Dynamical Systems

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Scattering and weak disjointness in topological dynamical systems

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Abstract

There are several ways to classify topological dynamical systems. Here, we use scattering and weak disjointness to achieve this goal. Among some results in classifying the systems, we give an affirmative answer, in a special case, to whether scattering implies strong scattering or not?

Keywords: $\mathcal{F}$-mixing, scattering, $\mathcal{F}$-scattering, weak disjointness.

Mathematics Subject Classification: 54H20, 37B20.

1 Introduction

A topological dynamical system (TDS) is a pair (X, T) such that X is a compact metric space and T is a homeomorphism.

In a TDS, the *return times set* is defined to be $N(U, V) = \{n \in \mathbb{Z} : T^n(U) \cap V \neq \emptyset\}$ where U and V are *opene* (non-empty and open) sets. A TDS (X, T) is *transitive* if for any two opene sets U and V, we have $N(U, V) \neq \emptyset$; and it is *weak mixing* if the product system $(X \times X, T \times T)$ is transitive. A TDS is *strong mixing* if $N(U, V)$ is cofinite for any opene sets U, V.

Let $\mathcal{F}$ be a family of nonempty subsets of $\mathbb{Z}$. This means $\mathcal{F}$ is hereditary upward: if $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2$, then $F_2 \in \mathcal{F}$. The *dual* of $\mathcal{F}$, denoted by $\mathcal{F}^*$, is defined to be all subsets of $\mathbb{Z}$ meeting all sets in $\mathcal{F}$:

$$\mathcal{F}^* = \{G \subset \mathbb{Z} : \ G \cap F \neq \emptyset, \ \forall F \in \mathcal{F}\}.$$

A family $\mathcal{F}$ is called *partition regular* if $F \in \mathcal{F}$ is partitioned to finite sets $F = F_1 \cup \cdots \cup F_k$, then there is i such that $F_i \in \mathcal{F}$. A non-empty family closed under finite intersections is called a *filter*. It is known that if $\mathcal{F}$ is partition regular then $\mathcal{F}^*$ is a filter. A filter which is partition regular is called an *ultrafilter*. An ulterafilter $p \in \beta\mathbb{Z}$ is called *idempotent* if $p + p = p$.

For a family $\mathcal{F}$ and $k \in \mathbb{Z}$, the *shifted family* is defined as $\mathcal{F} + k = \{F + k : \ F \in \mathcal{F}\}$ where $F + k = \{n + k : \ n \in F\}$.

Definition 1.1. *Let $\mathcal{F}$ be a family. Then $\mathcal{F}_+ := \bigcup_{k \in \mathbb{Z}} (\mathcal{F} + k)$ and $\mathcal{F}_\bullet := \bigcap_{k \in \mathbb{Z}} (\mathcal{F} + k)$.*

We have $\mathcal{F}_\bullet \subset \mathcal{F} \subset \mathcal{F}_+$ and both $\mathcal{F}_\bullet$ and $\mathcal{F}_+$ are shift invariant with $\mathcal{F}_\bullet^* = (\mathcal{F}_+)^*$ [2]. A set $F \subset \mathbb{Z}$ is called *IP-set* if it contains $FS(A) = \{a_{i_1} + a_{i_2} + \cdots + a_{i_n}, \ \ i_j < i_{j+1}\}$ for a subset of integers $A = \{a_n\}_{n \geq 1}$. Equivalently, F is an IP-set if and only if it is a member of an idempotent. Let $\mathcal{IP}$ be the family of all IP-sets and $\mathcal{IP}^*$ its dual family. A subset of integers is called *central* or *C-set* if it is a member of a minimal idempotent and $\mathcal{C}$ is the family of all C-sets.

A set $F \subset \mathbb{Z}$ is called *Δ-set* if there exists a sequence of integers $S = (s_n)_{n \in \mathbb{N}}$ such that the difference set $\Delta(S) = \{s_i - s_j; \ \ i > j\} \subset F$. Let Δ be the family of all Δ-sets and denote its dual family by Δ^*. Any IP-set is a Δ-set for let $S = \{a_1, \, a_1 + a_2, \, a_1 + a_2 + a_3, \cdots\}$.

A subset $E \subset \mathbb{Z}$ is called *thick*, if it contains arbitrarily long intervals $[a, b] = \{a, a+1, \cdots, b\}$. It is called *syndetic*, if its complement is not a thick set. Equivalently, it has bounded gaps. A subset $E \subset \mathbb{Z}$ is called *thickly syndetic*, if $\{n \in \mathbb{Z} : n + j \in E, 0 \leq j \leq N\}$ is syndetic. A subset of integers is called *piecewise syndetic*, if it is the intersection of a thick set and a syndetic set. Equivalently, its complement is not thickly syndetic. It means that $\mathcal{PS}$ is the dual family of thickly syndetic sets.

We call $d^*(A) = \limsup_{(M-N)\to\infty} \frac{|A\cap[N,M]|}{M-N+1}$ the *upper Banach density* of a set $A \subset \mathbb{Z}$.

2 Main Result

The classical different mixing concepts for a topological dynamical systems are mixing, mild mixing and weak mixing. They can be characterized in terms of $\mathcal{F} = \{N(U, V) : U$ and V are opene sets in $X\}$. A TDS is mixing, mild mixing or weak mixing if and only if $\mathcal{F}$ is cofinite, $(\mathcal{IP} - \mathcal{IP})^*$ or thick respectively [4]. To extend the concept, let $\mathcal{F}$ be any family of subsets of $\mathbb{N}$.

Definition 2.1. *A TDS (X, T) is called* $\mathcal{F}$-transitive *if for any two opene sets $U, V \subset X$ we have $N(U, V) \in \mathcal{F}$ and it is called* $\mathcal{F}$-mixing *if the product system $(X \times X, T \times T)$ is $\mathcal{F}$-transitive.*

It is easy to see that (X, T) is $\mathcal{F}$-mixing if and only if it is weak mixing and $\mathcal{F}$-transitive if and only if $N(U, V) \cap N(U, U) \in \mathcal{F}$ for any two opene sets U, V. Now we may define other mixings by considering the structure of $\mathcal{F} = \{N(U, V) :\ U, V \text{opene in X}\}$. A motivation for this is the following hierarchy:

$$\mathcal{I}^* \ \subset\ \Delta^* \ \subset\ \mathcal{IP}^* \ \subset \mathcal{C}^*. \tag{1}$$

As an application, we will define Δ^*, $\mathcal{IP}^*$ and $\mathcal{C}^*$-mixings and by constructing examples, the fact that they are different families will be shown. First a theorem:

Theorem 2.2. *A TDS (X, T) is $\mathcal{F}_\bullet$-transitive if and only if it is $\mathcal{F}$-transitive. In particular, suppose $\mathcal{F}$ is a filter. Then (X, T) is $\mathcal{F}_\bullet$-mixing if and only if it is $\mathcal{F}$-mixing.*

By definition $\mathcal{I}^*$ is a filter. Also Δ, $\mathcal{IP}$ and $\mathcal{C}$ are union of ultrafilters, so their dual families are filters. Therefore, all the families appearing in 1 are filters and mixing can be defined for them.

Let $\mathcal{U} = \{U_1, U_2, \cdots, U_n\}$ be an open cover of X. We call this cover *non-trivial* if U_i is not dense in X for $1 \leq i \leq n$. For any finite open cover $\mathcal{U}$, let $N(\mathcal{U})$ be the minimal cardinality of a sub-cover of $\mathcal{U}$. For such cover and an infinite sequence $A = \{a_1, a_2, \cdots\} \subset \mathbb{Z}$ the *complexity function* of $\mathcal{U}$ along a sequence A is defined to be

$$\mathcal{C}_A(\mathcal{U}) = \lim_{n\to\infty} N(\bigvee_{i=1}^{n} T^{-a_i}\mathcal{U}).$$

This complexity function is a tool to classify transitive systems. See [5, 6] for more details.

Let $\mathcal{U}$ be a non-trivial finite cover and $A \subseteq \mathbb{Z}_+$ such that $\mathcal{C}_A(\mathcal{U}) = \infty$. Then (X, T) is called *full-scattering*, *strongly scattering* or *scattering* if A is an arbitrary sequence, $d^*(A) > 0$ or $A = \mathbb{Z}$ respectively. It is easy to see that

Lemma 2.3. *Let (X, T) be a TDS. Then (X, T) is scattering if and only if it is $(PS - PS)^*$-transitive if and only if it is $(C - C)^*$-transitive.*

Definition 2.4. *[7] Let (X, T) be a TDS and $\mathcal{F}$ a family. A point $x \in X$ is* $\mathcal{F}$-transitive point *if for any opene set U of X, $N(x, U) \in \mathcal{F}$. Denote the set of all $\mathcal{F}$-transitive points by $Trans_{\mathcal{F}}(X, T)$. The system (X, T) is called $\mathcal{F}$-point transitive if it has some $\mathcal{F}$-transitive points.*

Theorem 2.5. *An $(\mathcal{F} - \mathcal{F})^*$-transitive TDS is weak disjoint from any $\mathcal{F}$-point transitive TDS.*

Proof. Assume (X, T) is an $(\mathcal{F}-\mathcal{F})^*$-transitive and so $(\mathcal{F}-\mathcal{F})^*_\bullet$-transitive. Let (Y, S) be a transitive TDS with an $\mathcal{F}$-transitive point y. Let U_1, U_2 be opene subsets of X and V_1, V_2 two opene subsets of Y. We want to show that $N(U_1\times V_1, U_2\times V_2)\neq\emptyset$. Since (Y, S) is transitive there exists $k\in\mathbb{Z}$ such that $T^{-k}V_1\cap V_2\neq\emptyset$. Set $V_0=T^{-k}V_1\cap V_2$. Then $N(V_0, V_0)+k\subset N(V_1, V_2)$ and

$$\begin{aligned} N(U_1\times V_1, U_2\times V_2) &= N(U_1, U_2)\cap N(V_1, V_2) \\ &\supset N(U_1, U_2)\cap(N(V_0, V_0)+k). \end{aligned}$$

By the assumption, y is an $\mathcal{F}$-transitive point, so $N(V_0, V_0)\supseteq N(y, V_0)-N(y, V_0)$ and thereby, $N(V_0, V_0)\in\mathcal{F}-\mathcal{F}$. On the other hand, (X, T) is an $(\mathcal{F}-\mathcal{F})^*_\bullet$-transitive and so $N(U_1, U_2)$ has non-empty intersection with any shift of an $(F-F)$-set where $F\in\mathcal{F}$. Hence $(X\times Y, T\times S)$ is transitive. □

Corollary 2.6. *An scattering system is weak disjoint from any $\mathcal{PS}$-point transitive system.*

Proof. By Theorem 2.3,this result is immediate. □

Definition 2.7. *A system (X, T) is called $\mathcal{F}$-scattering if for any open cover $\mathcal{U}$ of X and any $A\in\mathcal{F}$ we have $\mathcal{C}_A(\mathcal{U})=\infty$.*

Is there any relation between $\mathcal{F}$-scattering, $\mathcal{F}^*$-mixing or $(\mathcal{F}-\mathcal{F})^*$-transitive systems? In [6], the authors showed that a minimal TDS is $\mathcal{F}^*$-mixing if and only if it is $\mathcal{F}$-scattering and when a TDS is $\mathcal{F}$-scattering and weak mixing then it is $(\mathcal{F}-\mathcal{F})^*$-transitive .

Theorem 2.8. *If (X, T) is $\mathcal{F}$-scattering, then it is weak disjoint from all $\mathcal{F}$-point transitive systems.*

In [7] the author showed that a dynamical system (X, T) is an E-system if and only if it is $\mathcal{F}_{(d^*>0)}$-point transitive and (X, T) is a M-system if and only if it is $\mathcal{PS}$-point transitive. Then by [6, Lemma 2.4, 2.5Dawoud Ahmadi Dastjerdi and Maliheh Dabbaghian Amiri] we have

Theorem 2.9. *Let $\mathcal{F}\in\{\mathcal{PS}, \mathcal{F}_{d^*>0}\}$. If (X, T) is weak disjoint from all $\mathcal{F}$-point transitive systems, then it is $\mathcal{F}$-scattering.*

Whether scattering implies strongly scattering is an open problem asked in [5, 3]. If (X, T) is scattering, so it is weak scattering and thereby, $N(U, V)\cap B\neq\emptyset$ for any opene sets $U, V\subset X$ and Bohr set B. By Theorem 2.10, if this intersection has positive upper Banach density then (X, T) is strongly scattering.

