all $\psi \in Y_0$.

(v) $h(Y_d)$ is a finite set of nonisolated points of $M(B(X, E))$.

(vi) the functional $\delta_0 \circ \hat{T}$ is discontinuous on $B(X, E)$ for each $\psi \in Y_d$.

We now bring our main theorem, giving sufficient conditions for a separating bijection $T : B(X, E) \to B(Y, F)$ to be biseparating.

Theorem 2.4. Let $X, Y$ be compact Hausdorff spaces, $E, F$ be commutative unital Banach algebras and $B(X, E), B(Y, F)$ be hyper normal semisimple unital Banach algebras. Let $T$ be a separating bijection from $B(X, E)$ onto $B(Y, F)$. Then $\hat{T}$ is a weighted composition operator in the form $\hat{T} \hat{f}(\psi) = k(\psi) \hat{f}(h(\psi))$ for all $f \in B(X, E)$ and for all $\psi \in M(B(Y, F))$, where $k \in B(Y, F)$ is a nonvanishing function and $h$ is a homeomorphism from $M(B(Y, F))$ onto $M(B(X, E))$. In particular, $T$ is biseparating.

In the next theorem we present sufficient conditions for the algebra $B(X, E)$ to be semisimple.

Theorem 2.5. Let $X$ be a compact Hausdorff space, $E$ be a commutative unital Banach algebra and $B(X, E)$ contain the constant functions. Let every character on $B(X, E)$ be of the form $\psi \circ \delta_x$ for some $\psi \in M(E)$ and $x \in X$, where $\delta_x$ is the evaluation homomorphism on $B(X, E)$. Then $B(X, E)$ is semisimple if and only if $E$ is semisimple.

It is worth mentioning that the algebra $C(X, E)$ satisfies the conditions of the above theorem and hence it is semisimple if and only if $E$ is semisimple. Also, in [5] it has been shown that for a compact metric space $(X, d)$ and a commutative unital Banach algebra, the vector-valued Lipschitz algebras $Lip^\alpha(X, E)$ and $\ellip^\alpha(X, E)$ satisfy the conditions of the above theorem and hence they are both semisimple if and only if $E$ is semisimple.

We recall that the vector-valued Lipschitz algebra $Lip^\alpha(X, E)$ of order $\alpha$ $(0 < \alpha \leq 1)$, consists of those functions $f : X \to E$ for which $p_{\alpha}(f) := \sup_{x,y \in X} \frac{||f(x) - f(y)||}{d(x, y)^\alpha} < \infty$ and it is a Banach algebra equipped with the norm $||f||_\alpha = ||f||_X + p_{\alpha}(f)$, $f \in Lip^\alpha(X, E)$.

Similarly, the vector-valued Lipschitz algebra $\ellip^\alpha(X, E)$ of order $\alpha$ $(0 < \alpha < 1)$ is the (closed) subalgebra of $Lip^\alpha(X, E)$, consisting of those functions $f : X \to E$ for which $\frac{||f(x) - f(y)||}{d(x, y)^\alpha} \to 0$ as $d(x, y) \to 0$ in $X$. 


**References**


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Compactness of Volterra operators on weighted Bergman spaces

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Abstract
We study compactness of Volterra operators between weighted Bergman spaces by reducing this problem to the compactness of suitable multiplication operators between weighted Bergman spaces.

Keywords: Weighted Bergman spaces, Volterra operators, Weighted composition operators, Multiplication operators, Compact operators.

Mathematics Subject Classification: 47B38, 46E15, 32A36.

1 Introduction
Let $H(D)$ denote the set of all analytic functions on the open unit disc $D$ of the complex plane. For each $0 < p < \infty$ and $-1 < \alpha < \infty$, the **weighted Bergman space** $A^p_\alpha$ is the space of all functions $f \in H(D)$ for which

$$\int_D |f(z)|^p (1 - |z|^2)^\alpha \, dm(z) < \infty,$$

where $dm$ is the normalized Lebesgue area measure on $D$. When $1 \leq p < \infty$, the space $A^p_\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\alpha,p} = \left( \int_D |f(z)|^p (1 - |z|^2)^\alpha \, dm(z) \right)^{1/p}.$$


For $g \in H(D)$, the **Volterra operator** $V_g$, which is also known as **Riemann-Stieltjes operator**, is given by

$$(V_g f)(z) = \int_0^z f(\xi)g'(\xi) \, d\xi, \quad z \in D,$$

for each $f \in H(D)$. Various properties of Volterra operators on different spaces of analytic functions have been studied by many authors (see, for example, [1], [6] and the references therein).

Let $\psi, \varphi \in H(D)$ such that $\varphi(D) \subseteq D$. The **weighted composition operator** $\psi C_\varphi$, induced by $\varphi$ and the weight $\psi$, is the operator given by $\psi C_\varphi(f) = \psi(f \circ \varphi)$, for $f \in H(D)$, that is,

$$\psi C_\varphi(f)(z) = \psi(z)f(\varphi(z)), \quad z \in D.$$

Indeed, weighted composition operators are generalizations of both composition operators and multiplication operators. If the weight $\psi$ in the weighted composition operator $\psi C_\varphi$ is the constant function $1$, then we get the **composition operator** $C_\varphi$ given by $C_\varphi(f) = f \circ \varphi$, for $f \in H(D)$, that is,

$$C_\varphi(f)(z) = f(\varphi(z)), \quad z \in D.$$
Also, if the inducing map $\varphi$ in the weighted composition operator $\psi C \varphi$ is the identity map $\varphi(z) = z$, then we get the multiplication operator $M_\psi$ given by $M_\psi(f) = \psi f$, for $f \in H(\mathbb{D})$, that is,

$$M_\psi(f)(z) = \psi(z)f(z), \quad z \in \mathbb{D}.$$ 

Note that, by applying the above notations, we get the following decomposition for the weighted composition operator

$$\psi C \varphi = M_\psi \circ C \varphi.$$ 

Weighted composition operators play an important role in the study of operators between different Banach function spaces. For example, it is known that isometries in many Banach spaces of analytic functions are just weighted composition operators. For more general information on composition operators, multiplication operators and weighted composition operators we refer to the monographs [2], [5] and the references therein.

In this paper, we study compactness of Volterra operators between weighted Bergman spaces by reducing this problem to the compactness of suitable multiplication operators between weighted Bergman spaces.

2 Main Result

Let $D$ denote the differentiation operator given by $Df = f'$, for each $f \in H(\mathbb{D})$. Also, by the integration operator $I$ we mean the operator given by

$$(If)(z) = \int_0^z f(\xi)\,d\xi,$$

for each $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. Note that for all $f \in H(\mathbb{D})$ we have

$$(D \circ I)f = f,$$

and

$$(I \circ D)f = f - f(0).$$

This shows that one can deal with the differentiation operator and integration operator as the inverse of each other.

By applying the above mentioned notations, we get the following decomposition for the Volterra operator

$$V_g = I \circ M_{g'}.$$ 

Using this decomposition, we will be able to study certain properties of Volterra operators by focusing on differentiation, integration, and multiplication operators on suitable underlying spaces.

Using Proposition 1.11 [4], we get the following result which plays a crucial role in the proof of our next theorems.

**Theorem 2.1.** Let $1 \leq p < \infty$ and $-1 < \alpha < \infty$. Then

(i) the differentiation operator $D : A^p_\alpha \rightarrow A^{p+\alpha}_\alpha$ is (well-defined) bounded,

(ii) the integration operator $I : A^{p+\alpha}_\alpha \rightarrow A^p_\alpha$ is (well-defined) bounded.

Applying Theorem 5.1, we get the following interesting results.

**Theorem 2.2.** Let $g \in H(\mathbb{D})$, $1 \leq p < \infty$ and $-1 < \alpha < \infty$. Then the Volterra operator $V_g : A^p_\alpha \rightarrow A^p_\alpha$ is bounded if and only if the multiplication operator $M_{g'} : A^p_\alpha \rightarrow A^{p+\alpha}_\alpha$ is bounded.

In the next theorem, we show that the result of Theorem 1.2 is also valid for the compactness of Volterra operators.
Theorem 2.3. Let \( g \in H(\mathbb{D}) \), \( 1 \leq p < \infty \) and \(-1 < \alpha < \infty\). Then the Volterra operator \( V_g : A^p_\alpha \to A^p_\alpha \) is compact if and only if the multiplication operator \( M_g : A^p_\alpha \to A^{p+\alpha}_\alpha \) is compact.

Before giving a characterization for the compactness of Volterra operators between weighted Bergman spaces, we recall the definition of Bloch spaces.

Definition 2.4. For each \( 0 < \alpha < \infty \), the Bloch type space \( B^\alpha \), of order \( \alpha \), consists of all functions \( f \in H(\mathbb{D}) \) for which
\[
\sup_{z \in \mathbb{D}} (1 - |z|^\alpha) |f'(z)| < \infty.
\]
The space \( B^\alpha \) is a Banach space with the norm
\[
\|f\|_{B^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|)^\alpha |f'(z)|.
\]
When \( \alpha = 1 \), we have the classical Bloch space \( B^1 \).

Various types of operators on Bloch type spaces, including Volterra operators and weighted composition operators, have been studied by many authors in the recent years. For more information on these spaces see [7] and the references therein.

Now, we are in the position to give the desired characterization for the compactness of Volterra operators on weighted Bergman spaces. Here, it is worth mentioning to note that our method leading to the next result is based on the consideration of a suitable multiplication operator (given in Theorem 0.7).

Theorem 2.5. Let \( g \in H(\mathbb{D}) \), \( 1 < p < \infty \) and \(-1 < \alpha < \infty\). Then, the Volterra operator \( V_g : A^p_\alpha \to A^p_\alpha \) is compact if and only if \( g \in B^1 \).

References


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Biseparating maps on Fréchet function algebras

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Abstract
Let $A$ and $B$ be Fréchet function algebras on compact Hausdorff spaces $X$ and $Y$, respectively. A linear map $T : A \rightarrow B$ is called separating or disjointness preserving, whenever $\text{coz}(f) \cap \text{coz}(g) = \emptyset$ implies $\text{coz}(Tf) \cap \text{coz}(Tg) = \emptyset$, for all $f, g \in A$. Moreover, $T$ is called biseparating if it is bijective and both $T$ and $T^{-1}$ are separating.

If $A$ and $B$ are normal and strongly regular, then we show that every biseparating map $T : A \rightarrow B$ is a weighted composition operator in the form $Tf(y) = h(y)f(\varphi(y))$, where $\varphi$ is a homeomorphism from $Y$ onto $X$ and $h$ is a nonvanishing continuous scalar-valued function on $Y$. In particular, $T$ is automatically continuous.

Keywords: Automatic continuity, Banach function algebra, Fréchet function algebra, separating, disjointness preserving, biseparating, cozero preserving, normal algebra, strongly regular algebra, Ditkin’s condition.

Mathematics Subject Classification: 47B38, 47B48, 47B33, 46J10.

1 Introduction

Definition 1.1. Let $X$ be a non-empty topological space. The subalgebra $A$ of $C(X)$ is a function algebra on $X$ if $A$ contains the constants and separates the points of $X$. The algebra $A$ is a Fréchet function algebra ($\text{Ff}$-algebra) or a Banach function algebra ($\text{Bf}$-algebra) on $X$ if $A$ is a function algebra which is also a Fréchet algebra or a Banach algebra, respectively, with respect to some topology.

If the norm of a $\text{Bf}$-algebra $A$ on a compact Hausdorff space $X$ turns out to be the uniform norm, $\|f\|_X = \sup_{x \in X}|f(x)|$, then $A$ is called a uniform Banach function algebra, or briefly, a uniform algebra on $X$.

Note that every $\text{Ff}$-algebra is semisimple and every $\text{Bf}$-algebra on a compact Hausdorff space $X$ is an $\text{Ff}$-algebra on $X$.

Every unital commutative semisimple Fréchet algebra may be considered as a $\text{Ff}$-algebra on its spectrum so that the class of $\text{Ff}$-algebras and the class of unital commutative semisimple Fréchet algebras are the same.

Definition 1.2. Assume that $A$ and $B$ are spaces of complex functions on topological spaces $X$ and $Y$, respectively. A linear map $T : A \rightarrow B$ is called separating or disjointness preserving, whenever...
Example 1.3. Some examples of Banach and Fréchet function algebras.