Theorem 2.10. *Let (X, T) be a TDS. If for any opene sets $U, V\subset X$ and any Bohr set B, $N(U, V)\cap B$ has positive upper Banach density, then (X, T) is strongly scattering.*

Proof. By the definition of strong scattering it is sufficient to show that $N(U, V)\cap(A-A)\neq\emptyset$ where $d^*(A)>0$. For any set A with positive upper Banach density there exist a Bohr set B and a subset of integers N with $d^*(N)=0$ such that $B\subset(A-A)\cup N$ [1, Corollary 5.3Dawoud Ahmadi Dastjerdi and Maliheh Dabbaghian Amiri]. Set $C:=N(U, V)\cap B$. By the assumption $d^*(C)>0$, then $C\not\subset N$. So $(A-A)\cap C\neq\emptyset$ and therefore, $N(U, V)\cap(A-A)\neq\emptyset$. □

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Relative semi attractors

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Abstract

In this paper we are going to have a new approach to $\omega-$limit set and attractor set. Relative semi dynamical system and some of its properties is presented. Relative semi attractor set is considered, that this definition is a natural generalization of the concept of attractor in the work of Huang Tusen [9].
Keywords: $\omega_\mu-$limit set, relative semi-attractor, observer, invariant

Mathematics Subject Classification: 34D45

1 introduction

Let X be a compact metric space and $f = f : X \times R \longrightarrow X$ be a flow on X. $\omega(Y, f)$ is defined to be as a maximal invariant set in closure of $Y.[0, \infty)$ and is called $\omega-$limit of Y. A set A is called an attractor set if there exist a neighborhood N of A such that $A = \omega(N, f)$.
Let X denote a compact metric space with metric ρ. The metric on $X \times X$ is denoted by ρ also.
For $Y \subseteq X$, $Cl(Y)$ denote the closure of Y.
For $Y \subseteq X$ and $J \subseteq R$, $Y \cdot J \equiv f(Y \times J)$. A set Y is invariant under f if $Y \cdot R = Y$.

Definition 1.1. *[4]. A dynamical system is a pair $(\{f^s\}_{s\geq 0}, X)$, where X is a stat space and $f^s : X \to X$ is a family of evolution operators satisfies the following properties:*
i)$f^0 = id$, where id is the identity map on X.
ii)$f^{t+s} = f^t o f^s$. It means that

$$f^{t+s}(x) = f^t(f^s(x))$$

for all $x \in X$ and $t, s \geq 0$.

Definition 1.2. *[2]. For $Y \subseteq X$, $\omega(Y, f)$ is defined to be the maximal invariant set in the closure of $Y \cdot [0, +\infty]$. This set is called the $\omega-$limit set of Y.*

The set $\omega(Y, f)$ can otherwise be expressed as $\bigcap\{Cl(Y \cdot [t, +\infty))|t \geq 0\}$. For $Y = \{x\}$, $\omega(x, f)$ is the usual $\omega-$limit set of x.

Definition 1.3. *[1]. A set A is called an attractor for f if A admits a (closed) neighborhood N such that $A = \omega(N, f)$.*

Let $\{X, \tau\}$ is a topological space and $f = f : X \times R \longrightarrow X$ a flow, we define $f^s(x) = f(s, x)$, so $(\{f^s\}_{s\geq 0}, X)$ is a semi dynamical system.
In this study we assume that the relative sets are members of L^X, where L is $[0, 1]$ and X is a non-empty set.

Definition 1.4. *[6]. An observer of a set X is a function $\mu : X \longrightarrow [0, 1]$.*

Definition 1.5. *Let $\mu \in L^X$ and τ_μ be a collection of the elements of L^X which are contained in $\mu(\lambda \in \tau_\mu)$ implies $\lambda \subseteq \mu$). If τ_μ has the following properties: [8]*
i) $\mu, \chi_\emptyset \in \tau_\mu$, where $\chi_\emptyset(x) = 0$, for all $x \in X$.
ii) If $\lambda_1, \lambda_2 \in \tau_\mu$, then $\lambda_1 \cap \lambda_2 \in \tau_\mu$, where $\lambda_1 \cap \lambda_2 : X \longrightarrow [0,1]$ is defined by $(\lambda_1 \cap \lambda_2)(x) = \min\{\lambda_1(x), \lambda_2(x)\}$.
iii) If $\{\lambda_\gamma \mid \gamma \in \Gamma\} \subset \tau_\mu$, then $\bigcup_{\gamma\in\Gamma} \lambda_\gamma \in \tau_\mu$ where $\bigcup \lambda_\gamma : X \longrightarrow [0,1]$ is defined by $\bigcup \lambda_\gamma(x) = \sup\{\lambda_\gamma(x) \mid \gamma \in \Gamma\}$. Then τ_μ is called a $\mu-$relative topology for μ and elements of τ_μ are called $\mu-$open sets.

Let $\lambda \subset \mu$ and let the function $\lambda^\mu : X \to [0,1]$ be defined by $\lambda^\mu(x) = \mu(x) - \lambda(x)$. Then λ is called relative closed set if λ^μ is relative open set, i.e. $\lambda^\mu \in \tau_\mu$ [6].

Definition 1.6. *[8]. Suppose (μ, τ_μ) and (γ, τ_γ) are two relative topological spaces, where $\mu \in [0,1]^X$ and $\gamma \in [0,1]^Y$. Then a mapping $f : X \longrightarrow Y$ is called (μ,γ)-continuous if: $f^{-1}(\eta) \cap \mu \in \tau_\mu$, for all $\eta \in \tau_\gamma$, where $f^{-1}(\eta)(x) = \eta(f(x))$.*

Let (X, τ_μ) be a relative topological space and $(\{f^s\}_{s\geq 0}, X)$ be a collection of functions on X with conditions:
i) $f^0(x) = x$; for every $x \in X$.
ii) $f^{s+t}(x) = f^s(f(x,t)) = f(x, s+t)$; for every $s, t \geq 0$.
Then $(\{f^s\}_{s\geq 0}, X, \tau_\mu)$, is called a relative semi dynamical system.

2 Main Result

In this section, the invariant sets for $(\{f^s\}_{s\geq 0}, X, \tau_\mu)$ and their properties are considered at first, and what comes next is the concept of closure of a relative set.

2.1 Relative invariant sets

Definition 2.1. *Let $\eta \subseteq \mu$ be an observer in τ_μ. We define the observer of $f^s(\eta)$ to be $f^s(\eta)(x) := \sup(\{\eta(y) \mid f^s(y) = x\}, \{0\})$, for $s \in \mathbb{R}^+$.*

Definition 2.2. *Let $\eta \subseteq \mu$, η is invariant under f if $f^s(\eta) \subseteq \eta$.*

Lemma 2.3. *Any finite union of relative closed sets is a relative closed set.*

We recall [5] that the "intersection of all relative closed sets γ such that $\lambda \subseteq \gamma$ is called the closure of λ and denoted by $\overline{\lambda}$.

Lemma 2.4. *If λ_1, λ_2 are two observer map then $\overline{\lambda_1 \cup \lambda_2} = \overline{\lambda_1} \cup \overline{\lambda_2}$.*

In order to prove proposition 3.8, we need the following lemma.

Lemma 2.5. *Let $\{\eta_i\}_{i\in N} \subseteq \tau_\mu$ then $f^s(\bigcup_{i\in N} \eta_i) = \bigcup_{i\in N} f^s(\eta_i)$.*

The first question which comes into mind is that, what sets are relative invariant

Proposition 2.6. *If η_1, η_2 are two invariant observers. Then their union and intersection are invariant observers.*

We need to define the limit set for relative semi dynamical systems. $\omega-$limit set and their properties are defined in various ways, for example, see [9].

Definition 2.7. *Let τ_μ be a relative topological space. For $\eta \subseteq \mu$, $\omega_\mu(\eta, f)$ is defined to be $\bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} f^s(\eta)}$. For $\eta(x_0) = \chi_{x_0} \in [0,1]$, we have $\omega_\mu(\eta(x_0), f) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} f^s(\eta(x_0))}$.*

In the following proposition, some properties of $\omega_\mu-$limit sets are presented.

Proposition 2.8. *i) Let $\eta_i \subseteq \mu$, $\omega_\mu(\bigcup_{i=1}^n \eta_i, f) = \bigcup_{i=1}^n \omega_\mu(\eta_i, f)$.*
ii) If $\eta_1 \subset \eta_2$, then $\omega_\mu(\eta_1, f) \subset \omega_\mu(\eta_2, f)$.

2.2 The relative semi-attractor

The notion of attractor set have extended from different ways [3, 9, 1, 2]. We would like to present a relative extension of it.

Definition 2.9. *If* $(\{f^s\}_{s\geq 0}, X, \tau_\mu)$ *be a relative semi dynamical system.* $\lambda \subseteq \mu$ *is called relative semi-attractor if there exists relative open set* η*, such that* $\lambda \subseteq \eta \subseteq \mu$ *and* $\omega_\mu(\eta, f) = \lambda$.

This definition is the natural generalization of the concept of attractor which was introduced at first by Huang Tusen [9]. If $\mu = \chi_X$, $\tau_\mu = \{\chi_U \mid U \in \tau\}$ then A is an attractor, if there exists a closed set as N such that $\chi_A = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} f^s(\chi_N)}$.
We first prove the following lemma in order to prove the claim we just made.

Lemma 2.10. *Let* $(\{f^s\}_{s\geq 0}, X)$ *be a semi dynamical systems and* $(\{f^s\}_{s\geq 0}, X, \tau_\mu)$ *be a relative semi dynamical systems, where* $\tau_\mu = \{\chi_U | U \subseteq X$ *is an open set in metric space* $(X, \rho)\}$*. Then the following properties are hold:*
i) $A \subseteq B$ *if and only if* $\chi_A \subseteq \chi_B$ *in relative topology* (X, τ_χ).
ii) The set A *is an open(closed)set if and only if* χ_A *is a relative open(relative closed) set.*
iii) If $Cl(B) = C$ *then* $\overline{\chi_B} = \chi_C$.
iv) Let $N \subseteq X$*, if* $N \cdot [t, +\infty) = C$ *then* $\bigcup_{s\geq t} f^s(\chi_N) = \chi_C$.

Theorem 2.11. *Let* $(\{f^s\}_{s\geq 0}, X)$ *be a semi dynamical systems and* $(\{f^s\}_{s\geq 0}, X, \tau_\mu)$ *be a relative semi dynamical systems, where* $\tau_\mu = \{\chi_U | U \subseteq X$ *is an open set in metric space* $(X, \rho)\}$*. Then* $A \subseteq X$ *is an attractor for semi dynamical system* $(\{f^s\}_{s\geq 0}, X)$ *if and only if* χ_A *is a relative semi attractor for relative semi dynamical system* $(\{f^s\}_{s\geq 0}, X, \tau_\mu)$.

In the following example, a $(\{f^s_{s\geq 0}, X, \tau_\mu\})$ containing a relative attractor is given.

Example 2.12. *Let* $X = [0, \infty)$ *and* $\mu(x) = e^{-x}$*. Suppose* $\eta_n : X \longrightarrow [0, 1]$ *be observer map with* $\eta_n(x) = e^{-nx}$ *for all* $n \geq 2$ *and* $\lambda : X \longrightarrow [0, 1]$ *is an observer map defined as*

$$\lambda(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

Then $\tau_\mu = \{\eta_n, \lambda, \mu, \chi_\emptyset | n \geq 2\}$ *is a relative topology. Let* $f^s : X \longrightarrow X$ *be a map define by* $f^s(x) = 2^{-s}x$ *we show that* $\omega_\mu(\eta_n, f) = \lambda$ *for every* $n \geq 2$.

Theorem 2.13. *Let* $\lambda_1 \cup \lambda_2 = \lambda$ *and* λ_1, λ_2 *be both relative semi attractors for* f*, then* λ *is a relative semi attractor for* f.

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Global stability of disease-free equilibrium of HIV transmission model without health education program

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Abstract

In this paper, a HIV transmission model without health education program is considered. The disease-free equilibria is found and its local as well as global stabilities are investigated. Using a Lyapunov function and LaSalle's invariant set theorem, we proved that the disease-free equilibrium is globally asymptotically stable.

Keywords: HIV/AIDS, Reproduction number, Stability, Equilibria

Mathematics Subject Classification: 37N25

1 Introduction

The extensive spread of human immune deficiency virus (HIV) and the associated acquired immune deficiency syndrome (AIDS) continues around the world since its recognition in the early 1980s.

A number of mathematical models have been designed and used to study the impact of preventive control strategies on the spread of HIV/AIDS in given populations. Some of these studies have shown that a change in risky behaviour is necessary to prevent raging HIV/AIDS prevalence, even in the presence of a vaccine and/or treatment. Recently Hussaini et. al [1], have investigated the impact of public health educational campaigns on the transmission dynamics of HIV/AIDS in some populations. In this paper, we investigate the dynamical behaviour of education-free model.

1.1 Model description

Hussaini et al. constructed a HIV transmission model [1], which is described by

$$
\begin{aligned}
\frac{dS}{dt} &= \Pi(1-p) - \xi S - [\lambda + (1-k)\tilde{\lambda}]S - \mu S, \\
\frac{d\tilde{S}}{dt} &= \Pi p + \xi S - (1-\epsilon)[\lambda + (1-k)\tilde{\lambda}]\tilde{S} - \mu \tilde{S}, \\
\frac{dI}{dt} &= [\lambda + (1-k)\tilde{\lambda}]S - \sigma I - \mu I - \psi_1 I, \\
\frac{dA}{dt} &= \sigma I - \psi_2 A - \mu A - \delta A, \\
\frac{d\tilde{I}}{dt} &= (1-\epsilon)[\lambda + (1-k)\tilde{\lambda}]\tilde{S} + \psi_1 I - \tilde{\sigma}\tilde{I} - \mu \tilde{I}, \\
\frac{d\tilde{A}}{dt} &= \tilde{\sigma}\tilde{I} + \psi_2 A - \mu \tilde{A} - \tilde{\delta}\tilde{A},
\end{aligned}
\tag{1}
$$

where

$$\lambda = \frac{\beta(I+\eta A)}{N}, \quad \tilde{\lambda} = \frac{\beta(\tilde{I}+\tilde{\eta}\tilde{A})}{N}.$$

The associated variables and parameters are described in the following Table.

	Parameters and variables
	Variables Description
N	Adult population
S	Uneducated susceptible individuals
$\tilde{S}$	Educated susceptible individuals
I	Uneducated infecteds with no AIDS symptoms
$\tilde{I}$	Educated infecteds with no AIDS symptoms
A	Uneducated infecteds with AIDS symptoms
$\tilde{A}$	Educated infecteds with AIDS symptoms
λ	Force of infection of uneducated individuals
$\tilde{\lambda}$	Force of infection of educated individuals
	Parameters Description
Π	Recruitment rate of susceptibles
μ	Natural mortality rate
$\delta, \tilde{\delta}$	Disease-induced mortality rates
p	Fraction of educated newly-recruited individuals
ξ	Rate of educating susceptibles
ψ_1, ψ_2	Education rates of individuals in I and A classes
β	Effective contact rate
$\eta, \tilde{\eta}$	Modification parameters
ϵ	Efficacy of education in preventing infection
$1-k$	Reduction in transmissibility of educated individuals
$\sigma, \tilde{\sigma}$	Progression rates to AIDS classes

1.2 Analysis of HIV transmission without health education program

In this section, all education-related parameters and variables are set to zero in order to understand the dynamical behaviour of education-free sub-model. By setting $I = A = p = \xi = \epsilon = k = \sigma = \psi_1 = \psi_2 = 0$ in (13), education-free model is obtained as follows:

$$\begin{aligned} \frac{dS}{dt} &= \Pi - \frac{\beta(I+\varsigma A)}{N}S + \mu S, \\ \frac{dI}{dt} &= \frac{\beta(I+\varsigma A)}{N}S - (\sigma+\mu)I, \\ \frac{dA}{dt} &= \sigma I - (\mu+\delta)A, \end{aligned} \tag{2}$$

where $N = S + I + A$ and

$$D = \{(S, I, A) \in \mathbb{R}^3_+ : N \leq \frac{\Pi}{\mu}\}.$$

1.3 Local stability of disease-free equilibrium

The disease-free equilibrium of (2) is

$$E_0 = (S^*, I^*, A^*) = (\frac{\Pi}{\mu}, 0, 0).$$

1.4 Analytical Derivation of R_0

By using the next generation method [2], the matrices F and V for the new infection terms and the remaining transfer terms respectively, are given by

$$F = \begin{bmatrix} \beta\frac{S^*}{N^*} & \beta\eta\frac{S^*}{N^*} \\ 0 & 0 \end{bmatrix},$$

and

$$V = \begin{bmatrix} \sigma+\mu & 0 \\ -\sigma & \mu+\delta \end{bmatrix}.$$

It follows that the reproduction number, denoted by R_0, is given by

$$R_0 \quad = \quad \rho(Fv^{-1}) = \frac{\beta(\mu+\delta+\eta\sigma)}{(\sigma+\mu)(\mu+\delta)}.$$

Theorem 1.1. *The disease-free equilibrium, E_0, of the education-free model is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$.*

1.5 Global stability of disease-free equilibrium

Theorem 1.2. *The disease-free equilibrium, E_0, of the education-free model is globally asymptotically stable whenever $R0 < 1$.*

Proof. Consider the following Liapunov function

$$L \quad = \quad \beta I(\mu+\delta+\sigma\eta) + \beta\eta A(\mu+\sigma).$$

□

2 Main Result

In this study a HIV transmission model without health education program analysed. Using a Lyapunov function and LaSalle's invariant set theorem, we proved the global asymptotic stability of the disease-free equilibrium.

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$(S,\, S')$-gap Shifts

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Abstract
In this note, we introduce a coded system and call it $(S,\, S')$-gap Shift which is an extension of S-gap shifts and the asymmetric-RLL (d_1, k_1, d_0, k_0) constrained systems. Then the dynamical properties will be investigated and an explicit formula for the entropy will be given.

Keywords: S-gap shift, shift of finite type, sofic, almost-finite-type, periodic-finite-type, entropy.

Mathematics Subject Classification: 37B10, 37B40

1 Introduction

Recall that the Maximum Transition Run (MTR) constrained systems [8] are used to improve timing and detection performance in storage channels. The MTR $(j,\, k)$ code limits the run of 0 to be at most k and the run of 1 at most j. As an extension of MTR codes, consider the asymmetric-RLL(d_1, k_1, d_0, k_0) constrained systems which is the set of binary sequences whose runs of 1's have length between d_1 and k_1 and the runs of 0's between d_0 and k_0. The S-gap shifts in abstract symbolic dynamical system, may be considered as an extension of the run-length limited constrained systems [7] when the run of 0 is restricted to a subset $S \subseteq \mathbb{N} \cup \{0\}$. We further extend these concepts and introduce here the $(S,\, S')$-gap shifts as an extension of S-gap shifts on one side and also an extension of the asymmetric-RLL(d_1, k_1, d_0, k_0) constrained systems when the run of 0 and the run of 1 are restricted to subsets of positive integers S and S' respectively. To be more specific, fix two increasing sequences S and S' in $\mathbb{N}$. Define $X(S,\, S')$ to be the set of all binary sequences for which 0's and 1's occur infinitely often in each direction such that the run of 0 is an integer in S and the run of 1 is an integer in S'. When S (resp. S') is infinite, we need to allow points that begin or end with an infinite string of 0 (resp. 1).

In the course of our investigation, we will be needing two sequences obtained from the difference of two successive s_n in S and two successive s'_n in S'. That is, let $\Delta(S) = \{d_n\}_n$ and $\Delta(S') = \{d'_n\}_n$ where $d_1 = s_1$, $d'_1 = s'_1$, $d_n = s_n - s_{n-1}$ and $d'_n = s'_n - s'_{n-1}$, $n \geq 2$.

Our main notations and main definitions are taken from the classic book of symbolic dynamics by Lind and Marcus [7]. A shift space X has the *specification with variable gap length* if there exists $N \in \mathbb{N}$ such that for all $u, v \in \mathcal{B}(X)$, there exists $w \in \mathcal{B}(X)$ with $uwv \in \mathcal{B}(X)$ and $|w| \leq N$ [5]. A word $v \in \mathcal{B}(X)$ is *synchronzing* if whenever uv and vw are in $\mathcal{B}(X)$, we have $uvw \in \mathcal{B}(X)$. An irreducible shift space X is a *synchronized system* if it has a synchronzing word [3].

Set $\mathcal{F}$ to be a finite collection of words over a finite alphabet $\mathcal{A}$ where each $w_j \in \mathcal{F}$ is associated with a non-negative integer index n_j. Write

$$\mathcal{F} = \{w_1^{(n_1)}, w_2^{(n_2)}, ..., w_{|\mathcal{F}|}^{(n_{|\mathcal{F}|})}\}$$

and associate with the indexed list $\mathcal{F}$ a period T, where T is a positive integer satisfying $T \geq \max\{n_1, n_2, ..., n_{|\mathcal{F}|}\} + 1$. A shift space X is a shift of *periodic-finite-type* (PFT) if there exists a pair $\{\mathcal{F}, T\}$ with $|\mathcal{F}|$ and T finite so that $X = X_{\{\mathcal{F},T\}}$ is the set of bi-infinite sequences that can be shifted such that the shifted sequence does not contain a word $w_j^{n_j} \in \mathcal{F}$ starting at any index m with $m \mod T = n_j$. An strictly PFT shift cannot be represented as an SFT.

2 Dynamical Properties of $(S,\,S')$-gap Shifts

As in the S-gap shifts, $(S,\,S')$-gap shifts are highly chaotic, for they are transitive with dense periodic points. Note that $(S,\,S')$-gap shifts are all synchronized. This can be deduced directly by showing that 10 and 01 are synchronizing words. These systems are coded systems: there is a countable collection of words such that the sequences which are concatenations of these words are a dense subset in $X(S,\,S')$ [4, §3.5.4Dawoud Ahmadi Dastjerdi and Somayeh Jangjoo]. Now we investigate the dynamical properties of $(S,\,S')$-gap shifts in terms of S and S' and give an sketch of proof for any theorem.

Theorem 2.1. *The followings are equivalent.*

1. *$X(S,\,S')$ is mixing;*
2. *$\gcd(S+S') = \gcd\{s+s' :\; s \in S,\, s' \in S'\} = 1$;*
3. *$X(S,\,S')$ is totally transitive.*

Proof. We only prove $2 \Rightarrow 1$. All other cases are more or less similar to the results for S-gap shifts in [1, 6]. If $\gcd(S+S') = \gcd\{s+s' :\; s \in S,\, s' \in S'\} = 1$, then for all sufficiently large n, there is a word of length n of the form $10^{m_1}1^{n_1}0^{m_2}1^{n_2}\cdots 0^{m_k}1^{n_k}0$. But 10 is synchronizing and $X(S,\,S')$ is irreducible and so the result follows. □

Theorem 2.2. *$X(S,\,S')$ is*

1. *SFT if and only if S and S' are finite or cofinite.*
2. *sofic if and only if $\Delta(S)$ and $\Delta(S')$ are eventually periodic.*
3. *AFT if and only if $\Delta(S)$ and $\Delta(S')$ are eventually constant.*
4. *specification with variable gap length if and only if $\sup_i |s_{i+1} - s_i| < \infty$ and $\sup_i |s'_{i+1} - s'_i| < \infty$.*

Proof. The proof for (1) is obvious. (2) and (3) are proved as S-gap shifts [1, Theorem 3.4, Theorem 3.6Dawoud Ahmadi Dastjerdi and Somayeh Jangjoo]. For proving (4), we use the Jung's routines for a similar assertion for S-gap shifts [5]. □

Theorem 2.3. *Suppose $X(S,\,S')$ is not SFT. Then it is PFT if and only if it is AFT and non-mixing.*

Proof. For the sufficiency, we refer the reader to the similar proof for S-gap shifts in [1, Theorem 3.8Dawoud Ahmadi Dastjerdi and Somayeh Jangjoo]. To prove the necessity, suppose $\mathcal{G} = (G, \mathcal{L})$ is the minimal right-resolving presentation of $X(S,\,S')$ and let $p = \mathrm{per}(A_{\mathcal{G}})$. Then $\mathcal{O} = \{(10)^i, (01)^i :\, i \in \{0, 1, \cdots, p-3, p-1\}\}$ where $\mathcal{O}$ is the collection of all periodic first offenders [2]. Since $\mathcal{O}$ is finite $X(S,\,S')$ is PFT [2, Corollary 14Dawoud Ahmadi Dastjerdi and Somayeh Jangjoo]. □

Let S and S' be the subsets of $\mathbb{N}$. If the multiplicity in $S+S'$ is imoprtant we will show it by $\{\{S+S'\}\}$. For a synchronized system X, fix a synchronizing word $w \in \mathcal{B}(X)$. Let $C_n(X)$ be the set of words $v \in \mathcal{B}_n(X)$ such that $wvw \in \mathcal{B}(X)$. Then the synchronized entropy $h_{\rm syn}(X)$ is defined by

$$h_{\rm syn}(X) = \limsup_{n\to\infty} \frac{1}{n} \log |C_n(X)|.$$

This value is independent of w. Jung [5] showed that for specification with variable gap length shifts $h(X) = h_{\mathrm{syn}}(X)$. We exploit this to compute the entropy for $(S,\, S')$-gap shifts.