Let $X$ be a perfect compact plane set which is a finite union of regular sets. The algebra of all complex functions $f$ on $X$ which are $n$-times differentiable and $f^{(k)} \in C(X)$, $f^{(k)} \in \text{lip}(X, \alpha)$, $f^{(k)} \in \text{lip}(X, \alpha)$ for each $k, 0 \leq k \leq n$, is denoted by $D^n(X)$, $\text{Lip}^n(X, \alpha)$, $\text{lip}^n(X, \alpha)$, respectively. The algebra of all complex functions $f$ with derivatives of all orders such that $f^{(k)} \in \text{lip}(X, \alpha)$, $f^{(k)} \in \text{lip}(X, \alpha)$ for all $k \in \mathbb{N}$, is denoted by $D^\infty(X)$, $\text{Lip}^\infty(X, \alpha)$, $\text{lip}^\infty(X, \alpha)$, respectively.

It is well known (see [2, 7]) that $D^n(X)$ with the norm $\|f\|_n = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X}{k!}$, and $\text{Lip}^n(X, \alpha)$ and $\text{lip}^n(X, \alpha)$ with the norm $\|f\|_n = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X}{k!}$, $p_\alpha(f^{(k)})$, are natural $F_f$-algebras on $X$.

In addition the algebras $D^\infty(X)$, $\text{Lip}^\infty(X, \alpha)$ and $\text{lip}^\infty(X, \alpha)$ are natural $F_f$-algebras on $X$ by [6, Theorem 3.4].

Definition 1.4. Let $A$ be an $F_f$-algebra on a compact Hausdorff space $X$.

(i) $A$ is regular if for every closed subset $S$ of $X$ and each $x \in X \setminus S$ there exists $f \in A$ such that $f(x) = 1$ and $f(S) \subseteq \{0\}$, and it is normal if for any disjoint closed subsets $E$ and $F$ of $X$ there exists $f \in A$ such that $f(E) \subseteq \{1\}$ and $f(F) \subseteq \{0\}$. A semisimple commutative $F_f$-algebra is regular or normal if $A$ is regular or normal on $MA$, respectively.

(ii) $A$ satisfies Ditkin’s condition if for every $f \in A$ and $x \in X$ with $f(x) = 0$, there exists a sequence $\{f_n\}$ in $A$ and open neighborhoods $V_n$ of $x$ such that $f_n|V_n = 0$ for all $n \in \mathbb{N}$, and $f_n f \to f$ as $n \to \infty$.

(iii) $A$ is strongly regular if for every $f \in A$ and $x \in X$ with $f(x) = 0$, there exists a sequence $\{f_n\}$ in $A$ and open neighborhoods $V_n$ of $x$ such that $f_n|V_n = 0$ for all $n \in \mathbb{N}$, and $f_n f \to f$ as $n \to \infty$.

It is clear that if $A$ satisfies Ditkin’s condition then $A$ is strongly regular, but the converse is not true, as the following example shows:

Example 1.5. [1, Example 4.5.33]. Let $A$ be a commutative Banach algebra. Moreover, $A = M^\#$ (the unitization of $A$) is a strongly regular $F_f$-algebra on $Z_\infty$, the compactification of $Z$, but $A$ does not satisfy the Ditkin’s condition.
Example 1.6. Some algebras which satisfy the Ditkin’s condition.

(i) Let $G$ be a locally compact abelian group. Then $\mathcal{F}(L^1(G))$ (Fourier transform of $L^1(G)$) satisfies the Ditkin’s condition by [1, Theorem 4.5.18M. S. Hashemi, T. Ghasemi Honary and M. Najafi Tavani].

(ii) For a compact metric space $(X, d)$, the Lipschitz algebra $\text{lip}(X, \alpha)$ satisfies the Ditkin’s condition by [1, Theorem 4.4.30M. S. Hashemi, T. Ghasemi Honary and M. Najafi Tavani].

(iii) Let $X$ be a non-empty, locally compact space. Then $C_0(X)$ satisfies the Ditkin’s condition by [1, Theorem 4.2.1M. S. Hashemi, T. Ghasemi Honary and M. Najafi Tavani].

2 Main Result

With a similar argument as in [4, Lemma 2.1M. S. Hashemi, T. Ghasemi Honary and M. Najafi Tavani] and [4, Theorem 2.2M. S. Hashemi, T. Ghasemi Honary and M. Najafi Tavani], we have:

Lemma 2.1. Let $X$ and $Y$ be compact Hausdorff spaces, $A$ and $B$ be normal Ff-algebras on $X$ and $Y$, respectively, and let $T : A \rightarrow B$ be a biseparating map. Then there exists a homeomorphism $\phi$ from $Y$ onto $X$.

Theorem 2.2. Let $(A, (p_n))$ and $(B, (q_n))$ be strongly regular and normal Ff-algebras on the compact Hausdorff spaces $X$ and $Y$, respectively. Then every biseparating map $T : A \rightarrow B$ is a weighted composition operator

$$Tf(y) = h(y)f(\phi(y)), \quad (f \in A, y \in Y).$$

Here $\phi$ is a homeomorphism from $Y$ onto $X$ and $h$ is a nonvanishing continuous scalar-valued function on $Y$. In particular, $T$ is automatically continuous.

Definition 2.3. Let $A$ and $B$ be Ff-algebras on compact Hausdorff spaces $X$ and $Y$, respectively. A linear map $T : A \rightarrow B$ is cozero preserving, whenever $\text{coz}(f) \subseteq \text{coz}(g)$ yields $\text{coz}(Tf) \subseteq \text{coz}(Tg)$.

Juan J. Font in [5], has studied the automatic continuity of cozero preserving maps between Fourier algebras. Now we study the automatic continuity of these maps between Bf-algebras as well as Ff-algebras.

Theorem 2.4. Let $A$ and $B$ be Bf-algebras on compact Hausdorff spaces $X$ and $Y$, respectively, and let $T : A \rightarrow B$ be a cozero preserving linear map, which preserves the identity. Then $T$ is automatically continuous.

By a method similar to [4, Lemma 3.3M. S. Hashemi, T. Ghasemi Honary and M. Najafi Tavani], we can obtain new results for cozero preserving maps as follows:

Theorem 2.5. Let $A$ and $B$ be Ff-algebras on the compact Hausdorff spaces $X$ and $Y$ such that $A$ is regular. If $T : A \rightarrow B$ is a cozero preserving injection, then $T^{-1}$ is a separating map on the range of $T$.

Corollary 2.6. Let $A$ and $B$ be strongly regular normal Ff-algebras on compact Hausdorff spaces $X$ and $Y$, respectively, and let $T : A \rightarrow B$ be a linear bijection. Then the following statements are equivalent:

(i) $T$ is separating and cozero preserving;
(ii) $T$ is biseparating;
(iii) $T$ and $T^{-1}$ are cozero preserving;
(iv) $T$ and $T^{-1}$ are weighted composition operators.
References


Fourier algebra as a multi-Banach space

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Abstract
In this paper we give the multi-Banach structure of the Fourier algebras and generalize some known results in $L^1$-algebras. Multi-Banach spaces is a new concept in functional analysis introduced by G. H. Dales.

Keywords: multi-norm, multi-Banach space, Fourier algebra, Fourier-Stieltjes algebra.

1 Introduction
We shall be concerned with the Banach algebras which are the group algebra $L^1(G)$ and the Fourier algebra $A(G)$ of a locally compact group $G$. For the definition and basic properties of these algebras, see [D] or [Fol]. The Fourier algebra $A(G)$ of a locally compact group $G$ was introduced by Eymard in [E]. For an abelian locally compact group $G$, the Fourier transform yields an isometric isomorphism of $L^1(G)$ onto $A(\hat{G})$, where $\hat{G}$ is the Pontryagin dual group of the group $G$. Another group algebra defined by Eymard in [E] is the Fourier-Stieltjes algebra $B(G)$ of a locally compact group $G$. For an abelian locally compact group $G$, the Fourier-Stieltjes transform yields an isometric isomorphism of $M(G)$, the algebra of complex regular Borel measures on group $G$, onto $B(\hat{G})$. The Fourier-Stieltjes algebra is the closure of $C_c(G) \cap B(G)$ in $B(G)$, where $C_c(G)$ is the algebra of all continuous compact support functions on the group $G$.

The notion of multi-Banach spaces was introduced by Dales in [D-P]. In this work we want to find a multi-structure on the Fourier algebras such that gives us the multi-structure of the $L^1$-algebras discussed in [D-P], in the abelian case. Also we would like to generalize some results to the Fourier algebras.

2 Fourier Algebra As a Multi-Banach Space
For the definition and basic results about multi-Banach spaces please see [D-P].

Let $G$ be a locally compact group. Somehow the Fourier algebras are the non-commutative version of the $L^1$-algebras. In this section we are interested in studying Fourier algebras as multi-Banach spaces. To generalize some known results in the abelian case to the general case. First we remind the multi-structure of commutative $L^1$-algebras. For more results see [D-P].

Let $G$ be a locally compact group. Since $A(G)$ is a closed ideal of Fourier-Stieltjes algebra, $B(G)$, if we can find a multi-structure on $B(G)$ we can induced it on $A(G)$ as discussed in [D-P, 2.22(i)]. Consider $\Sigma_G$ as the set of all unitary irreducible representation of the group $G$. For each $\pi \in \Sigma$, $\mathcal{H}_\pi$ denotes the Hilbert space corresponding to $\pi$. Each unitary representation

$$\pi : G \longrightarrow B(\mathcal{H}_\pi)$$
induces a non-degenerate $\ast$-representation on $L^1(G)$, by

$$f \mapsto \int f(x)\pi(x)\,dx$$

the right hand side is a Bochner integral in the weak sense.

Now for each $\Sigma_1 \subseteq \Sigma$ we define

$$\|f\|_{\Sigma_1} := \sup_{\pi \in \Sigma_1, \|\pi\| \leq 1} |||\pi(f)|||,$$

where $|||\pi(f)|||$ denotes the operator norm of $\pi(f) \in B(H)$. The map $f \mapsto ||f\|_{\Sigma_1}$ is a semi-norm on the linear space $L^1(G)$. By $N_{\Sigma_1}$ we mean all $f \in L^1(G)$ such that for all $\pi \in \Sigma_1$, $\pi(f) = 0$. Now define

$$\|\hat{f}\| := \|f\|_{\Sigma_1} \quad f \in L^1(G)$$

on the space $L^1(G)/N_{\Sigma_1}$, where $\hat{f}$ denotes the class of $f \in L^1(G)$ in $L^1(G)/N_{\Sigma_1}$. By [E] this norm satisfies the $C^\ast$-algebra norm condition. The completion of $L^1(G)/N_{\Sigma_1}$ with this norm is denoted by $C^\ast_{\Sigma_1}(G)$, which is a $C^\ast$-algebra. When $\Sigma_1 = \Sigma$, completion of $L^1(G)$ with this norm is denoted by $C^\ast(G)$, which is the full group $C^\ast$-algebra on $G$. By [E], the algebra $B_{\Sigma_1}(G)$, is the dual space of $C^\ast_{\Sigma_1}(G)$ and for $u \in B_{\Sigma_1}(G)$ we have,

$$\|u\|_{B_{\Sigma_1}(G)} = \sup_{f \in L^1(G), \|f\|_{\Sigma_1} \leq 1} \int f(x)u(x)\,dx.$$ 

By [E, Proposition 1.15], for each $\Sigma_1 \subseteq \Sigma$ there is an onto $\ast$-homomorphism

$$\varphi_{\Sigma_1} : C^\ast(G) \to C^\ast_{\Sigma_1}(G) \quad f \mapsto \hat{f}$$

with the kernel $N_{\Sigma_1} = \{f \in C^\ast(G), \pi(f) = 0, \text{ for all } \pi \in \Sigma_1\}$. Now let $\Sigma_1, \ldots, \Sigma_n$ be a partition of $\Sigma$. Consider $C^\ast_{\Sigma_1}(G), \ldots, C^\ast_{\Sigma_n}(G)$. Then thanks equality, $N_{\Sigma_1} \cap N_{\Sigma_2} = N_{\Sigma_1 \cup \Sigma_2}$ for each $\Sigma_1$ and $\Sigma_2$ subsets of $\Sigma_G$.

$$C^\ast(G) \cong_{iso} C^\ast_{\Sigma_1}(G) \oplus \ldots \oplus C^\ast_{\Sigma_n}(G),$$

where the right hand side is $\ell^\infty$-direct sum.

So by going on the dual spaces we will have

$$B(G) = B_{\Sigma_1}(G) \oplus \ldots \oplus B_{\Sigma_n}(G), \quad (1)$$

where the right hand side is $\ell^1$-direct sum.

We know that the set of all irreducible unitary representations of a locally compact group $G$, that is denoted by $\Sigma_G$, is the subset off all irreducible representation of the group algebra $L^1(G)$, which is denoted by $\Sigma_{L^1(G)}$ so the hull-kernel topology can be induced on it.