Theorem 2.4. *The entropy of an $(S,\, S')$-gap shift is $\log\lambda$ where λ is the unique non-negative solution of*

$$\sum_{s+s'\in\{\!\{S+S'\}\!\}} x^{-(s+s')} = \sum_{n\in S+S'}\left(\sum_{k+l=n}\chi_S(k)\chi_{S'}(l)\right)x^{-n} = 1.$$

Proof. Since $X = X(S,\, S')$ is a synchronized system, we will compute $h_{\mathrm{syn}}(X)$. Let $C_i = C_i(X)$. By an induction arguement on n, we will have,

$$\begin{aligned} C_n \quad &= \sum_{k+l=n+2}\chi_S(k)\chi_{S'}(l) + \sum_{p\in S+S'}\left(\sum_{k+l=p}\chi_S(k)\chi_{S'}(l)\right)C_{n-p} + \\ &\chi_{S\cap S'}(1)\left(\chi_S(n-1)\chi_{S'}(n-1) + \sum_{k+l=n}\chi_S(k)\chi_{S'}(l)\right). \end{aligned}$$

Now for some subsequence $\{n_k\}_{k\in\mathbb{N}}$, C_{n_k} is asymptotic to λ^{n_k} as $k\to\infty$ where $\lambda = 2^{h_{\mathrm{syn}}(X)} > 1$. Therefore,

$$\lim_{n\to\infty}\frac{C_{n_k-p}}{C_{n_k}} = \lambda^{-p}.$$

Then $\sum_{p\in S+S'}\left(\sum_{k+l=p}\chi_S(k)\chi_{S'}(l)\right)\frac{1}{\lambda^p} = 1$. So, $h_{\mathrm{syn}}(X) = \log\lambda$ where λ is the non-negative solution of the equation

$$\sum_{s+s'\in\{\{S+S'\}\}} x^{-(s+s')} = 1.$$

If X has specification with variable gap length, then $h(X) = h_{\mathrm{syn}}(X)$ [5]. Therefore, suppose $X(S,\, S')$ does not have such a property. Let $S_n = \{s_1,\, s_2, \cdots,\, s_n\}$ and $S'_n = \{s'_1,\, s'_2, \cdots,\, s'_n\}$, $n\in\mathbb{N}$. If one of S or S', say S, is finite, then there is an $N\in\mathbb{N}$ such that $S_n = S$ for all $n\geq N$. Then $h(X(S_n, S'_n)) = h_{\mathrm{syn}}(X(S_n,\, S'_n))$ and the sequence $\{h(X(S_n, S'_n))\}_n$ is an increasing sequence such that $h(X(S_n, S'_n)) \longrightarrow h(X(S, S'))$. Since for all n, $X(S_n, S'_n)$ has specification with variable gap length, we are done. □

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Dynamics of geodesic flow on $\langle z+2, -\frac{1}{z}\rangle\backslash\mathcal{H}$

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Abstract
Let $\mathcal{H} = \{z = x + iy : y > 0\}$ and $G = \langle z+2, -\frac{1}{z}\rangle$, $z \in \mathbb{C}$. This group acts on the upper half plane, $\mathcal{H}$, and the associated quotient surface is topologically a sphere with two cusps. We conjugate the geodesic flow on this surface to a special flow over the symbolic space of geometric codes associated to this flow. We will show that for $k \geq 1$, a subsystem with codes from $\mathbb{Z} \setminus \{0, \pm 1, \pm 2, \cdots, \pm k\}$ is a TBS. We also give bounds for the entropy of these subsystems.
Keywords: geodesic flow, geometric code, arithmetic code, topological entropy

Mathematics Subject Classification: 37D40, 37B40, 20H05

1 Introduction

Let $\mathcal{H} = \{z = x + iy : y > 0\}$ and $G = \langle z+2, -\frac{1}{z}\rangle$ with generators $T(z) = z+2$ and $S(z) = -\frac{1}{z}$. The group G acts on $\mathcal{H}$ discontinuously with a Dirichlet fundamental domain $|z| \geq 1, |\text{Re}z| \leq 1\}$. The associated quotient space $G\backslash\mathcal{H}$ is a finite area Riemann surface with one elliptic point, as its only singular point, and two cusps. This surface is topologically a sphere with two punctures and we denote it by M^{c2} (a sphere with two cusps). One of our goals is to study the dynamics of geodesic flow on M^{c2} which will not go to the cusps in either directions. Lifting the geodesics to TM^{c2}, the unit tangent bundle of M^{c2}, gives the geodesic flow as an invariant set on M^{c2}. In this note, we introduce geometric codes for geodesic flow on M^{c2}. Basically, these codes are bi-infinite sequences of nonzero integers which tell how a geodesic enters F infinitely many times in past and future. In fact, these codes together with the length of geodesic between two successive return of geodesic to F reveals the dynamics of geodesic flow. For then we can construct a special flow, conjugate to our flow, whose base space is the symbolic space obtained by these codes and its height function is the aforementioned length. We first introduce parameter space and then we will obtain the codes from that space. Then we give an upper and a lower bound for the topological entropy of subsystems with codes in $\mathbb{Z} \setminus \{n : |n| \leq k, k \geq 2\}$.

2 Geometric code for geodesics on M^{c2}

We apply Morse method to have the geometric codes of the geodesic flow on M^{c2}. Label the circular side of F by s and the sides $x = -1$ and $x = 1$ by t^{-1} and t respectively. We consider the oriented geodesics which enter F via side s and call them *reduced geodesics*. Any geodesic on $\mathcal{H}$ is G-equivalent to a reduced geodesic. If $\gamma = (w, u)$ is a reduced geodesic with repelling and attracting endpoints w and u respectively, then $|w| > 1$ and $|u| < 1$. By Morse method we start from an initial point of a reduced geodesic on s and move in the direction of the geodesic and count the number of times that the geodesic hits sides t or t^{-1}. A bi-infinite sequence of non-zero integers will be assigned to γ called the *geometric code* of γ. Denote the geometric code of γ by $[\gamma] = [..., n_{-1}, n_0, n_1, ...]$.

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Consider the wu coordinate in the plane. The lines $w = \pm 1$ and $u = \pm 1$ partition the plane to 9 regions. Let $\mathbb{T}_n \subset \mathbb{T}$ be the square whose opposite vertices are $(2n-1, -1)$ and $(2n+1, 1)$, $n \in \mathbb{Z} \setminus \{0\}$. Now we show how the parameter space evolves geometric codes. Start with a reduced geodesic $\gamma = (w, u) \in \mathbb{T}_{n_0}$. Let $ST^{-n_0}(w, u) \in \mathbb{T}_{n_1}$ and set $\mathbb{T}_{n_0,n_1} = ST^{-n_0}(\mathbb{T}_{n_0}) \cap \mathbb{T}_{n_1}$. Inductively, let $\mathbb{T}_{n_0, n_1, \ldots, n_k} := ST^{-n_{k-1}}(\mathbb{T}_{n_0, n_1, \ldots, n_{k-1}}) \cap \mathbb{T}_{n_k}$ containing the reduced geodesic $ST^{-n_{k-1}}ST^{-n_{k-2}}...ST^{-n_0}(w, u)$. Note that $\mathbb{T}_{n_0, n_1, \ldots, n_k}$ contains all geodesics having n_i as the ith entry in their geometric code, $0 \le i \le k$.

Let $\mathcal{A}$ be a set of countable alphabets. Consider the space $\Sigma \subseteq \Sigma_{\mathcal{A}} = \{x = (x_i)_{i=-\infty}^{\infty}, x_i \in \mathcal{A}\}$ and the shift map $\sigma : \Sigma \to \Sigma$ defined by $\sigma(x_i) = x_{i+1}$. The symbolic dynamical system (Σ, σ) is called two-sided *countable topological Bernoulli scheme* (TBS) if $\Sigma = \Sigma_{\mathcal{A}}$.

Let $\mathcal{B} \subseteq \mathbb{Z} \setminus \{0, \pm 1\}$ and $\Sigma_{\mathcal{B}}$ be the space of geometric codes whose alphabets are from $\mathcal{B}$.

Theorem 2.1. *The space $\Sigma_{\mathcal{B}}$ is a TBS.*

Proof. Consider the region $\mathbb{T}_{n_0}$, $n_0 \in \mathcal{B}$. Since $ST^{-n_0}(\mathbb{T}_{n_0}) \cap \mathbb{T}_{n_1}$ is nonempty for all $n_1 \in \mathcal{B}$ it follows that $[n_0, n_1]$ is an admissible block. □

Corollary 2.2. *Let $\cdots, n_{-1}, n_0, n_1, \cdots$ be a bi-infinite sequence in $\mathbb{Z}\setminus\{0\}$ whose tails are neither all 1 nor all -1. Then this sequence represents a geometric code of an oriented geodesic on M^{c2} not going to cusps in positive or negative times.*

3 Entropy

In this section we use the method in [1] to give bounds for topological entropy on our subsystems which are all TBS. The subsystems we have chosen are those whose alphabets are in $\mathbb{Z}\backslash\{0, \pm1, \pm2, ..., \pm k\}$ and those with alphabets in $\mathbb{N}\backslash\{1, 2, ..., k\}$, $k \ge 2$.

Define the family $T_{\ell,\Sigma} = \{T^s_{\ell,\Sigma}\}_{s\in\mathbb{R}}$ to be the *special flow* constructed over the *base space* Σ and *height function* ℓ. For $k \in \mathbb{N} \setminus \{1\}$, let $\mathcal{A}_k = \{n : |n| \ge k\}$ and $h(T_{\ell,\Sigma_{\mathcal{A}_k}})$ the topological entropy of $T_{\ell,\Sigma_{\mathcal{A}_k}}$.

Theorem 3.1. *Let $\zeta(\cdot)$ be the Riemann zeta function. Then $x_l < h(T_{\ell,\Sigma_{\mathcal{A}_k}}) < x_u$ where for $\alpha \in \{l, u\}$, x_α is the unique solution of*

$$2c_\alpha^{-2x}\left(\zeta(2x) - \sum_{n=1}^{k-1}\frac{1}{n^{2x}} + \frac{1}{k^{2x}}\right) = 1. \tag{1}$$

Here $c_u = 2 - \frac{1}{k[2k]} = 2 - \frac{1}{k(k+\sqrt{k^2-1})}$ and $c_l = 2 + \frac{1}{k[2k]}$.

Theorem 3.2. *Let $x = [\gamma]$ be the geometric code of γ with repelling and attracting points $w = w(x)$ and $u = u(x)$ respectively. Then $\ell(x) = 2\ln(w(x)) + \ln(g(x)) - \ln(g(\sigma x))$ where $g(x) = \frac{(w(x)-u(x))\sqrt{w(x)^2-1}}{w(x)^2\sqrt{1-u(x)^2}}$.*

Proof. With almost no change, the lines of proof is similar to the proof of [2, Theorem 4Dawoud Ahmadi Dastjerdi and Sanaz Lamei]. Just let z_1 and z_1' be the intersection of $\gamma = (w, u)$ with $|z| = 1$ and $|z - 2n_1| = 1$ respectively. □

For any subsystem $\Sigma_{\mathcal{B}}$ of Σ denote the positive continued functions like $f(x)$ depending on the zero coordinate and satisfying the condition $\sum_{k=1}^{\infty} f(\sigma^k(x)) = \sum_{k=1}^{\infty} f(\sigma^{-k}(x)) = \infty$ by $\mathcal{F}_0(\Sigma_{\mathcal{B}})$.

Let H be a directed graph with vertex set $V = \mathcal{A}$ and the edge set E. The path $\tau = (v_0, ..., v_n)$ is called a *simple v-cycle* if $v_0 = v_n = v$ and $v_i \ne v$ for $1 \le i \le n-1$. Let $C(H; v)$ be the set of all simple v-cycles in the graph H. Let $f \in \mathcal{F}_0(\Sigma_{\mathcal{B}})$ and $F_{f,V}(x) = \sum_{v\in V} x^{f(v)}$ be a series for $x \ge 0$ and set $\phi_{H,f,w}(x) = \sum_{\tau\in C(H;w)} x^{f^*(\tau)}$, $x \ge 0$ be the *generating function* with respect to the special flow $T_{f,\Sigma}$ where $f^*(\tau) = \sum_{i=0}^{n} f(v_i)$, $\tau = (v_0, ..., v_n)$.

Remark 3.3. *Let $(\Sigma_{\mathcal{B}}, \sigma)$ be a 1-step topological Markov chain and $f \in \mathcal{F}_0(\Sigma_{\mathcal{B}})$. Then by [1, Remark 1Dawoud Ahmadi Dastjerdi and Sanaz Lamei], $h(T_{f,\Sigma}) = -\ln(\hat{x}_f)$ where $\hat{x}_f$ is either the unique solution of $\phi_{H,f,v}(x) = 1$ or $\hat{x}_f = r(\phi_{H,f,v})$.*

Lemma 3.4. *Let $c > 1$ and $f(x) = 2\ln(cn_0)$, $|n_0| \geq k \geq 2$. Then $h(T_{f,\Sigma_{\mathcal{A}_k}}) = -\hat{x}_f$ where $\hat{x}_f$ is the unique solution of $\phi_{H_{\mathcal{A}_k},f,v_k}(x) = 1$.*

Proof. Since $f(x) = 2\ln(cn_0)$ and $|n_0| \geq k$ so $f \in \mathcal{F}_0(\Sigma_{\mathcal{A}_k})$. Let $H_k := H_{\mathcal{A}_k}$ be a complete graph with vertex set $V(H_k) = \mathcal{A}_k$ and edge set $E(H_k)$. Apply [1, Lemma 1Dawoud Ahmadi Dastjerdi and Sanaz Lamei] for $m = 1$ to have $\phi_{H_k,f,v_k}(x) = \alpha_{00}(x) + \alpha_{10}(x)A_1(x)$. This implies $h(T_{f,\Sigma_{\mathcal{A}_k}}) = -\ln(\hat{x}_f)$ where $\hat{x}_f$ is either the unique solution of $\phi_{H_k,f,v_k}(x) = 1$ or $\hat{x}_f = r(\phi_{H_k,f,v_k}) = r(A_1)$. We want to show that for our case $\hat{x}_f$ is the unique solution of $\phi_{H_k,f,v_k}(x) = 1$. For $0 \leq x < r(F_{f,V(H_k)})$ set

$$\tilde{x}_0 = \begin{cases} r(F_{f,V(H_k)}), & \text{if } M(x) \text{ is invertible} \\ \inf\{x : 0 \leq x < r(F_{f,V(H_k)}), \det M(x) = 0\}, & \text{otherwise.} \end{cases} \quad (2)$$

From [1, Theorem 2Dawoud Ahmadi Dastjerdi and Sanaz Lamei] we have $\lim_{x\to\tilde{x}_0^-} \phi_{H_k,f,v_k}(x) = \infty$ which means $\phi_{H_k,f,v_k}(x) = 1$ has a solution in $0 < x < r(F_{f,V(H_k)})$. We will show that this is indeed the case. We achieve this if $\det M(x) = 0$ in $0 < x < r(F_{f,V(H_k)})$. But for $T_{f,\Sigma_{\mathcal{A}_k}}$, $\det M(x) = 1 - 2\sum_{n=k}^{\infty} x^{2\ln cn}$. So by setting $\ln\frac{1}{x} = s$, we have

$$\frac{c^{2s}}{2} = \sum_{n=k}^{\infty} \frac{1}{n^{2s}}. \quad (3)$$

Now (3) has a unique solution on $\frac{1}{2} < s < \infty$ or $M(x)$ has a unique solution on $0 < x < \frac{1}{\sqrt{e}}$. □

Proof of Theorem 1a. For $x = (..., n_0, n_1, ...)$

$$c_u|n_0| = |2n_0| - \frac{1}{[2k]} \leq |w(x)| = |2n_0 - \cfrac{1}{2n_1 - \cfrac{1}{2n_2 - \cfrac{1}{\ddots}}}| \leq |2n_0| + \frac{1}{[2k]} = c_l|n_0|, \quad (4)$$

where $c_l = 2 + \frac{1}{k[2k]}$ and $c_u = 2 - \frac{1}{k[2k]}$. Let $f_\alpha(x) = 2\ln c_\alpha|n_0|$ where $\alpha \in \{l, u\}$. Then by Abramov formula, $h(T_{f_l,\Sigma_{\mathcal{A}_k}}) \leq h(T_{f,\Sigma_{\mathcal{A}_k}}) \leq h(T_{f_u,\Sigma_{\mathcal{A}_k}})$.

We have $\phi_{H_k,f,v_k}(x) = \frac{x^{f(v)}}{1-x^{f(v)}-F_{f,V(H_k)}(x)}$ when $1 - x^{f(v)} - F_{f,V(H_k)}(x) > 0$ [4] and H_k is the complete graph introduced in the proof of Lemma 13. See also [1, Remark 2Dawoud Ahmadi Dastjerdi and Sanaz Lamei]. By the above lemma, $\hat{x}_l$ is the unique solution of $\phi_{H_k,f_l,v_k}(x) = 1$ and by letting $x_l = -\ln\hat{x}_l$, we have x_l is the solution of

$$2c_l^{-2x}\left(\zeta(2x) - \sum_{n=1}^{k-1}\frac{1}{n^{2x}} + \frac{1}{k^{2x}}\right) = 1.$$

□

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The homology of Smale spaces

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Abstract
Motivated by knowing the K-theory of the C^*algebra related to the SFT spaces, Krieger gave a description of the invariants (homolomy) more directly related to the dynamics of the SFT(zero-dimensional). Putnam extended the definition of Krieger's invariants so that they were defined for all Smale spaces which are irreducible. Our aim in this paper is to study and compare these invariants.
Keywords: Smale Spaces, Homology, Shift of Finite Type Spaces

Mathematics Subject Classification: 37D99

1 Introduction

David Ruelle defined Smale spaces, including the basic sets for Smale's Axiom A systems, extended Axiom A systems in terms of smooth maps of Riemannian manifolds to the topological case [2, 6].

Definition 1.1. *Suppose that (X, φ) is a dynamical system. (X, φ) is called a Smale space if there exist constants ε_X and $\lambda > 1$ and a continuous map from*

$$\triangle_{\varepsilon_X} = \{(x, y) \in X \times X \quad | \quad d(x, y) \leq \varepsilon_X \quad \}$$

to X (denoted with $[,]$) such that:
B 1 $[x, x] = x$, B 2 $[x, [y, z]] = [x, z]$,
B 3 $[x, y], z] = [x, z]$, B 4 $[\varphi(x), \varphi(y)] = [x, y]$,
C 1 $d(\varphi(x), \varphi(y)) \leq \lambda d(x, y), [x, y] = y$,
C 2 $d(\varphi^{-1}(x), \varphi^{-1}(y) \leq \lambda d(x, y), [x, y] = x$.
Whenever both sides of an equation are defined.

Examples of Smale spaces include solenoids, substitution tiling spaces, the basic sets for Smale's Axiom A systems and shifts of finite type where the last classes are in homology defined for Smale's spaces[3].

Motivated by the K-theory of the C^*-algebras involved, Krieger gave a description of the invariants more directly related to the dynamics for Shift of finite type. They are ordered abelian groups having certain properties which are summarized by saying they are dimension groups. In fact, there are two such invariants associated to each shift of finite type, which Krieger called the past and the future dimension groups. Krieger's starting point was the realization that two C^*-algebras could be constructed from a shift of finite type. This construction is quite ingenious, but since that time, more sophisticated techniques have been developed and, in modern language, these are the C^*-algebras associated with stable and unstable equivalence[8].Krieger's construction of algebras from a shift of finite type was extended to the general setting of Smale spaces by David Ruelle. Unlike the situation for shifts of finite type, the K-theory of these algebras is not immediately clear. Putnam defined a new homology to compute these K-theory. In fact Putnam defined this homology based on Bowen's Theorem and Krieger's dimension group invariant for shifts of finite, where the existence of such a theory was conjectured by Rufus Bowen[2, 4, 5].

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Lemma 1.2. (Bowen) *Let (X,φ) be an irreducible Smale space. Then there exists a shift of finite type (Σ,σ) and a finite-to-one factor map $\Sigma \to X$.* [5]

Definition 1.3. *Let (X,φ) and (Y,ψ) be Smale spaces and let $\pi : (Y,\psi) \to (X,\varphi)$ be a map. We say that π is s-resolving, u-resolving (or s-bijective, u-bijective) if, for any y in Y, its restriction to $Y^s(y)$, $Y^u(y)$ is injective (bijective).*

Definition 1.4. *Let (X,φ) be a Smale space. We say that $\pi = (Y,\psi,\pi_s,Z,\zeta,\pi_u)$ is an s/u-bijective pair for (X,φ) if*
1. (Y,ψ)aAnd (Z,ζ) are Smale space, 2. $\pi_s : (Y,\psi) \to (X,\varphi)$ is an s-bijective factor map, 3. $Y^u(y)$ is totally disconnected, for every y in Y, 4. $\pi_u : (Z,\zeta) \to (X,\varphi)$ is a u-bijective factor map, 5. $Z^u(z)$ is totally disconnected, for every z in Z.

One drawback in bowen's theorem is that the constructed factor map does not possess any of the resolving properties above. Therefore, Putnam established a complete version that satisfies resolving properties.

Theorem 1.5. *If (X,φ) is a non-wandering Smale space, then there exists an s/u-bijective pair for (X,φ).*

Definition 1.6. *Let $\pi = (Y,\psi,\pi_s,Z,\zeta,\pi_u)$ be an s/u-bijective pair for the Smale space (X,φ). For each $L, M \geq 0$, we define*

$$\Sigma_{L,M}(\pi) = \{(y_0,...,y_L,z_0,...,z_M)| \quad \pi_s(y_l) = \pi_u(z_m), 0 \leq l \leq L, 0 \leq m \leq M\}.$$

For convenience, we also let $\Sigma(\pi) = \Sigma_{0,0}(\pi)$.

Theorem 1.7. *If $\pi = (Y,\psi,\pi_s,Z,\zeta,\pi_u)$ be an s/u-bijective pair for the (X,φ). then for all $L, M \geq 0$, $\Sigma_{L,M}(\pi)$ is a shift of finite type.*

Let (Σ,σ) be a shift of finite type. We will assign to (Σ,σ) two abelian groups, denoted $D^s(\Sigma,\sigma)$ and $D^u(\Sigma,\sigma)$. Let $\mathrm{D}^s(\Sigma,\sigma)$ be the collection of all non-empty, compact, open subsets of $\Sigma^s(e)$, over all e in Σ. Notice that if E is in $\mathrm{D}^s(\Sigma,\sigma)$, then any non-empty subset which is compact and open relative to E is also in $\mathrm{D}^s(\Sigma,\sigma)$. Let $\sim$ be the smallest equivalence relation on $\mathrm{D}^s(\Sigma,\sigma)$ such that $E \sim F$ if $[E,F] = E$ and $[F,E] = F$ (meaning that both sets are defined) and such that $E \sim F$ if and only if $\sigma(E) \sim \sigma(F)$. We let $[E]$ denote the equivalence class of E.

Definition 1.8. *Let (Σ,σ) be a shift of finite type. The group $D^s(\Sigma,\sigma)$ is defined to be the free abelian group on the $\sim$-equivalence classes of $\mathrm{D}^s(\Sigma,\sigma)$, modulo the subgroup generated by $[E \cup F] - [E] - [F]$, where E, F and $E \cup F$ are in $\mathrm{D}^s(\Sigma,\sigma)$ and E and F are disjoint.*

The group $D^u(\Sigma,\sigma)$ is defined to be the free abelian group on the $\sim$ equivalence classes of $\mathrm{D}^u(\Sigma,\sigma)$, modulo the subgroup generated by $[E \cup F] - [E] - [F]$, where E, F and $E \cup F$ are in $\mathrm{D}^u(\Sigma,\sigma)$ and E and F are disjoint.

Definition 1.9. *1. For each $L, M \geq 0$, we define*

$$C^s(\pi)_{L,M} = (D^s(\Sigma_{L,M}(\pi)),\sigma).$$

We define these groups to be zero for all other integral values of L, M. We also define

$$d^s(\pi)_{L,M} = \Sigma_{0\leq l\leq L}(-1)^l\delta^s_{l,} + \Sigma_{0\leq m\leq M+1}(-1)^{L+m}\delta^{s^*}_{,m}$$

On the summand $C^s(\pi)_{L,M}$. 2. For each $L, M \geq 0$, we define

$$C^u(\pi)_{L,M} = (D^u(\Sigma_{L,M}(\pi)),\sigma).$$

and zero for all other integral values on L, M. We also define

$$d^u(\pi)_{L,M} = \Sigma_{0\le l\le L+1}(-1)^l \delta^{u^*}_{l,} + \Sigma_{0\le m\le M}(-1)^{L+m}\delta^u_{,m}$$

on the summand $C^u(\pi)_{L,M}$. For each integer N, consider

$$\oplus_{L-M=N} d^s(\pi)_{L,M} : \oplus_{L-M=N} C^s(\pi)_{L,M} \to \oplus_{L-M=N-1} C^s(\pi)_{L,M},$$

And the obvious actions of the groups $S_{L+1,}$, $S_{,M+1}$ and $S_{L+1\times M+1}$ on $\Sigma_{L,M}(\pi)$, $C^s(\pi)_{L,M}$(and $C^u(\pi)_{L,M}$).

Definition 1.10. *Let $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ be an s/u-bijective pair for the (X, φ). Let $L, M \ge 0$,*

1. We define $D^s_{B,}(\Sigma_{L,M}(\pi))$ to be the subgroup of $D^s(\Sigma_{L,M}(\pi))$ generated by i) All elements b such that $b = b.(\alpha, 1)$ for some transposition α in S_{L+1}, ii) All elements of the form $a - sgn(\alpha)a(\alpha, 1)$ for a in $D^s(\Sigma_{L,M}(\pi))$ and α in S_{L+1}.

2. We define $D^s_{Q,}(\Sigma_{L,M}(\pi))$ to be the quotient of $D^s(\Sigma_{L,M}(\pi))$ by the subgroup $D^s_{B,}(\Sigma_{L,M}(\pi))$ and let Q denote the quotient map.

3. We define $D^s_{,A}(\Sigma_{L,M}(\pi))$ to be the subgroup of all elements a in $D^s(\Sigma_{L,M}(\pi))$ satisfying $a = sgn(\beta)a(\beta, 1)$ for all a in S_{M+1} and let J denote the inclusion map.

4-We define $D^s_{Q,A}(\Sigma_{L,M}(\pi))$ to be the image in $D^s_{Q,}(\Sigma_{L,M}(\pi))$ of $D^s_{,A}(\Sigma_{L,M}(\pi))$ under Q. We let Q_A denote the restriction of Q to $D^s_{,A}(\Sigma_{L,M}(\pi))$ and J_Q denote the inclusion of $D^s_{Q,A}(\Sigma_{L,M}(\pi))$ in $D^s_{Q,}(\Sigma_{L,M}(\pi))$.