Let $G$ be an abelian group. Then $L^1(G)$ is a commutative Banach algebra. The set $\Sigma_{L^1(G)}$ is just $\hat{G}$, the set of all characters on $G$, via Fourier transform. Group algebra $L^1(G)$ is a regular $\ell^1$-direct sum and so Hull-Kernel topology and Gelfand topology on $\hat{G}$ are the same. So the Borel sets are the same.

Take $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in B(G)$, and $\{\Sigma_1, \ldots, \Sigma_n\}$ a Borel partition on $\Sigma_G$. We know that $B(G) = B_{\Sigma_1}(G) \oplus \ldots \oplus B_{\Sigma_n}(G)$, by $P_i$, we mean the natural projection of $B(G)$ onto $B_{\Sigma_i}(G)$, $i=1,\ldots,n$. Set

$$r_{\{\Sigma_1, \ldots, \Sigma_n\}}((u_1, \ldots, u_n)) = \sup_{f \in L^1(G), \|f\|_{\Sigma_1} \leq 1} |\int (P_1u_1)f| + \cdots + \sup_{f \in L^1(G), \|f\|_{\Sigma_n} \leq 1} |\int (P_nu_n)f|$$

$$= \|P_1u_1\|_{B_{\Sigma_1}(G)} + \cdots + \|P_nu_n\|_{B_{\Sigma_n}(G)}$$

$$= \|P_1u_1 + \cdots + P_nu_n\|_{B(G)}.$$
The last equality holds by the equation 2.1. Finally we define \( \| (u_1, \ldots, u_n) \|_n = \sup \{ r \mid \{ \Sigma_n \} \} \{ (u_1, \ldots, u_n) \} \), \( u_1, \ldots, u_n \in B(G) \). Where supremum is taken over all Borel partitions of \( \Sigma \).

This is a valid decomposition and it is easy to see that, \( ((B(G))^n, \| \cdot \|_n, n \in \mathbb{N}) \) is a multi-normed space and each decomposition explained above is an orthogonal decomposition of \( B(G) \) with respect to this multi-norm. [D-P, Proposition 7.24]. We call this multi-norm the standard multi-norm on \( B(G) \).

**Theorem 2.1.** Let \( G \) be an abelian group, Then the \((1,1)\)-multi-norm on \( M(\hat{G}) \) and the standard multi norm on \( B(G) \) are the same.

The following result is another generalization of \( L^1 \)-theory in the concept of multi-Banach spaces.

**Theorem 2.2.** With the notation used before, for \((x_1, \ldots, x_n) \in A(G)^n\)

\[
\|(x_1, \ldots, x_n)\|_n = \sup \| (P_1(x_1), \ldots, P_n(x_n)) \|_n^{\text{max}},
\]

where the supremum is taken over all orthogonal decomposition of \( A(G) \).

The following theorem is the generalization of [R; Lemma 4.6.14].

**Theorem 2.3.** Let \( \lambda_1, \ldots, \lambda_n \in C^*(G) \) then \( \mu_{1,n}(\lambda_1, \ldots, \lambda_n) = \| \sum_{i=1}^n (\lambda_i \lambda_i^*) \|_2 \|.

2.1 Relationship between multi-bounded and complete order maps on the Fourier algebras

By [D-P, Theorem 5.29], for locally compact abelian groups \( G \) and \( H \)

\[
\mathcal{M}(L^1(G), L^1(H)) \cong B_r(L^1(G), L^1(H)).
\]

Where \( B_r(L^1(G), L^1(H)) \) is the Banach lattice of all regular maps from \( L^1(G) \) to \( L^1(H) \). And group algebras are considered with their lattice multi-structure.

In this section we generalize the above result to the non-abelian case.

Let \( A \) and \( B \) be operator spaces. Define \( L_{c,or}(A, B) \) to be the space of all linear maps \( T : A \to B \) such that \( T = T_1 - T_2 + i(T_3 - T_4) \), where \( T_i \)'s are bounded completely positive linear maps.

**Theorem 2.4.** Let \( G \) and \( H \) be locally compact groups, and let \( G \) be amenable.

(1) Then \( L_{c,or}(A(G), A(H)) \subseteq \mathcal{M}(A(G), A(H)) \).

(2) If \( \Phi : A(G) \to A(H) \) is a completely positive one-one map then it is multi-isometry.

With using order structure instrument we can prove some topological facts about those maps studied in the above theorem. We want to consider the relationship between completely bounded and multi-bounded maps between Fourier algebras.

**Proposition 2.5.** Let \( G \) and \( H \) be locally compact groups. Then \( L_{c,or}(A(G), A(H)) \subseteq \mathbb{C}B(A(G), A(H)) \). When \( G \) is amenable equality holds.

**Corollary 2.6.** Let \( G \) and \( H \) be locally compact group, and let \( G \) be amenable. Then each completely bounded linear map \( T : A(G) \to A(H) \) is multi-bounded.

**Remark 2.7.** It is nice to see that the containing in Corollary 2.4 is proper. Let \( G \) be a locally compact group. Then the map

\[
\tilde{\cdot} : A(G) \to A(G), \quad f \mapsto \tilde{f}(x) = f(x^{-1})
\]

is completely bounded if and only if \( G \) is abelian by-finite. But it is easy to see that this map is always a multi-isometry.
References


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Twisted partial actions on $C^*$-algebras

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Abstract
Let $U$ be an inverse semigroup and $S(U)$ be the universal inverse semigroup associated to $U$. We already know that actions of $S(U)$ correspond bijectively to partial actions of $U$. In this paper we will show that such a similar result hold for twists.