There are analogous definitions of $D^u_{,A}(\Sigma_{L,M}(\pi))$, $D^u_{Q,}(\Sigma_{L,M}(\pi))$ and $D^s_{Q,A}(\Sigma_{L,M}(\pi))$. We define
$C^s_{Q,}(\pi)_{L,M} = D^s_{Q,}(\Sigma_{L,M}(\pi))$, $C^s_{,A}(\pi))_{L,M} = D^s_{,A}(\Sigma_{L,M}(\pi))$, $C^s_{Q,A}(\pi)_{L,M} = D^s_{Q,A}(\Sigma_{L,M}(\pi))$

1: $d^s_{Q,}(\pi)_{L,M}$ be the map induced by $d^s(\pi))_{L,M}$ on the quotient $D^s_{Q,}(\Sigma_{L,M}(\pi))$,
2: $d^s_{,A}(\pi)_{L,M}$ to be the restriction of $d^s(\pi)_{L,M}$ to $D^s_{,A}(\Sigma_{L,M}(\pi)$ and
3: $d^s_{Q,A}(\pi)_{L,M}$ to be the restriction of $d^s(\pi)_{L,M}$ to $D^s_{Q,A}(\Sigma_{L,M}(\pi)$.

Definition 1.11. *Let $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ be an s/u-bijective pair for the Smale space (X, φ). 1-We define H^s_* to be the homology of the double complex $(C^s_{Q,A}(\pi)_{L,M}, d^s_{Q,A}(\pi)_{L,M})$ that is, for each integer N, we have*

$$H^s_N = Ker(\oplus s_{L-M=N} d^s_{Q,A}(\pi)_{L,M})/Im(\oplus_{L-M=N+1} d^s_{Q,A}(\pi)_{L,M}).$$

2-We define H^u_* to be the homology of the double complex $(C^u_{A,Q}(\pi)_{L,M}, d^u_{A,Q}(\pi)_{L,M})$ that is, for each integer N, we have

$$H^u_N = Ker(\oplus_{L-M=N} d^u_{A,Q}(\pi)_{L,M}/Im(\oplus_{L-M=N-1} d^s_{A,Q}(\pi)_{L,M}).$$

2 Main Result

The key point in these constructions is that the homology of these complexes is independent of s/u-bijective pair and depends only on (X, φ). We present a number of examples whose homology are computed by Putnam and Bazett.

Example 2.1. 1-Shift of finite type $H^s_N = \begin{cases} D^s(\Sigma, \sigma) & N = 0 \\ 0 & N \neq 0. \end{cases}$, $H^u_N = \begin{cases} D^u(\Sigma, \sigma) & N = 0 \\ 0 & N \neq 0. \end{cases}$

Example 2.2. 2-A hyperbolic toral automorphism $H_0^s(\mathbb{T}^2, \sigma) \cong \mathbb{Z}^2, \;\; H_1^s(\mathbb{T}^2, \sigma) \cong \mathbb{Z}$

Example 2.3. 3-m^∞-solenoid $H_N^s(X, \varphi) = \begin{cases} \mathbb{Z}[1/m] & N = 0, \\ \mathbb{Z} & N = 1, \\ 0 & N \neq 0, 1. \end{cases}$

$\mathbb{Z}[1/m]$ the subgroup of all rational numbers of the form l/m^j, where $l \in \mathbb{Z}$ and $j \geq 0$.

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A chaotic secure communication scheme using adaptive projective synchronization

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Abstract
Most secure communication schemes using chaotic dynamics are based on identical synchronization. In this paper, we present projective synchronization between two chaotic systems via adaptive linear feedback control. Then, we show the possibility of secure communication using adaptive projective synchronization. Finally, numerical simulations show that the obtained theoretic results are feasible and efficient.
Keywords: Projective synchronization, Adaptive control, Secure communication

Mathematics Subject Classification: 74H65

1 Introduction

Chaos is very interesting nonlinear phenomenon, exhibiting sensitive dependence on initial conditions. Because of this property, chaotic behavior and chaos synchronization are beneficial and desirable in many applications such as mixing processes, heat transfer, biological systems and secure communication, etc.

The idea to apply chaotic systems to secure communication has appeared since Pecora and Carroll proposed synchronization of chaotic systems [1]. Just at the start of last decade, the continuous as well as discrete chaotic dynamical systems have been used for the development of cryptosystems. The reason of applying chaos theory in cryptography lies in its intrinsic features. These properties of chaos include: sensitivity to initial condition and control parameters, ergodicity, random like behavior and mixing property etc; by the way, the iteration process is one-way. Thus, it is a natural idea to use chaos as a new source to construct new encryption systems. In general, the algorithms used for cryptographic applications are classified into two, namely, public key (asymmetric methods) cryptography and secret key (symmetric methods) cryptography. Also, based on the structure of the encryption algorithm we can classify cryptosystems into two categories, namely, stream cipher and block cipher [2, 3].

This paper is organized as follows. In Section 2, an adaptive scheme for projective synchronization [4] is proposed. In section 2, Chen system is illustrated to show the effectiveness of the proposed adaptive projective synchronization method. In section 3, a scheme of secure communication based on the adaptive projective synchronization of Chen chaotic system is presented. The unpredictability of the scaling factor in projective synchronization can additionally enhance the security of communication. We conclude the paper in section 4.

2 Adaptive scheme for projective synchronization

Partially linear system, such as Chen system, is defined by a set of ordinary differential equations in which the state vector can be broken into two parts (u, z), where the equation for z is nonlinearly related to the other variable, while the equation for the rate of change of the vector u is linearly related to u through a matrix $M(z)$ that can be depend on the variable z:

$$\begin{aligned} \dot{u} &= M(z).u, \\ \dot{z} &= f(u, z) \end{aligned} \tag{1}$$

We consider two arbitrary dimensional chaotic systems (1) coupled through the variable z:

$$\begin{aligned} \dot{u}_d &= M(z).u_d, \\ \dot{z} &= f(u_d, z) \\ \dot{u}_r &= M(z).u_r, \end{aligned} \tag{2}$$

the subscripts d and r stand for the drive system and response system, respectively, $u_d = (u_d^1, u_d^2, \ldots, u_d^n)^T \in \mathbb{R}^n$, $u_r = (u_r^1, u_r^2, \ldots, u_r^n)^T \in \mathbb{R}^n$, $z \in \mathbb{R}$ is a one-dimensional coupling variable, which is the same in both the drive system and the response system. The matrix $M(z) = [m_{ij}(z)]$ is only dependent on the variable z that is nonlinearly related to the variables in u_d.

Hypothesis 1. Each function $m_{ij}(z)$ is bounded and $m := \max_{i,j} \|m_{i,j}(z)\|, i, j = 1, 2, \ldots, n$, here $\|m_{ij}\| \geq 0$ denotes the maximum absolute values of the function $m_{i,j}$.

If there exists a constant $\alpha(\alpha \neq 0)$ such that $\lim_{t\to\infty} \|u_r - \alpha u_d\| = 0$, then the projective synchronization between the drive system and response systems is achieved, and we call α as "scaling factor". The coupled systems (2) can achieve projective synchronization, but the scaling factor α is unpredictable. For directing the scaling factor onto the desired value α, we design an adaptive feedback controller which is added to the response system of (2):

$$\begin{aligned} \dot{u}_d &= M(z).u_d, \\ \dot{z} &= f(u_d, z) \\ \dot{u}_r &= M(z).u_r - k(u_r - \alpha u_d), \\ \dot{k}_i &= \gamma_i e_i^2 \end{aligned} \tag{3}$$

where $k(u_r - \alpha u_d) = ke = (k_1 e_1, k_2 e_2, \ldots, k_n e_n)^T$, $e_i = u_r^i - \alpha u_d^i$, $i = 1, \ldots, n$ denote the projective synchronization errors of (4), $\gamma_i > 0$, $i = 1, \ldots, n$ are arbitrary constants.

Theorem 2.1. *Under Hypothesis 1, the two identical chaotic system (1) coupled through the variable z in the form as (4) with an adaptive feedback controller can achieve projective synchronization with scaling factor α. That is to say, the bounded solutions starting from arbitrary initial conditions of (4) has asymptotic behavior*

$$u_r - \alpha u_d \to 0 \quad \textit{and} \quad k \to k_0 \quad \textit{as} \quad t \to \infty$$

where k_0 is a constant vector depending on the initial conditions and α is a desired constant.

In order to illustrate the effectiveness of the adaptive feedback controller for projective synchronization of partially linear chaotic system, we select chaotic Chen system as example.

Chen dynamical system is described by

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= (c - a)x + cy - xz, \\ \dot{z} &= xy - bz. \end{aligned}$$

where $a = 35$, $b = 3$ and $c = 28$ are three positive real constants.

3 The application in secure communication

In this section, we will apply the adaptive scheme derived above to secure communication using the Chen system. Assume that $m(t)$ is the message signal, adding it to the right of the first equation for the transmitter (drive system), then we have

$$\begin{aligned} \dot{x} &= a(y-x) + m(t), \\ \dot{y} &= (c-a)x + cy - xz, \\ \dot{z} &= xy - bz. \end{aligned} \tag{4}$$

Select the output $x(t)$ of the system (4) as the transmitted signal, then construct the receiver as follows:

$$\begin{aligned} \dot{x}_1 &= a(y_1 - x_1) + p(t) - k_1(x_1 - \alpha x), \\ \dot{y}_1 &= (c-a)x_1 + cy_1 - x_1 z - k_2(y_1 - \alpha y), \\ \dot{p} &= q(x_1 - \alpha x), \\ \dot{k}_1 &= (x_1 - \alpha x)^2, \\ \dot{k}_2 &= (y_1 - \alpha y)^2, \end{aligned} \tag{5}$$

where q is a positive parameter, $e_1 = x_1 - \alpha x$, $e_2 = y_1 - \alpha y$.

When the transmitter system (4) and the receiver system (5) are synchronized, the message signal can be recovered at the receiver. This can derive the synchronization error $e_3 = p - \alpha m \to 0$ as $t \to \infty$, that is $p(t)/\alpha$ can recover the message signal $m(t)$.

In numerical simulations, we consider a sinusoidal information signal as

$$m(t) = A\sin(wt).$$

Numerical result is given in main paper.

4 Conclusion

In this paper we have discussed a method for constructing synchronized chaotic systems. Furthermore, one encoding-decoding schemes were investigated that is based on chaotic synchronization. This encryption method allows us to recover the information signal exactly. Numerical, and analytical examples of continuous systems were presented to illustrate the basic ideas and to indicate also possible directions of future research.

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On the shadowing property of the nonautonomous dynamical system

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Abstract
Various shadowing properties are considered for sequences of mapping on compact metric space, i.e. nonautonomous dynamical system. We study the relation of h-shadowing and limit shadowing and s-limit shadowing properties for The sequence $\{f_i\}_{i=1}^{\infty}$ on the compact space.

1 Shadowing and limit shadowing

Definition 1.1. *Let (X, d) be a metric space , and $f_i : X \longrightarrow X$ is map,for all $i \geq 1$. The finite or infinite sequence $\{x_0, x_1, x_2, ...\} \subseteq X$ is a δ-pseudo-orbit , for some $\delta > 0$ if $d(f_i(x_{i-1}), x_i) < \delta$, for all $i \geq 1$.*

Definition 1.2. *The sequence $\{f_i\}_{i=1}^{\infty}$ has the shadowing property if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every δ-pseudo-orbit $\{x_0, x_1, x_2, ...\} \subseteq X$ there is $y \in X$ such that for all $i \geq 1$, $d(f_i o f_{i-1} o ... o f_1(y), x_i) < \varepsilon$.*

Definition 1.3. *The sequence $\{f_{i=1}\}_{i=1}^{\infty}$ is h-shadowing if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every δ-pseudo-orbit $\{x_0, x_1, x_2, ...x_n\} \subseteq X$ there is $y \in X$ such that,*

$$d(f_i o f_{i-1} o ... o f_1(y), x_i) < \varepsilon \text{ for all } 1 \leq i \leq n \qquad \text{and} \qquad f_n o f_{n-1} o ... o f_1(y) = x_n.$$

Definition 1.4. *The sequence $\{x_i\}_{i=1}^{\infty}$ is an asymptotic pseudo-orbit if $d(f_i(x_{i-1}), x_i) \longrightarrow 0$ as $i \longrightarrow \infty$. $\{x_i\}_{i=1}^{\infty}$ is an asymptotic δ-pseudo-orbit if both conditions hold.*

Definition 1.5. *The sequence $\{f_i\}_{i=1}^{\infty}$ is limit shadowing if every asymptotic pseudo-orbit $\{x_i\}_{i=1}^{\infty}$ asymptotic shadowed by a point $y \in X$.*

Definition 1.6. *The sequence $\{f_i\}_{i=1}^{\infty}$ has s-limit shadowing property on $Y \subseteq X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that the following two condition hold:*

(1) for every δ-pseudo-orbit there is $y \in X$ such that $\{x_i\}_{i=1}^{\infty}$ is shadowed by y.

(2) for every δ-pseudo-orbit $\{x_i\}_{i=1}^{\infty}$, there is $y \in X$ such that $\{x_i\}_{i=1}^{\infty}$ asymptotically ε-shadowed by y.