Keywords: $C^*$-algebra, Inverse semigroup, Partial homomorphism, partial action

Mathematics Subject Classification: 46L55

1 Introduction
Throughout this paper, we will let $I(X)$ be the symmetric inverse semigroup on $X$. Let $H$ be a semigroup. By a partial homomorphism of $U$ in $H$, we mean a map $\pi : U \rightarrow H$ such that
(i) $\pi(s)\pi(t)\pi(t^*) = \pi(st)\pi(t^*)$,
(ii) $\pi(s^*)\pi(s)\pi(t) = \pi(s^*)\pi(st)$;
(iii) $\pi(s)\pi(s^*)\pi(s) = \pi(s)$.
for all $s, t \in U$.
It should be noted that in the definition of partial homomorphism, $H$ is not necessarily an inverse semigroup. In the case that $H$ is an inverse semigroup, by applying (iii) to both $s$ and $s^*$, together with the uniqueness of inverses in a semigroup we have $\pi(s^*) = \pi(s)^*$. Hence, if $H$ is an inverse semigroup, the definition of partial homomorphism is equivalent to axioms (i), (ii) and $\pi(s^*) = \pi(s)^*$.
If $H$ is the semigroup $B(\mathcal{H})$ of all bounded linear maps on a Hilbert space $\mathcal{H}$, then partial homomorphism $\pi : U \rightarrow H$ is called partial representation of $U$ on $H$.
For a given inverse semigroup $U$ and a set $X$, by a partial action of $U$ on $X$ we shall mean a partial homomorphism $\pi : U \rightarrow I(X)$.
If $B$ is a $C^*$-algebra, then a partial automorphism of $B$ is nothing but an *-isomorphism between closed two-sided ideals $I, J$ of $B$.
If $X$ is a topological space, and $\alpha : U \rightarrow I(X)$ is a partial action of $U$ on $X$, then $\alpha_s : D_{s^*} \rightarrow D_s$ is continuous and $D_{s^*}, D_s$ are open subsets of $X$. Obviously, $\alpha_s$ is a homeomorphism with inverse $\alpha_{s^*}$. This shows that $\alpha_s$ is a partial homeomorphism of $X$. Therefore, a partial action of $U$ on a topological space $X$ is just a partial homomorphism of $U$ into the inverse semigroup of all partial homeomorphisms of $X$.
In a similar way, let $B$ be a $C^*$-algebra and $\beta : U \rightarrow I(B)$ be a partial action of $U$ on $B$. Clearly, $\beta_s : D_{s^*} \rightarrow D_s$ is a partial automorphism between closed two-sided ideals $D_{s^*}, D_s$ of $B$. So, given a partial action of $U$ on a $C^*$-algebra $B$, we have a partial homomorphism from $U$ into the inverse semigroup of all partial automorphisms of $B$. Also, if $B$ is a commutative $C^*$-algebra, say $B = C_0(X)$, from the fact that partial automorphisms of $B$ correspond to partial homeomorphisms of $X$ we conclude that partial actions on $X$ correspond to partial actions on $B$.  

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2 Twisted partial actions on $C^*$-algebras

In [3], it is proved that partial actions of $U$ correspond bijectively to actions of $S(U)$. In this section, we are going to show that twisted partial actions of $U$ are correspond to twisted (global) actions of $S(U)$. Before we give the right definition of twisted partial actions, we need to know the following fact about multiplier algebra.

For a given $C^*$-algebra $B$, we write $M(B)$ for the multiplier algebra of $B$ and $UM(B)$ for the group of unitary multipliers of $B$.

**Definition 2.1.** A twisted partial action of a group $G$ on a $C^*$-algebra $B$ is a triple

$$\Theta = ([D_t]_{t \in G}, \{\theta_t\}_{t \in G}, \{w(r, s)\}_{(r,s) \in G \times G})$$

where, for each $t$ in $G, D_t$ is a closed two-sided ideal in $B, \theta_t$ is a $*$-isomorphism from $D_{t^{-1}}$ onto $D_t$, and for each $(r, s)$ in $G \times G, w(r, s)$ is a unitary multiplier of $D_r \cap D_{rs}$ such that for all $r, s$ and $t$ in $G$ the following postulates hold:

(a) $D_e = B$ and $\theta_e$ is the identity automorphism of $B$;
(b) $\theta_t(D_{t^{-1}} \cap D_s) = D_r \cap D_{rs}$;
(c) $\theta_t(\theta_s(a)) = w(r, s)\theta_s(a)w(r, s)^*$, for $a \in D_{s^{-1}} \cap D_{s^{-1}t^{-1}}$;
(d) $\theta_t(w(r, s)) = w(t, r)$;
(e) $\theta_t(aw(s, t))w(r, st) = \theta_r(a)w(r, s)w(rs, t)$, for $a \in D_{s^{-1}} \cap D_s \cap D_{st}$.

(see [2, Definition 2.1]).

**Definition 2.2.** A twisted action of an inverse semigroup $U$ on a $C^*$-algebra $B$ is a triple

$$\Theta = \{\{\beta_s\}_{s \in U}, \{\theta_t\}_{t \in U}, \{w(s, t)\}_{s, t \in U}\}$$

consisting of a family of closed two-sided ideals $D_s$ of $B$ whose linear span is dense in $B$, a family of $*$-isomorphisms $\beta_s : D_s \to D_s$, and a family $\{w(s, t)\}_{s, t \in U}$ of unitary multipliers $w(s, t) \in UM(D_s)$ such that for all $r, s, t \in U$ and $e, f \in E(U)$ (semilattice of unitary elements of $U$) we have

(i) $\beta_t \circ \beta_s = Ad_{w(r, s)} \circ \beta_{st}$;
(ii) $\beta_s(xw(s, t))w(r, st) = \beta_s(x)w(r, s)w(rs, t)$ for $x \in D_{r{s^{-1}}t^{-1}} \cap D_s$;
(iii) $w(e, f) = 1_{ef}$ and $w(r, r^*) = 1, \text{ where } 1_r \text{ is the unit of } M(D_r)$;
(iv) $w(s^*, e)w(s^*, s)x = w(s^*, s)x$ for all $x \in D_{s^*e}$.

(see [1, Definition 4.1].)

Now, we present a simultaneous generalization of [1, Def. 4.1] and [2, Def. 2.1].

**Definition 2.3.** Let $U$ be an inverse semigroup and let $B$ be a $C^*$-algebra. A twisted partial action of $U$ on $B$ is a pair $(\beta, w)$ where $\beta = \{\beta_s\}_{s \in U}$ is a family of partial automorphisms $\beta_s : D_s \to D_s$ of $B$ and $w = \{w(s, t)\}_{s, t \in U}$ is a family of unitary multipliers $w(s, t) \in UM(D_s \cap D_t)$, such that for all $r, s, t \in U$ and $e, f \in E(U)$:

(i) $\beta_t(D_{r{s^{-1}}t^{-1}} \cap D_s) = D_r \cap D_{rs}$;
(ii) $\beta_t(\beta_s(x)) = w(r, s)\beta_s(x)w(r, s)^*$ for all $x \in D_{r{s^{-1}}t^{-1}} \cap D_s$;
(iii) $\beta_t(xw(s, t))w(r, st) = \beta_t(x)w(r, s)w(rs, t)$ if $x \in D_{r{s^{-1}}t^{-1}} \cap D_s$, $D_{rs}$;
(iv) $w(e, f) = 1_{ef}$ and $w(r, r^*) = 1, \text{ where } 1_r \text{ is the unit of } M(D_r)$;
(v) $w(s^*, e)w(s^*, s)x = w(s^*, s)x$ for all $x \in D_{s^*e}$.

In order to prove the main theorem, we need the following Lemma.

**Lemma 2.4.** If $(\beta, w)$ is a twisted partial action of $U$ on $B$ as above, the following holds:

(i) $D_s \subseteq D_{ss^*}$ for all $s \in U$. Moreover, $D_s = D_{ss^*}$ for all $s \in U$ if and only if $(\beta, w)$ is a twisted action in the sense of Definition 2.2;
(ii) $\beta_e : D_e \to D_e$ is the identity map for all $e \in E(U)$;
(iii) $D_s \cap D_t = D_s \cap D_{ss^*} = D_{tt^*} \cap D_{ss^*} = D_{ss^*} \cap D_{tt^*}$ for all $s, t \in U$, where $e = ss^*tt^*$, moreover
Twisted partial actions on $C^*$-algebras

The following main result is a generalization of Theorem 4.2 and 4.3 in [4].

**Conclusion:** There is a bijective correspondence between twisted partial actions of $U$ on $B$ and twisted (global) actions of $S(U)$ on $B$.

\[ D_e \cap D_s = D_{es} \text{ for all } e \in E(U) \text{ and } s \in U. \]
\[ (iv) \beta_s^{-1}(D_e \cap D_{rs}) = D_{es} \cap D_{s^*r^*}; \]
\[ (v) D_s \subseteq D_t \text{ whenever } s \leq t. \]

**References**


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On ideally factored Banach algebras

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Abstract

Let $\mathcal{A}$ be a Banach algebra and $\varphi$ be a character on $\mathcal{A}$. The product $a \cdot b = \varphi(a)b$ makes $\mathcal{A}$ into a Banach algebra which is called ideally factored Banach algebra.

We show that an ideally factored Banach algebra is biflat and weakly amenable but not approximately amenable. Also we show that an ideally factored Banach algebra is biprojective but its unitization is not biprojective.

Keywords: Banach algebras, Contractible, Character, Biprojective.

Mathematics Subject Classification: Primary: 46H20; Secondary: 46H35, 46H25

1 Introduction

Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$–bimodule. We recall that $X^*$ is a Banach $\mathcal{A}$–bimodule under the following actions:

$$< x, ax^* > = < xa, x^*> \quad , \quad < x, x^*a > = < ax, x^*> \quad (a \in \mathcal{A}, \quad x \in X, \quad x^* \in X^*).$$

A derivation $D : \mathcal{A} \rightarrow X$ is a linear map such that

$$D(ab) = D(a)b + aD(b) \quad (a, b \in \mathcal{A})$$

For each $x \in X$, $\delta_x(a) = ax - xa$ is a derivation, which is called inner derivation. The first cohomology group $H^1(\mathcal{A}, X)$ is the quotient of the space of derivations by the inner derivations. The Banach algebra $\mathcal{A}$ is called contractible (amenable) if $H^1(\mathcal{A}, X) = \{0\}$, $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$–bimodule $X$. $\mathcal{A}$ is weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. A Banach algebra $\mathcal{A}$ is called essential if $\mathcal{A}^2$ is dense in $\mathcal{A}$, where $\mathcal{A}^2 = \text{span}(\mathcal{A} \cdot \mathcal{A})$. It is known that every weakly amenable Banach algebra is essential. A derivation $D : \mathcal{A} \rightarrow X$ is called approximately inner if there exists a net $(x_\alpha) \subseteq X$ such that for every $a \in \mathcal{A},$

$$D(a) = \lim_{\alpha} ax_\alpha - x_\alpha a$$

in this case, $\mathcal{A}$ is approximately amenable if for any $\mathcal{A}$–bimodule $X$, every derivation $D : \mathcal{A} \rightarrow X$ ($D : \mathcal{A} \rightarrow X^*$) is approximately inner.

For a Banach algebra $\mathcal{A}$, the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$ is a Banach $\mathcal{A}$–bimodule, where the module actions are specified by

$$a \cdot (b \otimes c) = ab \otimes c \quad \text{and} \quad (b \otimes c) \cdot a = b \otimes ca.$$ 

Now let $X$ and $Y$ be $\mathcal{A}$-bimodules. A linear map $T : X \rightarrow Y$ is an $\mathcal{A}$-bimodule homomorphism if for each $a \in \mathcal{A}$, $x \in X$, $T(a \cdot x) = a \cdot Tx$ and $T(x \cdot a) = Tx \cdot a$. We define the multiplication map $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$ by

$$\pi(a \otimes b) = ab \quad (a, b \in \mathcal{A}).$$
The $\pi$ becomes a bounded $A$-bimodule homomorphism. The Banach algebra $A$ is said to be biprojective if $\pi$ has a bounded right inverse which is an $A$-bimodule homomorphism, i.e., there is a bounded $A$-bimodule homomorphism $\rho : A \rightarrow A\hat{\otimes}A$ such that $\pi \rho = 1_A$ (see [2] for more details).

The dual map $\pi^*$ is also $A$-bimodule homomorphism. A Banach algebra $A$ is said to be biprojective if $\pi^*$ has a bounded right inverse which is an $A$-bimodule homomorphism, i.e., there is a bounded $A$-bimodule homomorphism $\rho : A \rightarrow A\hat{\otimes}A$ such that $\pi o \rho = 1_A$ (see [2] for more details).

The dual map $\pi^*$ is also $A$-bimodule homomorphism. A Banach algebra $A$ is said to be biflat if $\pi^*$ has a right inverse as a bounded $A$-bimodule homomorphism. It is easy to see that every biflat Banach algebra is weakly amenable [2].

Let $A$ and $B$ be Banach algebras. Then the space $A\hat{\otimes}B$ becomes a Banach algebra with the multiplication given by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2 \quad (a_1, a_2 \in A, \ b_1, b_2 \in B).$$

The projective tensor product $A\hat{\otimes}B$ has the universal property that for each Banach space $X$, if $\theta : A \times B \rightarrow X$ is a bounded bilinear map, then there is a unique bounded linear map $\hat{\theta} : A\hat{\otimes}B \rightarrow X$ such that $\theta(a, b) = \hat{\theta}(a \otimes b)$ ($a \in A, \ b \in B$).