Lemma 1.7. *Let (X, d) be a compact space, and $f_i : X \longrightarrow X$ be continuous,for all $i \geq 1$.then the sequence $\{f_i\}_{i=1}^{\infty}$ has shadowing property if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that every finite δ-pseudo-orbit is ε-shadowed.*

Lemma 1.8. *Let (X, d) be a compact space , and $f_i : X \longrightarrow X$ be continuous surjective ,for all $i \geq 1$. If The sequence $\{f_i\}_{i=1}^{\infty}$ has s-limit shadowing properties then $\{f_i\}_{i=1}^{\infty}$ has limit shadowing properties.*

Lemma 1.9. *Let (X, d) be a compact space, and $f_i : X \longrightarrow X$ be continuous ,for all $i \geq 1$. If The sequence $\{f_i\}_{i=1}^{\infty}$ has h-limit shadowing property then $\{f_i\}_{i=1}^{\infty}$ has s-limit shadowing properties.*

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The infinite level normal form of Hopf-zero singularity

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Abstract
The classical normal forms of Hopf-Zero singular vector fields are decomposed to conservative and nonconservative vector fields. These two family are Lie subalgebras for all Hopf-Zero singular vector fields. This decomposition is performed via an sl(2)-representation of normal form vector fields. Assuming a generic condition with regards to quadratic terms, the infinite level normal forms are divided into three cases and the results for one of the three is presented.
Keywords: Normal form theory, Hopf-Zero singularity, sl(2)-representation

Mathematics Subject Classification: 34C20; 34A34; 16W5; 68U99.

1 Introduction

Normal form theory is one of the most important tools for bifurcation and stability analysis of singular vector fields. It has also many other applications that includes revealing the hidden symmetries and classification of vector fields as well as introduction of families of vector fields with similar dynamics. The later contributes to the understanding of possible dynamical properties of vector fields while the former has important applications in real life problems. It can be used for decomposing normal form vector fields into conservative and nonconservative parts that plays a role in the understanding of its dynamics, see *e.g.,* [2, 4–9]. Furthermore, it has also been handy in many techniques common for the global analysis of dynamical systems. The history of normal form theory goes back to more than a century, yet it has been among one of the most topic of active research. There are many research results on different singularities such as Bogdanov-Takens and Hopf singularity. However, there are few results on the simplest normal forms of Hopf-Zero singularity systems while it appears in many engineering problems. Therefore, substantial interests exist for any new research in the field.

In this paper we deal with the simplest normal forms of Hopf-Zero singularity and introduce a sl(2)-representation for such vector fields. In this draft we are concerned with computing the simplest normal form for the Hopf Zero system associated with

$$\dot{x} = \text{h.o.t.},\ \dot{y} = z + \text{h.o.t.},\ \dot{z} = -y + \text{h.o.t.}, \tag{1}$$

where h.o.t. stands for nonlinear terms. There exist invertible changes of variables transforming v into the first level (classical) normal form

$$\begin{cases} \dot{x} = \sum_{i+2j=2}^{\infty} \alpha_j^i x^i (y^2+z^2)^j, \\ \dot{y} = z + \big(\sum_{i+2j=1}^{\infty} x^i (y^2+z^2)^j (\beta_j^i y + \gamma_j^i z\big), \\ \dot{z} = -y + \big(\sum_{i+2j=1}^{\infty} x^i (y^2+z^2)^j (\beta_j^i y - \gamma_j^i z\big), \end{cases} \tag{2}$$

It may be surprising to use the sl(2)-representation of vector fields in normal form computation for a non-nilpotent system while it has been essentially used for nilpotent systems, see *e.g.,* [2, 4, 13].

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This is a new feature of this paper that distinguishes our results from the existing results on normal forms of Hopf-Zero singularity in the literature. There are important contributions on the simplest normal forms of Hopf-Zero singularity [1, 3, 14]. These treat Hopf-Zero singularities with certain nonzero conditions for quadratic terms plus an infinite number of non-equalities between coefficients of its Taylor expansion. Thus, the results are still incomplete and requires further investigation. We have benefited from our introduced sl(2)-representation to obtain an appropriate grading in which puts a certain leading resonant monomial ahead of the other resonant monomials of the same degree. This leads to a much simpler computational problem besides the fact that we only require a nonzero condition with regards to quadratic terms. Our results do not infer from any existing results and the systems considered in this paper consist the most likely systems (*i.e.*, generic systems) of Hopf-Zero singularity systems appearing in real life problems.

The vector fields associated with systems (2) are the first level Hopf-Zero normal forms and denoted by $\mathcal{L}$. This constructs a Lie algebraic structure by the Lie bracket $[v,w]=vw-wv$ for vector fields v and w, see [10–12] for further details. Denote $\mathrm{ad}_M v := [M,v]$ and $\mathrm{ad}_M{}^k v := \mathrm{ad}_M{}^{k-1}\circ \mathrm{ad}_M v$, for any natural number k and $M \in \mathcal{L}$. Define $\mathrm{sl}(2) = \mathrm{span}\{M,N,H\}$, where $N := (y^2+z^2)\frac{\partial}{\partial x}$, $M := -\frac{xy}{(y^2+z^2)}\frac{\partial}{\partial y} - \frac{xz}{(y^2+z^2)}\frac{\partial}{\partial z}$, and $H := 2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$. Now we introduce

$$\begin{aligned}
F_k^l &:= \frac{(-1)^l(k-l+1)!}{2^l(k+2)!}\mathrm{ad}_M{}^l (y^2+z^2)^k N, \quad \text{for} \quad -1 \le l \le k,\\
E_k^l &:= \frac{(-1)^l(k-l)!}{2^l k!}\mathrm{ad}_M{}^l (y^2+z^2)^k H, \quad \text{for} \quad 0 \le l \le k,\\
\Theta_k^l &:= \frac{(-1)^l(k-l)!}{2^l k!}\mathrm{ad}_M{}^l (y^2+z^2)^k M, \quad \text{for} \quad 0 \le l \le k.
\end{aligned}$$

Lemma 1.1. *The associated vector field with the differential system (2) can be written by*

$$v^{(1)} := \Theta_0^0 + \frac{1}{2}F_0^{-1} + \sum \alpha_k^{(1)} F_k^l + \sum \beta_k^{(1)} E_k^l + \sum \gamma_k^{(1)} \Theta_k^l. \tag{3}$$

where the first summation is over $-1 \le l \le k$ *for* $k > 0$, *and the last summations are over* $0 \le l \le k$.

Proof. The proof follows the equalities

$$\begin{aligned}
F_k^l &:= (k+1-l)x^{l+1}(y^2+z^2)^{k-l}\frac{\partial}{\partial x} - \frac{(l+1)}{2}x^l(y^2+z^2)^{k-l}y\frac{\partial}{\partial y} \qquad (4)\\
&\quad -\frac{(l+1)}{2}x^l(y^2+z^2)^{k-l}z\frac{\partial}{\partial z}, \qquad (-1 \le l \le k)\\
E_k^l &:= x^{l+1}(y^2+z^2)^{k-l}\frac{\partial}{\partial x} + \frac{1}{2}x^l(y^2+z^2)^{k-l}y\frac{\partial}{\partial y} + \frac{1}{2}x^l(y^2+z^2)^{k-l}z\frac{\partial}{\partial z}, \quad (0 \le l \le k)\\
\Theta_k^l &:= x^l(y^2+z^2)^{k-l}z\frac{\partial}{\partial y} - x^l(y^2+z^2)^{k-l}y\frac{\partial}{\partial z}, \qquad (0 \le l \le k).
\end{aligned}$$

□

Lemma 1.2. *The following holds true.*

$$\begin{aligned}
[F_k^l, F_n^m] &= \big((m+1)(k+2)-(l+1)(n+2)\big)F_{k+n}^{l+m},\\
[F_k^l, E_n^m] &= \frac{(n+2)\big(m(k+2)-n(l+1)\big)}{(k+n+2)}E_{k+n}^{l+m} - \frac{k(k+2)}{k+n+2}F_{k+n}^{l+m},\\
[F_k^l, \Theta_n^m] &= (m(k+2)-n(l+1))\Theta_{k+n}^{l+m},\\
[E_k^l, E_n^m] &= (n-k)E_{k+n}^{l+m},\\
[E_k^l, \Theta_n^m] &= n\Theta_{k+n}^{l+m},\\
[\Theta_n^m, \Theta_k^l] &= 0.
\end{aligned}$$

Lemma 1.3. *There exists an invertible transformation transforming $v^{(1)}$ into*

$$v^{(2)} = \Theta_0^0 + \frac{1}{2}F_0^{-1} + \sum_{k=1}^{\infty} \alpha_k^{(2)} F_k^k + \sum_{k=1}^{\infty} \beta_k^{(2)} E_k^k + \sum_{k=1}^{\infty} \gamma_k^{(2)} \Theta_k^k. \tag{5}$$

Assume there exist $\alpha_k^{(2)} \neq 0$ and $\beta_k^{(2)} \neq 0$ and

$$r := \min\{k|\alpha_k^{(2)} \neq 0, k \geq 1\} \text{ and } s := \min\{k|\beta_k^{(2)} \neq 0, k \geq 1\}. \tag{6}$$

Furthermore, let $(r = s)$ and define a grading function by

$$\delta(F_k^l) : = \delta(E_k^l) = (k-l)\min\{r,s\} + k, \tag{7}$$

$$\delta(\Theta_k^l) : = \min\{r,s\}(k-l+1) + k + 1. \tag{8}$$

2 Main Result

Theorem 2.1. *If $\frac{\alpha_r^{(2)}}{\beta_s^{(2)}}$ is the non-algebraic number, then there exist invertible transformations that send the differential system given by equation (1) into the infinite level normal form system*

$$\begin{cases} \dot{x} = (y^2+z^2) + \beta_s^{(2)} x^{s+1} + \alpha_r^{(2)} x^{s+1} + \sum_{k=r+1}^{\infty} \alpha_k^{(3)} x^{k+1} + \sum_{k=s+1}^{\infty} \beta_k^{(3)} x^{k+1} \\ \dot{y} = z - \frac{\alpha_r^{(2)}(s+1)}{2} x^s y + \frac{1}{2}\beta_s^{(2)} x^s y + \sum_{k=r+1}^{\infty} -\frac{\alpha_k^{(3)}(k+1)}{2} x^k y + \sum_{k=s+1}^{\infty} \frac{\beta_k^{(3)}}{2} x^k y + \sum_{k=1}^{\infty} \gamma_k^{(3)} x^k z \\ \dot{z} = -y - \frac{\alpha_r^{(2)}(s+1)}{2} x^s z + \frac{1}{2}\beta_s^{(2)} x^s z - \sum_{k=r+1}^{\infty} \frac{\alpha_k^{(3)}(k+1)}{2} x^k z + \sum_{k=s+1}^{\infty} \frac{\beta_k^{(3)}}{2} x^k z - \sum_{k=1}^{\infty} \gamma_k^{(3)} x^k y \end{cases} \tag{9}$$

where $\beta_k^{(3)} = 0$ for $k \equiv -1 \mod (s+1)$or $k \equiv 2s \mod (s+1)$, and $\gamma_k^{(3)} = 0$ for $k \equiv -1$ $\mod (s+1)$.

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A note on non uniformly expanding point and stable ergodicity

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Abstract

In this note we show that if f is a C^2 conservative partial hyperbolic diffeomorphism and x is a stable point for which f is non uniformly expanding on it then f has positive central exponents. As a corollary if such f is accessible then f is stably ergodic.

Keywords: Non-uniformly expanding, stable ergodicity, hyperbolic time

Mathematics Subject Classification: 37D25

1 Introduction

Let M be a smooth compact manifold of dimension at least 2, and let m be a smooth volume measure, that we also call Lebesgue. The set of m-preserving C^r-diffeomorphisms endowed with the C^r topology is denoted by $\mathrm{Diff}^r_m(M)$. Let $f \in \mathrm{Diff}^1_m(M)$. By Oseledets theorem [3], for m-almost every point $x \in M$ there is a splitting $T_x = E^1(x) \oplus ... \oplus E^{l_x}(x)$, and there are numbers $\lambda_1(f,x) > ... > \lambda_{l_x}(f,x)$, called the Lyapunov exponents, such that

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \log \parallel Df^n(x)v \parallel = \lambda_i(f,x) \quad for \quad every \quad v \in E^i(x) \backslash \{0\}.$$

We denote by $R(f)$ the set of all points satisfying the Oseledets theorem. So $R(f)$ has a full measure set.
Let f be a C^2 diffeomorphism of a compact smooth Riemanian manifold M preserving a Lebesque measure m.
f is said to be non uniformly expanding on a set $H \subset M$ if there is $\lambda > 0$ such that for every $x \in H$,

$$\liminf_{n \longrightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \parallel Df(f^j(x))^{-1} \parallel < -\lambda.$$

Definition 1.1. *Given $\sigma < 1$, we say that n is a σ-hyperbolic time for a point $x \in M$ if for all $1 \le k \le n$,*

$$\Pi_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \le \sigma^k.$$

The following propositions is proved in [1].

Proposition 1.2. *Given $0 < \sigma < 1$, there exists $\delta > 0$ such that if n is a σ-hyperbolic time for x, then there exists a neighborhood V_n of x such that*

1- f^n maps V_n diffeomorphically on to the ball of radius δ around $f^n(x)$.