Suppose that $\sigma$ is a bounded endomorphism of a Banach algebra $A$. For a Banach $A$-bimodule $X$ a $\sigma$-derivation is a bounded linear map $D : A \rightarrow X$ satisfying

$$D(ab) = \sigma(a) D(b) + D(a) \sigma(b) \quad (a, b \in A).$$

A Banach algebra $A$ is called $\sigma$-contractible ($\sigma$-amenable) if for any $A$-bimodule $X$, every $\sigma$-derivation $D : A \rightarrow X$ ($D : A \rightarrow x^*$) is inner.

## 2 Basic Properties of $\varphi A$

In this section, we define $\varphi A$ and express some basic properties of this new Banach algebra.

**Definition 2.1.** Let $A$ be a Banach algebra and $\varphi \in \Phi_A$. For each $a, b \in A$ we define

$$a \cdot b := \varphi(a)b$$

It is elementary to show that $(A, \cdot)$ becomes a Banach algebra. We denote this Banach algebra by $\varphi A$ which is called ideally factored Banach algebra.

Amyari and Mirzavaziri have been shown that $\varphi A$ is not an amenable Banach algebra [8, Theorem 3.2].

**Proposition 2.2.** The following holds:

1. The space of all characters on $\varphi A$ is equal to $\{\varphi\}$ i.e., $\Phi_{\varphi A} = \{\varphi\}$.
2. The Banach algebra $\varphi A$ has a left identity. Left identity of $\varphi A$ is unique if and only if $A$ is one dimensional.
3. The Banach algebra $\varphi A$ has no right approximate identity.
4. $a \in \varphi A$ is quasi-invertible if and only if $\varphi(a) \neq 1$.
5. The Banach algebra $\varphi A$ is not approximately amenable.

Therefore for Banach algebras with $\text{dim}(A) \geq 2$, $\varphi A$ has many left identities and consequently doesn’t have right identity. In this paper, we suppose that every Banach algebra has dimension at least 2.
3 Biprojectivity and Weak Amenability of $\varphi A$

In this section, we express several interesting results of $\varphi A$ with using the known propositions. At the end of this section, we define an idempotent bounded endomorphism, $\sigma$, of $\varphi A$ and show that $\varphi A$ is $\sigma$-contractible Banach algebra.

**Proposition 3.1.** The Banach algebra $\varphi A$ is biprojective.

**Proposition 3.2.** Let $A$ be a biprojective Banach algebra. Then $A$ is biflat, weakly amenable and essential Banach algebra. Consequently, $\varphi A$ is biflat, weakly amenable and essential.

As regarded $\varphi A$ is biprojective and biflat Banach algebra, but it is not contractible and amenable Banach algebra. Consequently, existence of an identity and bounded approximate identity in the following proposition is essential.

**Proposition 3.3.** Let $A$ be a Banach algebra. Then the following holds.

1. $A$ is contractible if and only if $A$ is biprojective and unital.
2. $A$ is amenable if and only if $A$ is biflat and has a bounded approximate identity.

By the following Proposition, we are going to show that $\varphi A$ is $(2n - 1)$-weakly amenable.

**Proposition 3.4.** Let $A$ be a weakly amenable Banach algebra, such that $A$ is a closed ideal in $(A^{**}, \Box)$. Then $A$ is $(2n - 1)$-weakly amenable for each $n \in \mathbb{N}$.

**Corollary 3.5.** The Banach algebra $\varphi A$ is $(2n - 1)$-weakly amenable for each $n \in \mathbb{N}$.

Now suppose that $B$ is a Banach algebra. Applying the following proposition we concern the relationship between $\varphi A \hat{\otimes} B$ and $B$.

**Proposition 3.6.** [2, Proposition 2.8.64.] Let $A$ and $B$ be Banach algebras, and let $\theta : A \longrightarrow B$ be a continuous homomorphism with $\theta(A) = B$. Suppose that $A$ is amenable (contractible). Then $B$ is amenable (contractible).

**Proposition 3.7.** Suppose that $B$ is a Banach algebra such that $\varphi A \hat{\otimes} B$ is amenable (contractible). Then $B$ is amenable (contractible).

Let $A$ be a non unital Banach algebra. Then its unitization $A^\# = A \otimes \mathbb{C}$ is a unital Banach algebra with unit $(0, 1)$. $A$ is a closed ideal in $A^\#$ and we have;

a) $A$ is amenable if and only if $A^\#$ is amenable [5,23.11].

b) If $A$ is a biprojective Banach algebra. Then $A$ is contractible if and only if $A^\#$ is biprojective. [5,4.3.1].

c) If $A$ is a biflat Banach algebra. Then $A$ is amenable if and only if $A^\#$ is biflat [5,4.3.15].

Now we have the following corollary;

**Corollary 3.8.** a) $(\varphi A)^\#$ is not amenable.

b) $(\varphi A)^\#$ is not biprojective.

c) $(\varphi A)^\#$ is not biflat.

Note that as previous corollary, for a Banach algebra $A$, the biprojectivity and biflatness of $A$ and $A^\#$ is not equivalent.

**Proposition 3.9.** $(\varphi A^{**}, \Box)$ is biprojective.
Now suppose that $e$ is a left identity of $\varphi A$ and consider $\sigma : (\varphi A) \rightarrow (\varphi A)$ defined by $\sigma(a) = \varphi(a)e$. Clearly $\sigma$ is bounded endomorphism of $\varphi A$ and we have,

$$\sigma^2(a) = \sigma(\varphi(a)e) = \varphi(a)\sigma(e) = \varphi(a)\varphi(e)e = \varphi(a)e = \sigma(a)$$

Thus $\sigma$ is an idempotent bounded endomorphism of $\varphi A$. It is known that every biprojective Banach algebra is $\sigma$-biprojective for each bounded endomorphism $\sigma$ of $A$ [10 , corollary 5.3]. Our aim of expression the following proposition is to prove that $\varphi A$ is $\sigma$-contractible.

**Proposition 3.10.** Let $\sigma$ be an idempotent endomorphism of $A$. Then $A$ is $\sigma$-contractible if and only if it is $\sigma$-biprojective and $\sigma(A)$ is unital.

**Corollary 3.11.** Let $e$ be a left identity in $\varphi A$ and define $\sigma : (\varphi A) \rightarrow (\varphi A)$ by $\sigma(a) = \varphi(a)e$. Then $\varphi A$ is $\sigma$-contractible.

**References**