2- For all $1 \leq k < n$ and $y, z \in V_n$,

$$d(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{\frac{k}{2}} d(f^n(y), f^n(z)).$$

For hyperbolic time n the neighborhood V_n in proposition A is called a per-ball.
A point $x \in M$ is said to be stable point if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d(f^n(x), f^n(y)) < \epsilon$ for every positive integer n and every $y \in M$ with $d(x, y) < \delta$.

Lemma 1.3. *Let $x \in M$ be a stable point and f is non uniformly expanding on x. Then there is an open neighborhood V_x of x such that $V_x \subset V_n$ for every σ-hyperbolic time n.*

Proof. Suppose that δ is as proposition A. There is $0 < \delta_1 < \delta$ such that $d(f^k(x), f^k(y)) < \delta$ for every positive integer k and y with $d(x, y) < \delta_1$. Consider $V_x = \{y : d(x, y) < \delta_1\}$ so item 1 of proposition A show that for every pre-ball V_n, $V_x \subset V_n$. □

A diffepmprphism f is partially hyperbolic if the tangent bundle TM can be split into three df-invariant continuous subbundles (distributions)

$$TM = E^s \oplus E^c \oplus E^u.$$

The differential df contracts uniformly over $x \in M$ along the strongly stable subspace $E^s(x)$, it expands uniformly along the strongly unstable subspace $E^u(x)$, and it can either as nonuniform contraction or expansion with weaker rates along the central direction $E^c(x)$. The distribution $E^s(x)$ and $E^u(x)$ are integrable and their integrable manifolds form two transversal foliations of M, the strongly stable and strongly unstable foliations of M, which we denote by W^s and W^u respectively. For every $x \in M$ the leaves $W^s(x)$ and $W^u(x)$ of the foliations containing x are smooth immersed submanifolds of M, called the strongly stable and strongly unstable global manifolds at x. If $y \in W^s(x)$, then $d(f^n(x), f^n(y)) \longrightarrow 0$ with an exponential rate as $n \longrightarrow \infty$, and If $y \in W^u(x)$, then $d(f^n(x), f^n(y)) \longrightarrow 0$ with an exponential rate as $n \longrightarrow -\infty$.

Two points $p, q \in M$ are called accessible if there are points $p = z_0, z_1, ..., z_{l-1}, z_l = q$, $z_i \in M$ such that $z_i \in W^u(z_{i-1})$ or $z_i \in W^s(z_{i-1})$ for $i = 1, ..., l$. The collection of points $z_0, ..., z_l$ is called a us-path connecting p and q and is denoted by $[p, q] = [z_0, ..., z_l]$. Accessibility is an equivalence relation. The diffeomorphism f is said to have the accessibility property if the partition into accessibility classes is trivial(i.e. any two points $p, q \in M$ are accessible).
A partially hyperbolic diffeomorphism with invariant measure μ is said to have negatively central exponents (on a set A) if for μ-a.e. x (in the set A) we have $\chi(x, v) < 0$ for all nonzero $v \in E^c(x)$, where $\chi(x, v)$ is the Lyapunov exponent. The definition of positive central exponents is analogous. In [2] the following theorem has been proved.

Theorem 1.4. *Let f be a C^2 partially hyperbolic diffeomorphism of a compact smooth Riemanian manifold M preserving a smooth measure μ. Assume that f is accessible and has positive central exponents on a set of positive measure. Then f is stably ergodic.*

2 Main Result

Lemma 2.1. *There exists $C > 1$ such that given any per-ball V_n, then for all $y, z \in V_n$,*

$$\frac{1}{C} \leq \frac{\| Df^n_{|E^c_y} \|}{\| Df^n_{|E^c_z} \|} \leq C.$$

Proof. Since f is C^2, so $\log Df$ is (l,ζ)-Holder continuous, for some $l,\zeta>0$. Moreover by Lemma 1, the sum of all $d(f^j(y),f^j(z))^\zeta$ over $0\le j\le n$ is bounded by $2\delta_1/1-\sigma^{\frac{\zeta}{2}}$. So

$$\log\frac{\| Df^n_{|E^c_y}\|}{\| Df^n_{|E^c_z}\|}=\sum_{i=0}^{n-1}(\log\| Df_{|E^c_{f^i(y)}}\|-\log\| Df_{|E^c_{f^i(z)}}\|)$$

$$\le\sum_{i=0}^{n-1} ld(f^i(y),f^i(z))^\zeta\le 2l\delta_1/1-\sigma^{\frac{\zeta}{2}}.$$

Then it suffices to take

$$C=exp(2l\delta_1/1-\sigma^{\frac{\zeta}{2}}).$$

□

Lemma 2.2. *If f is non uniformly expanding for x, then x has infinitely many σ-hyperbolic times.*

Theorem 2.3. *Let f be a C^2 conservative partially hyperbolic diffeomorfhism and $x\in M$ is a stable point. If f is non uniformly expanding on x then f has positive central exponents.*

Proof. Since f is non uniformly expanding on x, so the Lyapunov exponent of x is positive on E^c. By Lemma 2 and 3 there is sequence $\{n_k\}$ of hyperbolic times of x such that for every $y\in V_{n_k}$

$$\frac{1}{C}\le\frac{\| Df^{n_k}_{|E^c_x}\|}{\| Df^{n_k}_{|E^c_y}\|}\le C.$$

We have

$$|\frac{1}{n_k}\log\| Df^{n_k}_{|E^c_x}\|-\frac{1}{n_k}\log\| Df^{n_k}_{|E^c_x}\|\,|<\frac{\log C}{n_k}.\quad (*)$$

Since x is stable point by Lemma 1, there exist a open neighborhood V_x of x such that for every hyperbolic time n, $V_x\subset V_n$. For every $y\in V_x$ if Lyapunov exponent of y on E^c exists then by (*), the Lyapunov exponent of y on E^c is the same Lyapunov of x so is positive. By Oseledec theorem and $m(V_x)>0$, f has positive central exponents. □

By the Theorem 1.4 and 2.3 we have the following theorem.

Theorem 2.4. *Let f be a C^2 conservative partially hyperbolic diffeomorfhism and $x\in M$ is a stable point. If f is accessible and non uniformly expanding on x then f is stably ergodic.*

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Limit shadowing and average shadowing property in linear dynamical systems

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Abstract
In this talk we state that linear dynamical systems having limit shadowing property or average shadowing property are hyperbolic.
Keywords: limit shadowing, average shadowing, hyperbolicity, linear dynamical systems

Mathematics Subject Classification: Primary: 54H20 Secondary: 54F99

1 Introduction

Let (X, d) be a metric space and let $f : X \to X$ be a homeomorphism(a discrete dynamical system on X). A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is called an orbit of f, denote by $o(x, f)$, if for each $n \in \mathbb{Z}$, $x_{n+1} = f(x_n)$ and we call it a δ-pseudo-orbit of f if,

$$d(f(x_n), x_{n+1}) \leq \delta, \forall n \in \mathbb{Z}.$$

The homeomorphism f is said to have the shadowing property if for each $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit $\{x_n\}_{n\in\mathbb{Z}}$ is ϵ shadowed by the orbit $\{f^n(y) : n \in \mathbb{Z}\}$, for some y in X, i.e for all $n \in \mathbb{Z}$ we have

$$d(f^n(y), x_n) < \epsilon.$$

Limit shadowing was studied by some authors for example see [7].

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is said to be an asymptotic pseudo orbit for f if

$$\underset{|i| \longrightarrow \infty}{d(f(x_i), x_{i+1})} \longrightarrow 0.$$

A homeomorphism f is said to have limit shadowing property if for every asymptotic pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$, there exists a $x \in M$ s.t.

$$\underset{|n| \longrightarrow \infty}{d(f^n(x), x_n)} \longrightarrow 0.$$

Fix a non-singular matrix A and consider the corresponding linear diffeomorphism

$$\varphi(x) = Ax$$

of the space $\mathbb{C}^n$. As usual, we say that a matrix is hyperbolic if its eigenvalues A; satisfy the inequalities $|\lambda_i| \neq 1$

In [3], Blank introduced the notion of average-shadowing property (see [4]) and some authors have studied about it (see [5, 6]). Fix $\delta > 0$ a sequence $\{x_i\}_{i=0}^{\infty}$ in M is called a δ-average pseudo orbit of $f \in \mathrm{Diff}^1(M)$ if there is a natural number $N = N(\delta) > 0$. such that for all $n \geq N$, and $k \in \mathbb{N}$,

$$\frac{1}{n}\sum_{i=1}^{n} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

We say that f has the average shadowing property if for every $\epsilon > 0$ there is a $\delta > 0$ such that every δ-average pseudo orbit $\{x_i\}_{i=0}^{\infty}$ is ϵ-shadowed in average by some $z \in M$, that is,

$$\lim_{n\to\infty} \sup \frac{1}{n}\sum_{i=1}^{n} d(f^i(z), x_i) < \epsilon.$$

2 Main Result

Lemma 2.1. *Let (X, d) be a metric space. Assume that for two dynamical systems f, g on X there exists homeomorphism h, such that $f \circ h = h \circ g$. Then f has limit shadowing property if and only if g has limit shadowing property.*

Lemma 2.2. *(Theorem 3.2.1 in [1]) Let A be a non hyperbolic matrix, and λ be an eigevvalue of A with $|\lambda| = 1$. then there exists a nonsingular $J = T^{-1}AT$ is a jordan form of A and the matrix J has the form*

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}.$$

where B is the nonsingular complex matrix with the form

$$\begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}$$

Theorem 2.3. *If diffeomorphism $\varphi(x) = Ax$ has the limit shadowing property, then A is hyperbolic.*

Proof. suppose that A is not hyperbolic. Then A has an eigenvalue λ such that $|\lambda| = 1$. we will get a contradiction. By lemma 2.1 and 2.2 we may assume that the first row of A is

$$(\lambda, 0, 0. \cdots, 0).$$

We Construct an asymptotic pseudo orbit sequence $\zeta = \{x_k \in \mathbb{C}^n : k \in \mathbb{Z}\}$ as follows. Denote by $x^{(i)}$ the its component of a vector $x \in \mathbb{C}^n$. Put $x_0^{(1)} = 1$ and $x_0^{(i)} = 0$ for $i = 2, \cdots, n$. For $k \geq 0$ we define $x_{k+1}^{(1)}$ as follows

$$x_{k+1}^{(1)} = Ax_k^{(1)}(1 + \frac{1}{2k|x_k^{(1)}|}).$$

for $k \leq 0$, consider

$$x_{k-1}^{(1)} = A^{-1}x_k^{(1)}(1 + \frac{1}{2k|k_k^{(1)}|}).$$

For $i = 2, \cdots, n$, we consider

$$\begin{cases} x_{k+1}^{(i)} = (Ax_k)^{(i)} & k \geq o \\ x_{k-1}(i) = (A^{-1}x_k)^{(i)} & k \leq 0. \end{cases}$$

We have

$$\begin{aligned}|Ax_k - x_{k+1}| &= |(Ax_k - x_{k+1})^{(1)}| \\ &= |\frac{Ax_k^{(1)}}{2k|x_k^{(1)}|}| \\ &= \frac{1}{2k}.\end{aligned}$$

Hence $\{x_k\}_{k\in\mathbb{Z}}$ is an asymptotic pseudo orbit. Since $\varphi(x) = Ax$ has limit shadowing property, So there exists a point $x \in \mathbb{C}^n$ such that

$$\begin{array}{cc}|A^m x - x_m| & \longrightarrow 0 \\ x \longmapsto +\infty & \end{array}.$$

Note that

$$(A^m x)^{(1)} = \lambda^m x^{(1)} \quad \forall m \in \mathbb{Z}.$$

So we have

$$|\lambda^m x^{(1)} - x_m^{(1)}| \longrightarrow 0. \tag{1}$$

It is easy to see that

$$|x_{k+1}^{(1)}| = |x_k^{(1)}| + \frac{1}{k}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergence so

$$|x_k^{(1)}| \longrightarrow \infty \quad as \quad k \longrightarrow +\infty.$$

But the value $|\lambda^m x^{(1)}|$ does not depend on m. So

$$\begin{array}{cc}|\lambda^m x^{(1)} - x_m^{(1)}| & \longrightarrow +\infty \\ m \longmapsto +\infty & \end{array}.$$

□

Lemma 2.4. *(lemma 2.2 in [2]) Let $\varphi(x)$ be a continuous map on X. For $\epsilon > 0$, there is a $\delta > 0$ such that for sequences $\{x_i\}_{i\in\mathbb{N}}$ if*

$$\limsup_{x\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} d(x_i, y_i) < \delta$$

then

$$\limsup \frac{1}{n} \sum_{i=1}^{n-1} d(\varphi(x_i), \varphi(y_i)) < \epsilon.$$

Lemma 2.5. *Let (X, d) be a metric space. Assume that for two dynamical systems f, g on X there exists homeomorphism h, such that $f \circ h = h \circ g$. Then f has average shadowing property if and only if g has average shadowing property.*

Theorem 2.6. *If diffeomorphism $\varphi(x) = Ax$ has the average shadowing property, then A is hyperbolic.*

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